Constrained optimisation

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Constrained optimisation

Numerical optimisation

Constrained optimisation

We discuss two strategies for solving constrained minimisation problems

The penalty method

- Problems with both equality and inequality constraints

The augmented Lagrangian method

- Problems with equality constraints only

The two methods allow the solution of relatively simple problems

- Basic tools for more robust and complex algorithms

Constrained optimisation (cont.)

Definition

Let $f : \mathcal{R}^n \to \mathcal{R}$ with $n \geq 1$ be a cost or objective function

The constrained optimisation problem

$$\min_{\mathbf{x} \in \Omega \subset \mathcal{R}^n} f(\mathbf{x}) \tag{1}$$

Ω is a closed subset determined by equality or inequality constraints

Given functions $h_i : \mathcal{R}^n \to \mathcal{R}$, for $i = 1, \dots, p$

$$\rightsquigarrow \quad \Omega = \big\{\mathbf{x} \in \mathcal{R}^n : h_i(\mathbf{x}) = 0, \text{ for } i = 1, \dots, p\big\} \tag{2}$$

Given functions $g_j : \mathcal{R}^n \to \mathcal{R}$, for $j = 1, \dots, g$

$$\rightsquigarrow \quad \Omega = \big\{\mathbf{x} \in \mathcal{R}^n : g_j(\mathbf{x}) \geq 0, \text{ for } j = 1, \dots, q\big\} \tag{3}$$

p and q are natural numbers

Constrained optimisation (cont.)

More generally,

$$\min_{\mathbf{x} \in \Omega \subset \mathcal{R}^n} f(\mathbf{x}) \quad (4)$$

Ω a closed subset determined by both equality and inequality constraints

$$\Omega = \Big\{\mathbf{x} \in \mathcal{R}^n : h_i(\mathbf{x}) = 0 \text{ for } \underbrace{i \in \mathcal{I}_h}_{i=1,\dots,p} \text{ and } g_j(\mathbf{x}) \geq 0 \text{ for } \underbrace{j \in \mathcal{I}_g}_{j=1,\dots,q}\Big\}$$

Constrained optimisation (cont.)

Definition

*Let $f : \mathcal{R}^n \rightarrow \mathcal{R}$ with $n \geq 1$ be a **cost** or **objective function***

*The general **constrained optimisation** problem*

$$\begin{aligned} &\min_{\mathbf{x} \in \mathcal{R}^n} f(\mathbf{x}) \\ &\textit{subjected to} \\ &h_i(\mathbf{x}) = 0, \quad \textit{for all } i \in \mathcal{I}_h \\ &g_j(\mathbf{x}) \geq 0, \quad \textit{for all } j \in \mathcal{I}_g \end{aligned} \quad (5)$$

The two sets $\mathcal{I}_h = \{1, 2, \dots, p\}$ and $\mathcal{I}_g = \{1, 2, \dots, q\}$

- ⇝ In Equation (3), we used $\mathcal{I}_h = \emptyset$
- ⇝ In Equation (2), we used $\mathcal{I}_g = \emptyset$

Constrained optimisation (cont.)

Suppose that $f \in \mathcal{C}^1(\mathcal{R}^n)$ and that h_i and g_j are class $\mathcal{C}^1(\mathcal{R}^n)$, for all i, j

Points $\mathbf{x} \in \Omega \subset \mathcal{R}$ that satisfy all the constraints are **feasible points**

- ⇝ The closed subset Ω is the set of all feasible points

Consider a point $\mathbf{x}^* \in \Omega \subset \mathcal{R}^n$,

$$f(\mathbf{x}^*) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega \quad (6)$$

Point $\mathbf{x}^*$ is said to be a **global minimiser** for the problem

Consider a point $\mathbf{x}^* \in \Omega \subset \mathcal{R}^n$,

$$f(\mathbf{x}^*) \leq f(\mathbf{x}), \quad \forall \mathbf{x} \in B_r(\mathbf{x}^*) \cap \Omega \quad (7)$$

Point $\mathbf{x}^*$ is said to be a **local minimiser** for the problem

- ⇝ $B_r(\mathbf{x}^*) \in \mathcal{R}^n$ is a ball centred in $\mathbf{x}^*$, radius $r > 0$

Constrained optimisation (cont.)

