# Probabilistic reasoning

## Artificial intelligence (CK0031/CK0248)

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## Outline





# Probability refresher



## Probability refresher

A key concept in the field of artificial intelligence is that of **uncertainty**

- ⇝ Through noise on measurements
- ⇝ Through the finite size of data

**Probability theory** provides a consistent modelling framework

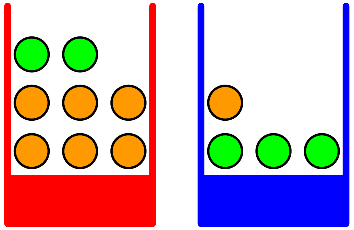
- Quantification and manipulation of uncertainty

Probability theory forms one of the central foundations of PRML

## Probability refresher (cont.)

Suppose that we are given two boxes, one red and one blue

- In the red box, there are 2 apples and 6 oranges
- In the blue box, there are 3 apples and 1 orange



We randomly select one box

We randomly pick a fruit (from that box)

1. We check the fruit
2. We replace it in its box

Suppose that we are asked to repeat the process *many* times

- 60% of the time, we pick the blue box
- 40% of the time, we pick the red box

For same reason, we are not equally likely to pick either box

We are equally likely to select any piece of fruit from the box



## Probability refresher (cont.)

The **identity of the box** that will be chosen is a **random variable** $B$

- This random variable can take only **two possible values**
- Either `r`, for red box or `b`, for blue box

The **identity of the fruit** that will be chosen is a **random variable** $F$

- This random variable can take only **two possible values**
- Either `a`, for apple or `o`, for orange



## Probability refresher (cont.)

### Definition

***Probability of an event***

*The* ***probability of an event*** *is defined to be the fraction of times that some event occurs out of the total number of trials*

- *In the limit that this number goes to infinity*

### Example

The probabilities associated to the two states of the random variable $B$

- The probability of picking the blue box is $6/10$
- The probability of picking the red box is $4/10$

We can write them formally

- ⇝ $p(B = \mathtt{r}) = 4/10$
- ⇝ $p(B = \mathtt{b}) = 6/10$

■



## Probability refresher (cont.)

By this definition, **probabilities must lie in the unit interval** $[0, 1]$

Consider the usual case in which the events are **mutually exclusive**

Consider the case in which events **include all possible outcomes**

- Then, the **probabilities** for such events **must sum to one**

## Probability refresher (cont.)

After we defined the random experiment and we can start asking questions

- What is the probability that the selection procedure picks an apple?
- Given that we have picked an orange, what is the probability that the box we chose was the blue one?
- ...

We can answer questions such as these, and much more complex ones

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We need first to define the **two elementary rules of probability**

⇝ The **product rule**

⇝ The **sum rule**

To derive these rules, consider the slightly more general example

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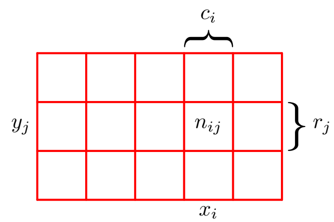
## Probability refresher (cont.)

Suppose that we model a problem with **two random variables** $X$ and $Y$




⇝ $X$ can take any of the values

$$x_i, \text{ with } i = 1, \dots, M$$

⇝ $Y$ can take any of the values

$$y_j, \text{ with } j = 1, \dots, L$$

Consider a **total of $N$ trials** in which we sample both variable $X$ and $Y$

⇝ Let $n_{ij}$ be the number of such trials in which $X = x_i$ and $Y = y_j$

⇝ Let $c_i$ be the number of trials in which $X$ takes the value $x_i$ (irrespective of the value that $Y$ takes)

⇝ Let $r_j$ be the number of trials in which $Y$ takes the value $y_j$ (irrespective of the value that $X$ takes)

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In the diagram, $M = 5$ and $L = 3$

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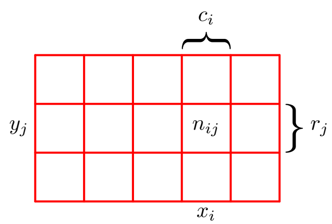
## Probability refresher (cont.)

The probability that $X$ will take the value $x_i$ and $Y$ will take the value $y_j$

$$\rightsquigarrow \quad p(X = x_i, Y = y_j)$$

This is the **joint probability** of $X = x_i$ and $Y = y_j$




Number of points falling in cell $(i, j)$, as fraction of the total $N$ points

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} \qquad (1)$$

Implicitly, in the limit $N \rightarrow \infty$

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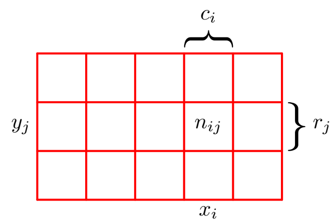
## Probability refresher (cont.)

The probability that $X$ takes the value $x_i$ irrespective of the value of $Y$

$$\rightsquigarrow \quad p(X = x_i)$$

This is the **marginal probability** of $X = x_i$




The number of points that are falling in column $i$, as fraction of the total $N$ points

$$p(X = x_i) = \frac{c_i}{N} \qquad (2)$$

Implicitly, in the limit $N \rightarrow \infty$

$$p(X = x_i) = \frac{c_i}{N} = \frac{\sum_{j=1}^{L} n_{ij}}{N} = \sum_{j=1}^{L} \underbrace{\frac{n_{ij}}{N}}_{p(X=x_i, Y=y_j)} \qquad (3)$$
$$= \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

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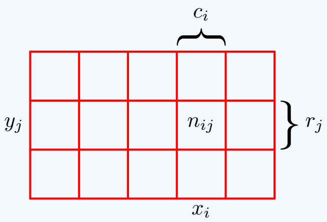
It is obtained by marginalising, summing out, the other variables ($Y$)

## Probability refresher (cont.)

**Definition**

***Sum rule***

*The marginal probability sets us for the* ***sum rule*** *of probability*




$$p(X = x_i) = \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$



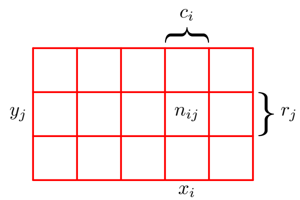
## Probability refresher (cont.)