A constraint is said to be **active** at $\mathbf{x} \in \Omega$ if it is satisfied with equality

- Active constraints at $\mathbf{x}$ are the $h_i(\mathbf{x}) = 0$ and the $g_j(\mathbf{x}) = 0$

Let Ω be a non-empty, bounded and closed set in $\mathcal{R}^n$

Weierstrass guarantees existence of a maximum and a minimum for f in Ω

- ⇝ The general constrained optimisation problem admits a solution

Constrained optimisation (cont.)

Example

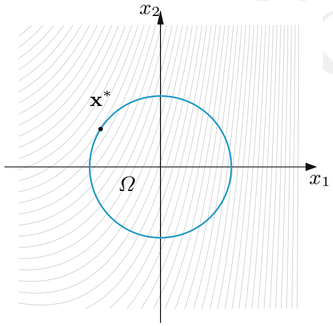
Consider the minimisation of function $f(\mathbf{x})$ under equality constraint $h_1(\mathbf{x})$

Let

$$f(\mathbf{x}) = 3/5x_1^2 + 1/2x_1x_2 - x_2 + 3x_1$$

Let

$$h_1(\mathbf{x}) = x_1^2 + x_2^2 - 1 = 0$$



Global minimiser $\mathbf{x}^*$ constrained to Ω

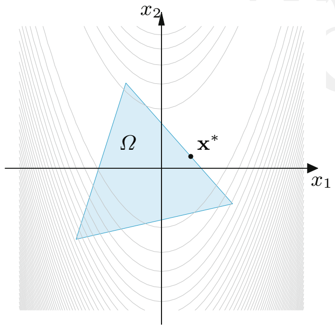
- Contour lines of the cost $f(\mathbf{x})$
- Admissibility set $\Omega \in \mathcal{R}^2$

Constrained optimisation (cont.)

Example

Minimise $f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, under inequality constraints

$$g_1(\mathbf{x}) = -34x_1 - 30x_2 + 19 \geq 0$$
$$g_2(\mathbf{x}) = +10x_1 - 05x_2 + 11 \geq 0$$
$$g_3(\mathbf{x}) = +03x_1 + 22x_2 + 08 \geq 0$$



Global minimiser $\mathbf{x}^*$ constrained to Ω

- Contour lines of the cost $f(\mathbf{x})$
- Admissibility set $\Omega \in \mathcal{R}^2$

Constrained optimisation (cont.)

Definition

Strongly convexity

*The condition for function $f : \Omega \subseteq \mathcal{R}^n \rightarrow \mathcal{R}$ to be **strongly convex** in Ω*

f is strongly convex if $\exists \rho > 0$ such that $\forall(\mathbf{x}, \mathbf{y}) \in \Omega$ and $\forall \alpha \in [0, 1]$

$$\underbrace{f\big[\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\big] \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})}_{Convexity} - \alpha(1-\alpha)\rho||\mathbf{x} - \mathbf{y}||^2 \qquad (8)$$

Strong convexity reduces to the usual convexity when $\rho = 0$

Constrained optimisation (cont.)

Proposition

Optimality conditions

Let $\Omega \subset \mathcal{R}^n$ be a convex set and let $\mathbf{x}^* \in \Omega$ be such that $f \in \mathcal{C}^1\big[B_r(\mathbf{x}^*)\big]$

Suppose that $\mathbf{x}^*$ is a local minimiser for constrained minimisation,

$$\nabla f(\mathbf{x}^*)^\top(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \Omega \qquad (9)$$

If f is convex in Ω and (9) is satisfied, then $\mathbf{x}^*$ is a global minimiser

Suppose that we require Ω to be closed and f to be strongly convex
$\rightsquigarrow$ *It can be shown that the minimiser $\mathbf{x}^*$ is also unique*

Constrained optimisation (cont.)

There are many algorithms for solving constrained minimisation problems

Many search for the stationary points of the **Lagrangian function**

⇝ The **KKT** or **Karush-Kuhn-Tucker points**

Definition

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x})$$

The ***Lagrangian function*** *associated with the constrained minimisation*

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) - \sum_{i \in \mathcal{I}_h} \lambda_i h_i(\mathbf{x}) - \sum_{j \in \mathcal{I}_g} \mu_j g_j(\mathbf{x}) \qquad (10)$$

$\boldsymbol{\lambda}$ *and* $\boldsymbol{\mu}$ *are* ***Lagrangian multipliers***

⇝ $\boldsymbol{\lambda} = (\lambda_i)$, *for* $i \in \mathcal{I}_h$

⇝ $\boldsymbol{\mu} = (\mu_i)$, *for* $j \in \mathcal{I}_g$

They are (weights) associated with the equality and inequality constraints

Constrained optimisation (cont.)

Definition

Karush-Kuhn-Tucker conditions

A point $\mathbf{x}^*$ *is said to be a KKT point for* $\mathcal{L}$ *if there exist* $\boldsymbol{\lambda}^*$ *and* $\boldsymbol{\mu}^*$ *such that the triplet* $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ *satisfies the Karush-Kuhn-Tucker conditions*

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \nabla f(\mathbf{x}^*) - \sum_{i \in \mathcal{I}_h} \lambda_i^* \nabla h_i(\mathbf{x}^*) - \sum_{j \in \mathcal{I}_g} \mu_j^* \nabla g_j(\mathbf{x}^*) = \mathbf{0}$$

$$\begin{aligned} h_i(\mathbf{x}^*) &= 0, \quad \forall i \in \mathcal{I}_h \\ g_i(\mathbf{x}^*) &\geq 0, \quad \forall j \in \mathcal{I}_g \\ \mu_j^* &\geq 0, \quad \forall j \in \mathcal{I}_g \\ \mu_j^* g_j(\mathbf{x}^*) &= 0, \quad \forall j \in \mathcal{I}_g \end{aligned}$$

Constrained optimisation (cont.)

Let $\mathbf{x}$ be some given point

Consider $\nabla h_i(\mathbf{x})$ and $\nabla g_j(\mathbf{x})$ associated with active constraints in $\mathbf{x}$

- Suppose that these gradients are linearly independent

Linear independence (**constraint**) **qualification** (**LI**(**C**)**Q**) in $\mathbf{x}$

Constrained optimisation (cont.)

Theorem

First-order KKT conditions

Let $\mathbf{x}^*$ *be a local minimum for the constrained problem*

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$
$$\text{subjected to}$$
$$h_i(\mathbf{x}) = 0, \forall i \in \mathcal{I}_h$$
$$g_j(\mathbf{x}) \geq 0, \forall j \in \mathcal{I}_g$$

Let functions f, h_i *and* g_j *be* $\mathcal{C}^1(\Omega)$ *and let the constraints be LIQ in* $\mathbf{x}^*$

There exist $\boldsymbol{\lambda}^*$ *and* $\boldsymbol{\mu}^*$ *such that* $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ *is a KKT point*

Constrained optimisation (cont.)

In the absence of inequality constraints, the Lagrangian takes the form

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i \in \mathcal{I}_h} \lambda_i \nabla h_i(\mathbf{x}^*)$$

The KKT conditions are known as **Lagrange (necessary) conditions**

$$\begin{aligned} \nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) &= \nabla f(\mathbf{x}^*) - \sum_{i \in \mathcal{I}_h} \lambda_i^* \nabla h_i(\mathbf{x}^*) = \mathbf{0} \\ h_i(\mathbf{x}^*) &= 0, \forall i \in \mathcal{I}_h \end{aligned} \quad (11)$$

Constrained optimisation (cont.)