Suppose that we consider only those observations for which $X = x_i$

Consider the fraction of such instances for which $Y = y_j$

$$\rightsquigarrow \quad p(Y = y_j | X = x_i)$$

This is the **conditional probability** of $Y = y_j$ given $X = x_i$




The number of points falling in cell $(i, j)$, as fraction of the number of points that fall in column $i$

$$p(Y = y_i | X = x_i) = \frac{n_{ij}}{c_i} \qquad (4)$$



## Probability refresher (cont.)

**Definition**

***Product rule***

*We derive the* ***product rule*** *of probability*

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \underbrace{\frac{n_{ij}}{c_i}}_{p(Y=y_j|X=x_i)} \underbrace{\frac{c_i}{N}}_{p(X=x_i)} = p(Y = y_j | X = x_i) p(X = x_i) \qquad (5)$$



## Probability refresher (cont.)

**Definition**

***The rules of probability***

- ***Sum rule***

$$p(X) = \sum_{Y} p(X, Y) \qquad (6)$$

- ***Product rule***

$$p(X, Y) = p(Y|X)p(X) \qquad (7)$$

To compact notation, we use $p(\star)$ for some distribution[1] over some RV $\star$

- $p(X, Y)$ is a joint probability, the probability of $X$ and $Y$
- $p(Y|X)$ is a conditional probability, the probability of $Y$ given $X$
- $p(X)$ is a marginal probability, the probability of $X$

[1] $p(\star = \cdot)$ or simply $p(\cdot)$ denotes the distribution evaluated for the particular value $\cdot$

## Probability refresher (cont.)

**Definition**

*Consider the product rule and the symmetry property $p(X, Y) = p(Y, X)$*

*We obtain a relationship between conditional probabilities*

$$\rightsquigarrow \quad p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)} \tag{8}$$

*Conditional probabilities $p(Y|X)$ over all values of $Y$ must sum to one*

*The relationship is called the **Bayes' rule***

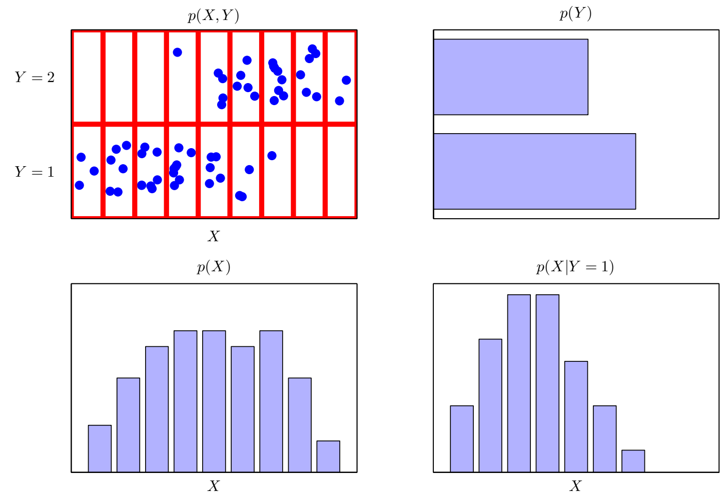
Using the sum rule, the denominator in Bayes' theorem can be explicitated

$$p(X) = \sum_Y p(X|Y)p(Y) \tag{9}$$

- Conditional probabilities $p(Y|X)$ over all values of $Y$ must sum to one
- The denominator in terms of the quantities in the numerator
- The denominator is a normalisation constant



## Probability theory (cont.)



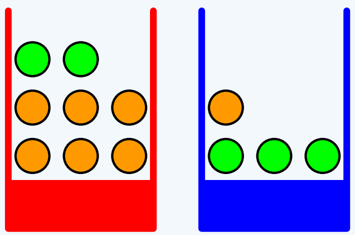



## Probability refresher (cont.)

**Example**

Consider the probability of selecting either the red or the blue box

$\rightsquigarrow$ $p(B = \mathtt{r}) = 4/10$ and $p(B = \mathtt{b}) = 6/10$



These two probabilities must satisfy $p(B = \mathtt{r}) + p(B = \mathtt{b}) = 4/10 + 6/10 = 1$

Suppose that we pick a box at random, say the blue box (that is, $B = \mathtt{b}$)

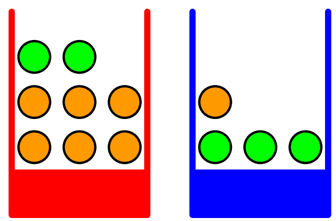
The probability of picking an apple is the fraction of apples in it

$\rightsquigarrow$ $p(F = \mathtt{a}|B = \mathtt{b}) = 3/4$



## Probability refresher (cont.)

We write all conditional probabilities for the type of fruit, given the box



$$p(F = \mathtt{a}|B = \mathtt{r}) = 1/4 \tag{10}$$
$$p(F = \mathtt{o}|B = \mathtt{r}) = 3/4 \tag{11}$$
$$p(F = \mathtt{a}|B = \mathtt{b}) = 3/4 \tag{12}$$
$$p(F = \mathtt{o}|B = \mathtt{b}) = 1/4 \tag{13}$$

Note that these conditional probabilities are (must be) normalised

$$p(F = \mathtt{a}|B = \mathtt{r}) + p(F = \mathtt{o}|B = \mathtt{r}) = 1 \tag{14}$$
$$p(F = \mathtt{a}|B = \mathtt{b}) + p(F = \mathtt{o}|B = \mathtt{b}) = 1 \tag{15}$$