Remark

Sufficient conditions for a KKT point $\mathbf{x}$ to be a minimiser of f in Ω

- ⇝ Knowledge about the Hessian of the Lagrangian is required

Alternatively, we need strict convexity hypothesis on f and the constraints

In general, it is possible to reformulate a constrained optimisation problem

- As an unconstrained optimisation problem

The idea is to replace the original problem by a sequence of subproblems in which the constraints are represented by terms added to the objective

- ⇝ **(Quadratic) Penalty function**
- ⇝ **Augmented Lagrangian**

The penalty method

Constrained optimisation

The penalty method

Consider solving the general constrained optimisation problem

$$\min_{\mathbf{x} \in \mathcal{R}^n} f(\mathbf{x})$$

subjected to

$$\begin{aligned} h_i(\mathbf{x}) &= 0, \quad \forall i \in \mathcal{I}_h \\ g_j(\mathbf{x}) &\geq 0, \quad \forall j \in \mathcal{I}_g \end{aligned}$$

We reformulate it as an unconstrained optimisation problem

Definition

The modified ***penalty function****, for a fixed* ***penalty parameter*** $\alpha > 0$

$$\mathcal{P}_\alpha(\mathbf{x}) = f(\mathbf{x}) + \frac{\alpha}{2} \sum_{i \in \mathcal{I}_h} {h_i}^2(\mathbf{x}) + \frac{\alpha}{2} \sum_{j \in \mathcal{I}_g} \Big[\max\{-g_j(\mathbf{x}), 0\}\Big]^2 \quad (12)$$

The method adds a multiple of the square of the violation of each constraint

- Terms are zero when $\mathbf{x}$ does not violate the constrain

The penalty method (cont.)

$$\mathcal{P}_\alpha(\mathbf{x}) = f(\mathbf{x}) + \frac{\alpha}{2}\sum_{i\in\mathcal{I}_h} {h_i}^2(\mathbf{x}) + \frac{\alpha}{2}\sum_{j\in\mathcal{I}_g}\Big[\max\{-g_j(\mathbf{x}),0\}\Big]^2$$

By making the coefficients larger, we penalise violations more severely

- This forces the minimiser closer to the feasible region

Consider the situation in which the constraints are not satisfied at $\mathbf{x}$

- The sums quantify how far point $\mathbf{x}$ is from the feasibility set Ω
- A large α heavily penalises such a violation

If $\mathbf{x}^*$ is a solution to the constrained problem, $\mathbf{x}^*$ is a minimiser of $\mathcal{P}$

Conversely, under some regularity hypothesis for f, h_i and g_i,

$$\lim_{\alpha\to\infty} \mathbf{x}^*(\alpha) = \mathbf{x}^*,$$

$\mathbf{x}^*(\alpha)$ denotes the minimiser of $\mathcal{P}_\alpha(\mathbf{x})$

As $\alpha >> 1$, $\mathbf{x}^*(\alpha)$ is a good approximation of $\mathbf{x}^*$

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The penalty method (cont.)

Example

Consider the minimisation of function $f(\mathbf{x})$ under equality constraint $h_1(\mathbf{x})$

Let

$$f(\mathbf{x}) = x_1 + x_2$$

Let

$$h_1(\mathbf{x}) = x_1^2 + x_2^2 - 2 = 0$$

Consider the quadratic penalty function

$$\mathcal{P}_\alpha(\mathbf{x}) = (x_1 + x_2) + \frac{\alpha}{2}(x_1^2 + x_2^2 - 2)^2$$

The minimiser is $(-1,-1)'$

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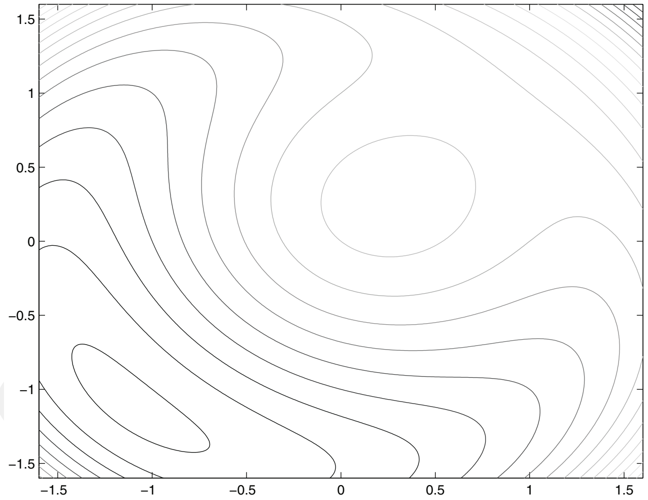
The penalty method (cont.)

The plot of the contour of the penalty function for $\alpha = 1$



There is a local minimiser near $(0.3, 0.3)'$

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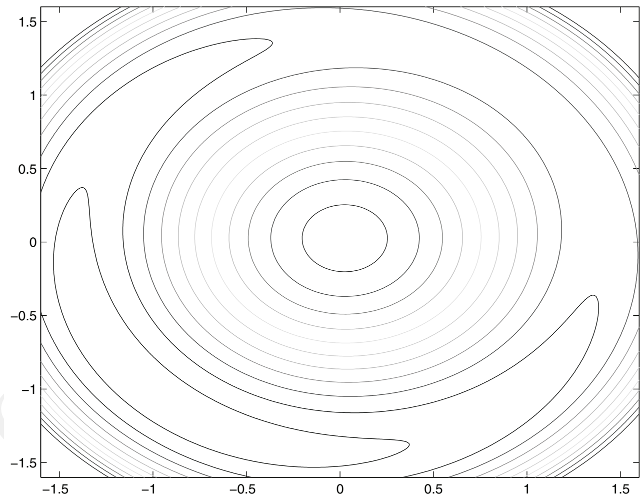
The penalty method (cont.)

The plot of the contour of the penalty function for $\alpha = 10$



Points outside the feasible region suffer a much greater penalty

The penalty method (cont.)

Not advised (instability) to minimise $\mathcal{P}_\alpha(\mathbf{x})$ directly for large values of α

Rather, consider an increasing and unbounded sequence of parameters $\{\alpha_k\}$

- For each α_k, calculate an approximation $\mathbf{x}^{(k)}$ of the solution $\mathbf{x}^*(\alpha_k)$ to the unconstrained optimisation problem $\min_{\mathbf{x}\in\mathcal{R}^n} \mathcal{P}_{\alpha_k}(\mathbf{x})$

$$\mathbf{x}^{(k)} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \mathcal{P}_{\alpha_k}(\mathbf{x})$$

- At step k, α_{k+1} is a chosen as a function of α_k (say, $\alpha_{k+1} = \delta\alpha_k$, for $\delta \in [1.5, 2]$) and $\mathbf{x}^{(k)}$ is used to initialise the minimisation at step $k+1$

In the first iterations there is no reason to believe that the solution to $\min_{\mathbf{x}\in\mathcal{R}^n} \mathcal{P}_{\alpha_k}(\mathbf{x})$ should resemble the correct solution to the original problem

- This supports the idea of searching for an inexact solution to $\min_{\mathbf{x}\in\mathcal{R}^n} \mathcal{P}_{\alpha_k}(\mathbf{x})$ that differs from the exact one, $\mathbf{x}^{(k)}$, a small ε_k