## Probability refresher (cont.)

Suppose tht we want to evaluate the overall probability of choosing an apple

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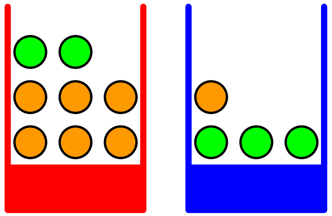
We can use the sum and product rules of probability[2]



$$p(B = \mathtt{r}) = 4/10$$
$$p(B = \mathtt{b}) = 6/10$$
$$p(F = \mathtt{a}|B = \mathtt{r}) = 1/4$$
$$p(F = \mathtt{o}|B = \mathtt{r}) = 3/4$$
$$p(F = \mathtt{a}|B = \mathtt{b}) = 3/4$$
$$p(F = \mathtt{o}|B = \mathtt{b}) = 1/4$$

$$\begin{aligned} p(F = \mathtt{a}) &= p(F = \mathtt{a}|B = \mathtt{r})p(B = \mathtt{r}) + p(F = \mathtt{a}|B = \mathtt{b})p(B = \mathtt{b}) \\ &= \frac{1}{4} \times \frac{4}{10} + \frac{3}{4} \times \frac{6}{10} = \frac{11}{20} \end{aligned} \tag{16}$$

It follows (from using the sum rule) that $p(F = \mathtt{o}) = 1 - 11/20 = 9/20$

[2] $P(X) = \sum_Y p(X, Y)$ with $p(X, Y) = p(Y|X)p(X) = p(Y, X) = p(X|Y)p(Y)$



## Probability refresher (cont.)

Suppose that we are told that a piece of fruit has been selected

- Say, it is an orange

We would like to know which box it came from

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The probability distribution over boxes conditioned on the fruit identity

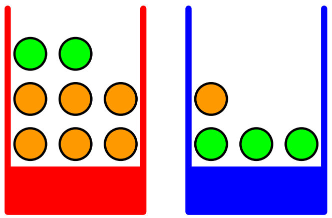
$$\rightsquigarrow \quad P(B|F)$$

The probability distribution over fruits conditioned on the box identity

$$\rightsquigarrow \quad P(F|B)$$



## Probability refresher (cont.)



$$p(B = \mathtt{r}) = 4/10$$
$$p(B = \mathtt{b}) = 6/10$$
$$p(F = \mathtt{a}|B = \mathtt{r}) = 1/4$$
$$p(F = \mathtt{o}|B = \mathtt{r}) = 3/4$$
$$p(F = \mathtt{a}|B = \mathtt{b}) = 3/4$$
$$p(F = \mathtt{o}|B = \mathtt{b}) = 1/4$$

We need to reverse the conditional probability (Bayes' rule)

$$p(B = \mathtt{r}|F = \mathtt{o}) = \frac{p(F = \mathtt{o}|B = \mathtt{r})p(B = \mathtt{r})}{p(F = \mathtt{o})} = \frac{3}{4} \times \frac{4}{10} \times \frac{20}{9} = \frac{2}{3} \tag{17}$$

It follows (sum rule) that $p(B = \mathtt{b}|F = \mathtt{o}) = 1 - 2/3 = 1/3$



## Probability refresher (cont.)

$$p(B|F) = \frac{p(F|B)p(B)}{p(F)}$$

The most complete information about the box is initially probability $p(B)$

- It is the probability before we observe the identity of the fruit
- ⇝ We call this the **prior probability**

Once we are told that the fruit is an orange, it became probability $p(B|F)$

- It is the probability after we observe the identity of the fruit
- ⇝ We call this the **posterior probability**

## Probability refresher (cont.)

$$\underbrace{p(B = \mathtt{r}|F = \mathtt{o})}_{2/3} = \frac{p(F = o|B = r)}{p(F = o)} \underbrace{p(B = r)}_{4/10}$$

The prior probability of selecting the red box is 4/10

- (blue is more probable)

The posterior probability of the red box is 2/3

- (red is more probable)



## Probability refresher (cont.)

Consider the joint distribution that factorises into the product of marginals

$$p(X, Y) = p(Y)p(X)$$

Random variables $X$ and $Y$ are said to be **independent**

$$p(X, Y) = p(Y|X)p(X)$$

The conditional distribution of $Y$ given $X$ is independent of the $X$ value

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)} = P(Y) \qquad \Longleftarrow P(X|Y) = P(X)$$



# Reasoning under uncertainty



# Probabilistic modelling

## Reasoning under uncertainty

## Probabilistic modelling

**Variables** will be denoted by using either upper case $X$ or lower case $x$

**Sets of variables** will be typically denoted by calligraphic symbols

⇝ For example, $\mathcal{V} = \{a, B, c\}$

The **domain of variable** $x$ is $\text{dom}(x)$, it denotes the **states** $x$ can take

- States will typically be represented using typewriter type fonts
- For example, $X = \mathtt{x}$



## Probabilistic modelling (cont.)

**Example**

Suppose that we are discussing an experiment about a two-face coin $c$

- For a coin $c$, $\text{dom}(c) = \{\mathtt{heads}, \mathtt{tails}\}$
- $p(c = \mathtt{heads})$ is the probability that variable $c$ is in state $\mathtt{heads}$

The meaning of $p(\mathtt{state})$ is often clear, without reference to a variable

⇝ The meaning of $p(\mathtt{heads})$ is clear from context

⇝ It is shorthand for $p(c = \mathtt{heads})$



## Probabilistic modelling (cont.)

When summing over a variable, as in $\sum_x f(x)$, all states of $x$ are included

$$\rightsquigarrow \quad \sum_x f(x) = \sum_{\mathtt{s} \in \text{dom}(x)} f(x = \mathtt{s})$$

Given variable $x$, its domain $\text{dom}(x)$ and a full specification of probability values for each of the states, $p(x)$, we say we have a **distribution** for $x$



## Probabilistic modelling (cont.)

**Events** will be considered as expressions about random variables

⇝ *Two heads in 6 coin tosses*

Two events are **mutually exclusive** if they cannot both be true

⇝ Events *Coin is heads* and *Coin is tails* are mutually exclusive

**Example**

One can think of defining a new variable named by the event

⇝ $p(\textit{The coin is tails})$ can be interpreted as $p(\textit{The coin is tails} = \mathtt{true})$

- $p(x = \mathtt{tr})$, the probability of event/variable $x$ being in state $\mathtt{true}$
- $p(x = \mathtt{fa})$, the probability of $x$ being in state $\mathtt{false}$

## Probabilistic modelling (cont.)

**Definition**

***Rules of probability for discrete variables (1)***

*Probability $p(x = \mathbb{x})$ of variable $x$ being in state $\mathbb{x}$ is represented by a value between 0 and 1*

- $p(x = \mathbb{x}) = 1$ *means that we are certain $x$ is in state $\mathbb{x}$*
- $p(x = \mathbb{x}) = 0$ *means that we are certain $x$ is NOT in state $\mathbb{x}$*

*Values in $[0, 1]$ represent the degree of certainty of state occupancy*

**Definition**

***Rules of probability for discrete variables (2)***

*The summation of the probability over all states is one*

$$\sum_{\mathbb{x} \in dom(x)} p(x = \mathbb{x}) = 1 \qquad (18)$$

***Normalisation condition***

- $\sum_x p(x) = 1$

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## Probabilistic modelling (cont.)