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The penalty method (cont.)

```
% PENALTY Constrained optimisation with penalty function
% [X,ERR,K]=PFUNCTION(F,GRAD_F,H,GRAD_H,G,GRAD_G,X_0,TOL,...
%                     KMAX,KMAXD,TYP)
%  Approximate a minimiser of the cost function F
%  under constraints H=0 and G>=0
%
% X0 is initial point, TOL is tolerance for stop check
% KMAX is the maximum number of iterations
% GRAD_F, GRAD_H, and GRAD_G are the gradients of F, H, and G
% H and G, GRAD_H and GRAD_G can be initialised to []
%
% For TYP=0 solution by FMINSEARCH M-function
%
% For TYP>0 solution by a DESCENT METHOD
%  KMAXD is maximum number of iterations
%  TYP is the choice of descent directions
%  TYP=1 and TYP=2 need the Hessian (or an approx. at k=0)
%  [X,ERR,K]=PFUNCTION(F,GRAD_F,H,GRAD_H,G,GRAD_G,X_0,TOL,...
%                      KMAX,KMAXD,TYP,HESS_FUN)
%  For TYP=1 HESS_FUN is the function handle associated
%  For TYP=2 HESS_FUN is a suitable approx. of Hessian at k=0
```

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The penalty method (cont.)

```
function [x,err,k]=pFunction(f,grad_f,h,grad_h,g,grad_g,...
                              x_0,tol,kmax,kmaxd,typ,varargin)

xk=x_0(:); mu_0=1.0;

if typ==1; hess=varargin{1};
 elseif typ==2; hess=varargin{1};
 else; hess=[]; end
if ~isempty(h), [nh,mh]=size(h(xk)); end
if ~isempty(g), [ng,mg]=size(g(xk)); end

err=1+tol; k=0; muk=mu_0; muk2=muk/2; told=0.1;

while err>tol && k<kmax
 if typ==0
  options=optimset('TolX',told);
  [x,err,kd]=fminsearch(@P,xk,options); err=norm(x-xk);
 else
  [x,err,kd]=dScent(@P,@grad_P,xk,told,kmaxd,typ,hess);
  err=norm(grad_P(x));
 end

 if kd<kmaxd; muk=10*muk; muk2=0.5*muk;
 else muk=1.5*muk; muk2=0.5*muk; end

 k=1+k; xk=x; told=max([tol,0.10*told]);
end
```

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The penalty method (cont.)

```
function y=P(x) % This function is nested inside pFunction

y=fun(x);
if ~isempty(h); y=y+muk2*sum((h(x)).^2); end
if ~isempty(g); G=g(x);
 for j=1:ng
  y=y+muk2*max([-G(j),0])^2;
 end
end
```

```
function y=grad_P(x) % This function is nested in pFunction

y=grad_fun(x);
if ~isempty(h), y=y+muk*grad_h(x)*h(x); end
if ~isempty(g), G=g(x); Gg=grad_g(x);
 for j=1:ng
  if G(j)<0
   y=y+muk*Gg(:,j)*G(j);
  end
 end
end
```

The augmented Lagrangian

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The augmented Lagrangian

Consider a minimisation problem with equality constraints only ($\mathcal{I}_g = \emptyset$)

$$\min_{\mathbf{x} \in \mathcal{R}^n} f(\mathbf{x})$$

subjected to

$$h_i(\mathbf{x}) = 0, \forall i \in \mathcal{I}_h$$

Definition

*Define the **augmented Lagrangian** objective function*

$$\mathcal{L}_A(\mathbf{x}, \boldsymbol{\lambda}, \alpha) = f(\mathbf{x}) - \sum_{i \in \mathcal{I}_h} \lambda_i h_i(\mathbf{x}) + \alpha/2 \sum_{i \in \mathcal{I}_h} {h_i}^2(\mathbf{x}) \qquad (13)$$

$\alpha > 0$ is a suitable coefficient

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The augmented Lagrangian (cont.)

$$\mathcal{L}_A(\mathbf{x}, \boldsymbol{\lambda}, \alpha) = f(\mathbf{x}) - \sum_{i \in \mathcal{I}_h} \lambda_i h_i(\mathbf{x}) + \alpha/2 \sum_{i \in \mathcal{I}_h} {h_i}^2(\mathbf{x})$$

Constrained optimisation using the augmented Laplacian is iterative

α_0 and $\boldsymbol{\lambda}^{(0)}$ are set arbitrarily, then build a sequence of parameters $\alpha_k \to \infty$

$\alpha_k \to \infty$ is st $\{(\mathbf{x}^{(k)}, \boldsymbol{\lambda}^{(k)})\}$ converges to a KKT point for the Lagrangian

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i \in \mathcal{I}_h} \lambda_i h_i(\mathbf{x})$$

At the k-th iteration, for a given α_k and for a given $\boldsymbol{\lambda}^{(k)}$, we compute

$$\mathbf{x}^{(k)} = \arg\min_{\mathbf{x} \in \mathcal{R}^n} \mathcal{L}_A\big[\mathbf{x}, \boldsymbol{\lambda}^{(k)}, \alpha_k\big] \qquad (14)$$

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The augmented Lagrangian (cont.)

We get multipliers $\boldsymbol{\lambda}^{(k+1)}$ from the gradient of the augmented Lagrangian

- We set it to be equal to zero

$$\nabla_{\mathbf{x}} \mathcal{L}_A\big[\mathbf{x}^{(k)}, \boldsymbol{\lambda}^{(k)}, \alpha_k\big] = \nabla f\big[\mathbf{x}^{(k)}\big] - \sum_{i \in \mathcal{I}_h} \Big\{\lambda_i^{(k)} - \alpha_k h_i\big[\mathbf{x}^{(k)}\big]\Big\} \nabla h_i\big[\mathbf{x}^{(k)}\big]$$

By comparison with optimality condition

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \nabla f(\mathbf{x}^*) - \sum_{i \in \mathcal{I}_h} \lambda_i^* \nabla h_i(\mathbf{x}^*) = \mathbf{0}$$

$$h_i(\mathbf{x}^*) = 0, \quad \forall i \in \mathcal{I}_h$$

We identify $\lambda_i^{(k)}$ as $\lambda_i^{(k)} - \alpha_k h_i\big[\mathbf{x}^{(k)}\big] \simeq \lambda_i^*$

We thus define,

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} - \alpha_k h_i\big[\mathbf{x}^{(k)}\big] \qquad (15)$$

We get $\mathbf{x}^{(k+1)}$ by solving with k replaced by $k+1$

$$\mathbf{x}^{(k)} = \arg\min_{\mathbf{x} \in \mathcal{R}^n} \mathcal{L}_A\big[\mathbf{x}, \boldsymbol{\lambda}^{(k)}, \alpha_k\big]$$

The augmented Lagrangian (cont.)

Given α_0 (typically, $\alpha_0 = 1$), given ε_0 (typically $\varepsilon_0 = 1/10$), given $\overline{\varepsilon} > 0$, given $\mathbf{x}_0^{(0)} \in \mathbb{R}^n$ and given $\boldsymbol{\lambda}_0^{(0)} \in \mathbb{R}^p$, for $k = 0, 1, \dots$ until convergence

Pseudo-code

Compute an approximated solution

$$\mathbf{x}^{(k)} = \underset{\mathbf{x}\in\mathbb{R}^n}{arg\ min}\ \mathcal{L}_A\big[\mathbf{x}, \boldsymbol{\lambda}^{(k)}, \alpha_k\big]$$

(Using the initial point $\mathbf{x}_0^{(0)}$ and tolerance ε_k)

If $\big|\big|\nabla_{\mathbf{x}}\mathcal{L}_A\big[\mathbf{x}^{(k)}, \boldsymbol{\lambda}^{(k)}, \alpha_k\big]\big|\big| \leq \overline{\varepsilon}$
  Set $\mathbf{x}^* = \mathbf{x}^{(k)}$ *(convergence)*
else
  Compute $\lambda_i^{(k+1)} = \lambda_i^{(k)} - \mu_k h_i\big[\mathbf{x}^{(k)}\big]$
  Choose $\alpha_{k+1} > \alpha_k$
  Choose $\varepsilon_{k+1} < \varepsilon_k$
  Set $\mathbf{x}_0^{(k+1)} = \mathbf{x}^{(k)}$
Endif

The augmented Lagrangian (cont.)