**Definition**

***Rules of probability for discrete variables (3)***

*Variable $x$ and variable $y$ can interact*

$$p(x = \mathbb{a} \textit{ OR } y = \mathbb{b}) = p(x = \mathbb{a}) + p(y = \mathbb{b}) - p(x = \mathbb{a} \textit{ AND } y = \mathbb{b}) \qquad (19)$$

*Or, more generally,*

$$p(x \textit{ OR } y) = p(x) + p(y) - p(x \textit{ AND } y) \qquad (20)$$

We use $p(x, y)$ for $p(x \text{ AND } y)$

- $p(x, y) = p(y, x)$
- $p(x \text{ OR } y) = p(y \text{ OR } x)$

An alternative **set notation**

$$p(x \text{ OR } y) \equiv p(x \cup y)$$
$$p(x, y) \equiv p(x \cap y) \qquad (21)$$

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## Probabilistic modelling (cont.)

**Definition**

***Marginals***

*Given a **joint distribution** $p(x, y)$, the distribution of a single variable*

$$p(x) = \sum_y p(x, y) \qquad (22)$$

*$p(x)$ is termed a **marginal** of the joint probability distribution $p(x, y)$*

***Marginalisation***

*Process of computing a marginal from a joint distribution (sum rule)*

$$p(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \sum_{x_i} p(x_1, \dots, x_n) \qquad (23)$$

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## Probabilistic modelling (cont.)

**Definition**

***Conditional probability/Bayes' rule***

*The probability of some event $x$ conditioned on knowing some event $y$*

- *The probability of $x$ given $y$*

$$p(x|y) = \frac{p(x, y)}{p(y)} \qquad (24)$$

*IFF $p(y) = 0$, otherwise $p(x|y)$ is not defined*

From this definition and $p(x, y) = p(y, x)$, we write the Bayes' rule

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} \qquad (25)$$

Bayes' rule follows from the definition of conditional probability

- Bayes' rule plays a central role in probabilistic reasoning
- It is used to invert probabilistic relations

$$p(y|x) \Leftrightarrow p(x|y)$$

## Probabilistic modelling (cont.)

### Remark

**Subjective probability**

Probability is a contentious topic and we do not debate it

- It is not the rules of probability that are contentious
- Rather what interpretation we should place on them

Suppose that potential repetitions of an experiment can be envisaged

- The frequentist definition of probability then makes sense
- (Probabilities defined wrt a potentially infinite repetitions)

In coin tossing, the interpretation of the probability of heads

- '*If I were to repeat the experiment of flipping a coin (at 'random'), the limit of the number of heads that occurred over the number of tosses is defined as the probability of a head occurring*'



## Probabilistic modelling (cont.)

A typical problem and scenario in an AI situation

### Example

A film enthusiast joins a new online film service

Based on a few films a user likes/dislikes, the company tries to estimate the probability that the user will like each of the $10K$ films in its offer collection

Suppose that we define probability as limiting case of infinite repetitions

- ⇝ This would not make much sense in this case
- ⇝ (we cannot repeat the experiment)

Suppose the user behaves in a manner that is consistent with other users

- We should be able to exploit the data from other users' ratings
- We make a reasonable 'guess' as to what this consumer likes



## Probabilistic modelling - Conditional probability

A **degree of belief** or **Bayesian** subjective interpretation of probability

- It sidesteps non-repeatability issues
- It is a framework for manipulating real values
- The framework is consistent with our intuition about probability



## Probabilistic modelling - Conditional probability (cont.)

### Example

Imagine a circular dart board, split into 20 equal sections, labels $1-20$

- A dart thrower hits any one of the 20 sections uniformly at random

The probability that a dart occurs in any one of the 20 regions is simple

$$p(\texttt{region i}) = 1/20$$

Suppose that someone tells that the dart has not hit the 20-region

What is the probability that the dart thrower has hit the 5-region?

## Probabilistic modelling - Conditional probability (cont.)

Conditioned on this information, only regions 1 to 19 remain possible

- There is no preference for the thrower to hit any of these regions
- The probability is 1/19

Formally,

$$p(\texttt{region 5}|\texttt{not region 20}) = \frac{p(\texttt{region 5}, \texttt{not region 20})}{p(\texttt{not region 20})} = \frac{p(\texttt{region 5})}{p(\texttt{not region 20})} = \frac{1/20}{19/20} = 1/19$$

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## Probabilistic modelling - Conditional probability (cont.)

**Remark**

A not-fully-correct interpretation of $p(A = \texttt{a}|B = \texttt{b})$

⇝ '*Given the event $B = b$ has occurred, $p(A = a|B = b)$ is the probability of the event $A = a$ occurring*'

In most contexts, no explicit temporal causality can be implied

The correct interpretation

⇝ '*$p(A = a|B = b)$ is the probability of $A$ being in state $a$ under the constraint that $B$ is in state $b$*'

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## Probabilistic modelling - Conditional probability (cont.)

The relationship between conditional distributions and joint distributions

Between the conditional $p(A = \texttt{a}|B = \texttt{b})$ and the joint $p(A = \texttt{a}, B = \texttt{b})$

- There is a normalisation constant

$p(A = \texttt{a}, B = \texttt{b})$ is not a distribution in $A$, $\sum_{\texttt{a}} p(A = \texttt{a}, B = \texttt{b}) \neq 1$

- To make it a distribution, we need to normalise

$$\rightsquigarrow \quad p(A = \texttt{a}|B = \texttt{b}) = \frac{1}{p(B = \texttt{b}) = \sum_{\texttt{a}} p(A = \texttt{a}, B = \texttt{b})} p(A = \texttt{a}, B = \texttt{b})$$

This is a distribution in $A$ (summed over $\texttt{a}$, does sum to 1)

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## Probabilistic modelling - Conditional probability (cont.)

**Definition**

***Independence***

*Variables $x$ and $y$ are independent if knowing the state (or value) of one variable gives no extra information about the other variable*

$$p(x, y) = p(x|y)p(y) = p(x)p(y) \tag{26}$$

*For $p(x) \neq 0$ and $p(y) \neq 0$, the independence of $x$ and $y$*

$$p(x|y) = p(x) \iff p(y|x) = p(y) \tag{27}$$

*If $p(x|y) = p(x)$ for all states of $x$ and $y$, then $x$ and $y$ are independent*

Suppose that for some constant $k$ and some positive functions $f(\cdot)$ and $g(\cdot)$

$$p(x, y) = kf(x)g(y) \tag{28}$$

Then, we say that $x$ and $y$ are independent and write $x \perp\!\!\!\perp y$

## Probabilistic modelling - Conditional probability (cont.)

### Example

Let $x$ denote the day of the week in which females are born

Let $y$ be the day in which males are born

$$\text{dom}(x) = \text{dom}(y) = \{\texttt{M}, \texttt{T}, \dots, \texttt{S}\}$$

It seems reasonable to expect that $x$ is independent of $y$

Suppose randomly select a woman from the phone book (Alice)

- ⇝ We find out that she was born on a Tuesday
- ⇝ $x = \texttt{T}$



## Probabilistic modelling - Conditional probability (cont.)

Knowing when Alice was born does not provide extra information

$$p(y|x) = p(y)$$

⇝ The probabilities of Bob's birthday remain unchanged

Does not mean that the distribution of Bob's birthdays is uniform

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The probability distribution of birthdays, $p(y)$ and $p(x)$, are non-uniform

- (Fewer babies are born on weekends, statistically)

Although nothing suggests that $x$ and $y$ are independent

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## Probabilistic modelling - Conditional probability (cont.)

### Example

Consider two binary random variables $x$ and $y$ (domain consists of 2 states)

Define the distribution such that $x$ and $y$ are always both in a certain state

$$p(x = \texttt{a}, y = 1) = 1$$
$$p(x = \texttt{a}, y = 2) = 0$$
$$p(x = \texttt{b}, y = 2) = 1$$
$$p(x = \texttt{b}, y = 1) = 0$$

Are $x$ and $y$ dependent?

Since

- $p(x = \texttt{a}) = 1$, $p(x = \texttt{b}) = 0$
- $p(y = 1) = 1$, $p(y = 2) = 0$

We have that $p(x)p(y) = p(x, y)$ for ALL states of $x$ and $y$

⇝ $x$ and $y$ are thus independent



## Probabilistic modelling - Conditional probability (cont.)

This may seem strange, as we know of the relation between $x$ and $y$

- They are always in the same joint state and yet independent

The distribution is concentrated in a single joint state

- Knowing the state of $x$ tells nothing more about the state of $y$
- (and viceversa)

This potential confusion comes from using the term 'independent'

- This may suggest that there is no relations between objects

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## Probabilistic modelling - Conditional probability (cont.)



**Remark**

To get the concept of statistical independence, ask whether or not knowing the state of $y$ tells something more than we knew about the state of $x$

- '*knew before*' means reasoning with the joint distribution $p(x, y)$
- To figure out what we can know about $x$, or equivalently $p(x)$

## Probabilistic modelling - Conditional probability (cont.)

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**Definition**

***Conditional independence***

*Sets of variables* $\mathcal{X} = \{x_1, \cdots, x_N\}$ *and* $\mathcal{Y} = \{y_1, \cdots, y_M\}$ *are independent of each other if, given all states of* $\mathcal{X}$, $\mathcal{Y}$ *and* $\mathcal{Z} = \{z_1, \cdots, z_P\}$, *we have*

$$p(\mathcal{X}, \mathcal{Y}|\mathcal{Z}) = p(\mathcal{X}|\mathcal{Z})p(\mathcal{Y}|\mathcal{Z}) \quad (29)$$

*(Provided that we know the state of set* $\mathcal{Z}$*)*

*We write* $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}|\mathcal{Z}$

Suppose that the conditioning set is empty, we write $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$ for $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}|\emptyset$

⇝ $\mathcal{X}$ is (conditionally) independent of $\mathcal{Y}$

If $\mathcal{X}$ and $\mathcal{Y}$ are not conditionally independent, then conditionally dependent

$$\mathcal{X} \top\!\!\!\top \mathcal{Y}|\mathcal{Z} \quad (30)$$

Similarly, $\mathcal{X} \top\!\!\!\top \mathcal{Y}|\emptyset$ can be written as $\mathcal{X} \top\!\!\!\top \mathcal{Y}$

## Probabilistic modelling - Conditional probability (cont.)

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Suppose that variable $x$ is conditionally independent of variable $y$, given $z$

⇝ Then, given $z$, $y$ contains no additional information about $x$

⇝ Then, given $z$, knowing $x$ does not add information about $y$

**Remark**

$$\mathcal{X} \top\!\!\!\top \mathcal{Y}|\mathcal{Z} \Longrightarrow \mathcal{X}' \top\!\!\!\top \mathcal{Y}'|\mathcal{Z}, \quad \text{for } \mathcal{X}' \subseteq \mathcal{X} \text{ and } \mathcal{Y}' \subseteq \mathcal{Y}$$

## Probability refresher - Conditional probability (cont.)

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**Remark**

**Independence implications**

It is tempting to think that if '$a$ *is independent of* $b$' and '$b$ *is independent of* $c$', then '$a$ *must be independent of* $c$'

$$\{a \perp\!\!\!\perp b, b \perp\!\!\!\perp c\} \quad \not\Longrightarrow \quad a \perp\!\!\!\perp c \quad (31)$$