The implementation of the algorithm

```
% ALGRNG Constrained optimisation with augmented Lagrangian
% [X,ERR,K]=ALGRNG(F,GRAD_F,H,GRAD_H,X_0,LAMBDA_0,...
%                  TOL,KMAX,KMAXD,TYP)
%  Approximate a minimiser of the cost function F
%  under equality constraints H=0
%
% X_0 is initial point, TOL is tolerance for stop check
% KMAX is the maximum number of iterations
% GRAD_F and GRAD_H are the gradients of F and H
%
% For TYP=0 solution by FMINSEARCH M-function
% FOR TYP>0 solution by a DESCENT METHOD
%  KMAXD is maximum number of iterations
%  TYP is the choice of descent directions
%  TYP=1 and TYP=2 need the Hessian (or an approx. at k=0)
```


The augmented Lagrangian (cont.)

```
function [x,err,k]=aLgrng(f,grad_f,h,grad_h,x_0,lambda_0,...
                          tol,kmax,kmaxd,typ,varargin)

mu_0=1.0;

if typ==1; hess=varargin{1};
 elseif typ==2; hess=varargin{1};
 else; hess=[]; end

err=1+tol+1; k=0; xk=x_0(:); lambdak=lambda_0(:);

if ~isempty(h); [nh,mh]=size(h(xk)); end

muk=mu_0; muk2=muk/2; told=0.1;

while err>tol && k<kmax
 if typ==0
  options=optimset ('TolX',told);
  [x,err,kd]=fminsearch(@L,xk,options); err=norm(x-xk);
 else
  [x,err,kd]=descent(@L,@grad_L,xk,told,kmaxd,typ,hess);
  err=norm(grad_L(x));
 end

 lambdak=lambdak-muk*h(x);
 if kd<kmaxd; muk=10*muk; muk2=0.5*muk;
 else muk=1.5*muk; muk2=0.5*muk; end

 k=1+k; xk=x; told=max([tol,0.10*told]);
```


The augmented Lagrangian (cont.)

```
function y=L(x) % This function is nested inside aLgrng

y=fun(x);
if ~isempty(h)
 y=y-sum(lambdak'*h(x))+muk2*sum((h(x)).^2);
end
```

```
function y=grad_L(x) % This function is nested inside aLgrng

y=grad_fun(x);
if ~isempty(h)
 y=y+grad_h(x)*(muk*h(x)-lambdak);
end
```

`lambda_0` contains the initial vector $\boldsymbol{\lambda}^{(0)}$ of Lagrange multipliers

The augmented Lagrangian (cont.)

Example

```
fun = @(x) 0.6*x(1).^2 + 0.5*x(2).*x(1) - x(2) + 3*x(1);
grad_fun = @(x) [1.2*x(1) + 0.5*x(2) + 3; 0.5*x(1) - 1];

h = @(x) x(1).^2 + x(2).^2 - 1;
grad_h = @(x) [2*x(1); 2*x(2)];

x_0 = [1.2,0.2]; tol = 1e-5; kmax = 500; kmaxd = 100;
p=1; % The number of equality constraints
lambda_0 = rand(p,1); typ=2; hess=eye(2);

[xmin,err,k] = aLagrange(fun,grad_fun,h,grad_h,x_0,...
                          lambda_0,tol,kmax,kmax,typ,hess)
```

Stopping criterion: A tolerance set 10^{-5}

The unconstrained minimisation by quasi-Newton descent directions

- (with `typ=2` and `hess=eye(2)`)