However, this does NOT necessarily hold true

## Probabilistic modelling - Conditional probability (cont.)

Consider the distribution

$$p(a, b, c) = p(b)p(a, c) \quad (32)$$

From this,

$$p(a, b) = \sum_c p(a, b, c) = p(b) \sum_c p(a, c) = p(b)p(a) \quad (33)$$

$p(a, b)$ is a function of $b$ multiplied by a function of $a$

$\rightsquigarrow$ $a$ and $b$ are independent

- One can show that also variables $b$ and $c$ are independent
- One can show that $a$ is not necessarily independent of $c$

$\rightsquigarrow$ (distribution $p(a, c)$ can be set arbitrarily)



## Probabilistic modelling - Conditional probability (cont.)

**Remark**

Similarly, it is tempting to think that if '*$a$ and $b$ are dependent*', and '*$b$ and $c$ are dependent*', then '*$a$ and $c$ must be dependent*'

$$\{a \perp\!\!\!\perp b, b \perp\!\!\!\perp c\} \implies a \perp\!\!\!\perp c \quad (34)$$

However, this also does NOT follow ($\star$)

**Remark**

Conditional independence

$$x \perp\!\!\!\perp y | z$$

does not imply marginal independence

$$x \perp\!\!\!\perp y$$



## Probabilistic modelling - Probability tables

Population of countries ($CNT$) England ($\texttt{E}$), Scotland ($\texttt{S}$) and Wales ($\texttt{W}$)

- England ($\texttt{E}$), 60776238
- Scotland ($\texttt{S}$), 5116900
- Wales ($\texttt{W}$), 2980700

*A priori* probability that a randomly selected person from the combined countries would live in England, Scotland or Wales is 0.88, 0.08 and 0.04

$$\begin{bmatrix} p(CNT = \texttt{E}) \\ p(CNT = \texttt{S}) \\ p(CNT = \texttt{W}) \end{bmatrix} = \begin{pmatrix} 0.88 \\ 0.08 \\ 0.04 \end{pmatrix} \quad (35)$$

- These are based on population

For simplicity, assume that only three mother tongues ($MT$) exist

- English ($\texttt{Eng}$)
- Scottish ($\texttt{Scot}$)
- Welsh ($\texttt{Wel}$)



## Probability modelling - Probability tables (cont.)

The conditional probabilities $p(MT|CNT)$ by residence $\texttt{E}$, $\texttt{S}$ and $\texttt{W}$

$$p(MT = \texttt{Eng}|CNT = \texttt{E}) = 0.95$$
$$p(MT = \texttt{Scot}|CNT = \texttt{E}) = 0.04$$
$$p(MT = \texttt{Wel}|CNT = \texttt{E}) = 0.01$$

$$p(MT = \texttt{Eng}|CNT = \texttt{S}) = 0.70$$
$$p(MT = \texttt{Scot}|CNT = \texttt{S}) = 0.30$$
$$p(MT = \texttt{Wel}|CNT = \texttt{S}) = 0.00$$

$$p(MT = \texttt{Eng}|CNT = \texttt{W}) = 0.60$$
$$p(MT = \texttt{Scot}|CNT = \texttt{W}) = 0.00$$
$$p(MT = \texttt{Wel}|CNT = \texttt{W}) = 0.40$$

That is,

$$\begin{bmatrix} p(\texttt{Eng}|\texttt{E}) & p(\texttt{Eng}|\texttt{S}) & p(\texttt{Eng}|\texttt{W}) \\ p(\texttt{Scot}|\texttt{E}) & p(\texttt{Scot}|\texttt{S}) & p(\texttt{Scot}|\texttt{W}) \\ p(\texttt{Wel}|\texttt{E}) & p(\texttt{Wel}|\texttt{S}) & p(\texttt{Wel}|\texttt{W}) \end{bmatrix} = \begin{pmatrix} 0.95 & 0.70 & 0.60 \\ 0.04 & 0.30 & 0.00 \\ 0.01 & 0.00 & 0.40 \end{pmatrix}$$

## Probabilistic modelling - Probability tables (cont.)

We can form the joint distribution $p(CNT, MT) = p(MT|CNT)p(CNT)$

$$\begin{bmatrix} p(\texttt{Eng}, \texttt{E}) & p(\texttt{Eng}, \texttt{S}) & p(\texttt{Eng}, \texttt{W}) \\ p(\texttt{Scot}, \texttt{E}) & p(\texttt{Scot}, \texttt{S}) & p(\texttt{Scot}, \texttt{W}) \\ p(\texttt{Wel}, \texttt{E}) & p(\texttt{Wel}, \texttt{S}) & p(\texttt{Wel}, \texttt{W}) \end{bmatrix}$$

We write joint country-language probability distribution as $(3 \times 3)$ matrix

- Columns indexed by country, rows indexed by mother tongue

$$\begin{pmatrix} 0.95 \times 0.88 & 0.70 \times 0.08 & 0.60 \times 0.04 \\ 0.04 \times 0.88 & 0.30 \times 0.08 & 0.00 \times 0.04 \\ 0.01 \times 0.88 & 0.00 \times 0.08 & 0.40 \times 0.04 \end{pmatrix} = \begin{pmatrix} 0.8360 & 0.056 & 0.024 \\ 0.0352 & 0.024 & 0.000 \\ 0.0088 & 0.000 & 0.016 \end{pmatrix}$$

The joint distribution contains all the information about the model



## Probabilistic modelling - Probability tables (cont.)

$$p(CNT, MT) = \begin{pmatrix} 0.8360 & 0.0560 & 0.0240 \\ 0.0352 & 0.0240 & 0.0000 \\ 0.0088 & 0.0000 & 0.0160 \end{pmatrix}$$

By summing the columns, we have the marginal distribution $p(CNT)$

$$p(CNT) = \sum_{MT \in \text{dom}(MT)} p(CNT, MT)$$

That is,

$$\begin{bmatrix} p(CNT = \texttt{E}) \\ p(CNT = \texttt{S}) \\ p(CNT = \texttt{W}) \end{bmatrix} = \begin{pmatrix} 0.8352 + 0.0352 + 0.0088 = 0.88 \\ 0.0352 + 0.0240 + 0.0000 = 0.08 \\ 0.0088 + 0.0000 + 0.0160 = 0.04 \end{pmatrix} \quad (36)$$



## Probabilistic modelling - Probability tables (cont.)

$$p(CNT, MT) = \begin{pmatrix} 0.8360 & 0.0560 & 0.0240 \\ 0.0352 & 0.0240 & 0.0000 \\ 0.0088 & 0.0000 & 0.0160 \end{pmatrix}$$

By summing the rows, we have the marginal distribution $p(MT)$

$$p(MT) = \sum_{CNT \in \text{dom}(CNT)} p(CNT, MT)$$

That is,

$$\begin{bmatrix} p(MT = \texttt{Eng}) \\ p(MT = \texttt{Scot}) \\ p(MT = \texttt{Wel}) \end{bmatrix} = \begin{pmatrix} 0.8360 + 0.0560 + 0.0240 = 0.916 \\ 0.0352 + 0.0240 + 0.0000 = 0.059 \\ 0.0088 + 0.0000 + 0.0160 = 0.025 \end{pmatrix} \quad (37)$$



## Probabilistic modelling - Probability tables (cont.)

We infer the conditional distribution $p(CNT|MT) \propto p(MT|CNT)p(CNT)$

$$p(CNT|MT) = \begin{pmatrix} 0.913 & 0.061 & 0.026 \\ 0.590 & 0.410 & 0.000 \\ 0.360 & 0.000 & 0.640 \end{pmatrix}$$

The $p(CNT|MT)$ by dividing entries of $p(CNT, MT)$ by their rowsum

## Probabilistic modelling - Probability tables (cont.)



Consider joint distributions over a larger set of variables $\{x_i\}_{i=1}^{D}$

- Suppose that each variable $x_i$ takes $K_i$ states

The table of the joint distribution is an array of $\prod_{i=1}^{D} K_i$ entries

Storing tables requires space exponential in the number of variables

- It rapidly becomes impractical for a large number $D$

## Probabilistic modelling - Probability tables (cont.)



**Remark**

A probability distribution assigns a value to each of the joint states

- $p(T, J, R, S)$ is equivalent to $p(J, S, R, T)$ or any reordering

The joint setting is a different index to the same probability

This is more clear in set theoretic notation $p(J \cap S \cap T \cap R)$

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# Probability reasoning

### Reasoning under uncertainty

## Probabilistic reasoning



The central paradigm of probabilistic reasoning

Identify all relevant variables $x_1, \ldots, x_N$ in the environment

Make a **probabilistic model**

$$p(x_1, \ldots, x_N)$$

**Reasoning** (or **inference**) is performed by introducing **evidence**

- Evidence sets variables in known states

Then, we compute probabilities of interest, conditioned on this evidence

## Probabilistic reasoning (cont.)

Probability theory with Bayes' rule make for a complete reasoning system

- Deductive logic emerges as a special case

We discuss examples in which the number of variables is still very small

Then, we shall discuss reasoning in networks of many variables

- A graphical notation will play a central role

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## Probabilistic reasoning (cont.)

**Example**

**Hamburgers and the KJ disease**

People with Kreuzfel-Jacob (KJ) disease almost inevitably ate hamburgers

$$p(\textit{Hamburger eater} = \texttt{tr} | KJ = \texttt{tr}) = 0.9$$

The probability of a person having KJ disease is very low

$$p(KJ = \texttt{tr}) = 1/100K$$

Assume (safely) that eating hamburgers is commonplace

$$p(\textit{Hamburger eater} = \texttt{tr}) = 0.5$$

What is the probability that a hamburger eater will have KJ disease?

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## Probabilistic reasoning (cont.)

$$\begin{aligned} p(\texttt{KJ}|\texttt{Hamburger eater}) &= \frac{p(\texttt{Hamburger eater}, \texttt{KJ})}{p(\texttt{Hamburger eater})} \\ &= \frac{p(\texttt{Hamburger eater}|\texttt{KJ})p(\texttt{KJ})}{p(\texttt{Hamburger eater})} \\ &= \frac{9/10 \times 1/100K}{1/2} = 1.8 \times 10^{-5} \end{aligned} \quad (38)$$

Suppose that $p(\texttt{Hamburger eater}) = 0.001$

$\rightsquigarrow$ $p(\texttt{KJ}|\texttt{Hamburger eater}) \approx 1/100$

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## Probabilistic reasoning (cont.)

**Example**

**Inspector Clouseau**

Inspector Clouseau arrives at the scene of a crime

The victim lies dead near the possible murder weapon, a knife

$\rightsquigarrow$ $K$, $\text{dom}(K) = \{\texttt{knife used}, \texttt{knife not used}\}$

The butler ($B$) and the maid ($M$) are the inspector's suspects

$\rightsquigarrow$ $B$ and $M$, $\text{dom}(B) = \text{dom}(M) = \{\texttt{murderer}, \texttt{not murderer}\}$

Prior beliefs that they are the murderer

$$\begin{aligned} p(B = \texttt{murderer}) = 0.6 \quad &\rightsquigarrow \quad p(B = \texttt{not murderer}) = 0.4 \\ p(M = \texttt{murderer}) = 0.2 \quad &\rightsquigarrow \quad p(M = \texttt{not murderer}) = 0.8 \end{aligned}$$

These beliefs are independent

$$p(B)p(M) = p(B, M)$$

## Probabilistic reasoning (cont.)

Still possible that both the butler and the maid killed the victim or neither

$$p(K = \texttt{knife used} | B = \texttt{not murderer}, M = \texttt{not murderer}) = 0.3$$
$$p(K = \texttt{knife used} | B = \texttt{not murderer}, M = \texttt{murderer}) = 0.2$$
$$p(K = \texttt{knife used} | B = \texttt{murderer}, M = \texttt{not murderer}) = 0.6$$
$$p(K = \texttt{knife used} | B = \texttt{murderer}, M = \texttt{murderer}) = 0.1$$

In addition, $p(K, B, M) = p(K|B, M)p(B, M) = p(K|B, M)p(B)p(M)$

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Assume that the knife is the murder weapon ($K = \texttt{tr}$)

What is the probability that the butler is the murderer

$$p(B = \texttt{murderer} | K = \texttt{tr})$$

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## Probabilistic reasoning (cont.)

- Let $b = \text{dom}(B)$ indicate the two states of $B$
- Let $m = \text{dom}(M)$ indicate the two states of $M$

$$\begin{aligned} p(B|K) &= \sum_{M \in m} p(B, M|K) = \sum_{M \in m} \frac{p(B, M, K)}{p(K)} = \frac{1}{p(K)} \sum_{M \in m} p(B, M, K) \\ &= \frac{1}{\sum_{\substack{B \in b \\ M \in m}} p(B, M, K)} \sum_{M \in m} p(K|B, M)p(B, M) \\ &= \frac{1}{\sum_{\substack{B \in b \\ M \in m}} p(K|B, M)p(B, M)} \sum_{M \in m} p(K|B, M)p(B, M) \\ &= \frac{1}{\sum_{\substack{B \in b \\ M \in m}} p(K|B, M)p(B)p(M)} \sum_{M \in m} p(K|B, M)p(B)p(M) \\ &= \frac{1}{\sum_{B \in b} \left[ p(B) \sum_{M \in m} p(K|B, M)p(M) \right]} p(B) \sum_{M \in m} p(K|B, M)p(M) \end{aligned} \tag{39}$$

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## Probabilistic reasoning(cont.)

Plugging in the values, we have

$$\begin{aligned} & p(B = \texttt{murderer} | K = \texttt{knife used}) \\ &= \frac{0.6 \left( \frac{2}{10} \times \frac{1}{10} + \frac{8}{10} \times \frac{6}{10} \right)}{0.6 \left( \frac{2}{10} \times \frac{1}{10} + \frac{8}{10} \times \frac{6}{10} \right) + 0.4 \left( \frac{2}{10} \times \frac{2}{10} + \frac{8}{10} \times \frac{3}{10} \right)} \\ &= 300/412 \simeq 0.73 \end{aligned} \tag{40}$$

Knowing that it was the knife strengthens our belief that the butler did it

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## Probabilistic reasoning (cont.)

**Remark**

The role of $p(K = \texttt{knife used})$ in the example can be cause of confusion

$$\begin{aligned} p(K = \texttt{knife used}) &= \sum_{B \in b} p(B) \sum_{M \in m} p(K = \texttt{knife used} | B, M)p(M) \\ &= 0.412 \end{aligned} \tag{41}$$

But surely also $p(K = \texttt{knife used}) = 1$, since this is given

Quantity $p(K = \texttt{knife used})$ relates to the **prior**

- The probability the model assigns to the knife being used
- (in the absence of any other info)

Clearly, if we know that the knife is used, then the **posterior**

$$\begin{aligned} & p(K = \texttt{knife used} | K = \texttt{knife used}) = \\ & \frac{p(K = \texttt{knife used}, K = \texttt{knife used})}{p(K = \texttt{knife used})} = \\ & \frac{p(K = \texttt{knife used})}{p(K = \texttt{knife used})} = 1 \end{aligned} \tag{42}$$

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## Probabilistic reasoning (cont.)

**Example**

**Who's in the bathroom?**

Consider a household with 3 persons: Alice, Bob, Cecil

Cecil wants to go to the bathroom but finds it occupied

- He goes to Alice's room and he sees she is there

Cecil knows that only either Bob or Alice can be in the bathroom

- He infers that Bob must be occupying it

We can arrive at the same conclusion mathematically

Define the events

$$\begin{cases} A: & \text{Alice is in her bedroom} \\ B: & \text{Bob is in his bedroom} \\ O: & \text{Bathroom is occupied} \end{cases} \quad (43)$$

We need to encode the available information

## Probabilistic reasoning (cont.)

If either Alice or Bob are not in their rooms, they must be in the bathroom

- (both may be there)

$$\begin{aligned} p(O = \texttt{tr} | A = \texttt{fa}, B) &= 1 \\ p(O = \texttt{tr} | B = \texttt{fa}, A) &= 1 \end{aligned} \quad (44)$$

- The first term expresses that the bathroom is occupied ($O = \texttt{tr}$) if Alice is not in her bedroom ($A = \texttt{fa}$), wherever Bob is ($B$)
- The second term expresses that the bathroom is occupied ($O = \texttt{tr}$) if Bob is not in his bedroom ($B = \texttt{fa}$), wherever Alice is ($A$)

---

$$\begin{aligned} p(B = \texttt{fa} | O = \texttt{tr}, A = \texttt{tr}) &= \frac{p(B = \texttt{fa}, O = \texttt{tr}, A = \texttt{tr})}{p(O = \texttt{tr}, A = \texttt{tr})} \\ &= \frac{\overbrace{p(O = \texttt{tr} | A = \texttt{tr}, B = \texttt{fa})}^{1} p(A = \texttt{tr}, B = \texttt{fa})}{p(O = \texttt{tr}, A = \texttt{tr})} \end{aligned} \quad (45)$$

## Probabilistic reasoning (cont.)

$$\begin{aligned} & p(O = \texttt{tr}, A = \texttt{tr}) \\ & \quad = \overbrace{p(O = \texttt{tr} | A = \texttt{tr}, B = \texttt{fa})}^{1} p(A = \texttt{tr}, B = \texttt{fa}) \\ & \qquad + \underbrace{p(O = \texttt{tr} | A = \texttt{tr}, B = \texttt{tr})}_{0} p(A = \texttt{tr}, B = \texttt{tr}) \end{aligned} \quad (46)$$

- If Alice is in her room and Bob is not, the bathroom must be occupied

$$p(O = \texttt{tr} | A = \texttt{tr}, B = \texttt{fa}) = 1$$

- If Alice and Bob are in their rooms, the bathroom cannot be occupied

$$p(O = \texttt{tr} | A = \texttt{tr}, B = \texttt{tr}) = 0$$

---

$$p(B = \texttt{fa} | O = \texttt{tr}, A = \texttt{tr}) = \frac{p(A = \texttt{tr}, A = \texttt{fa})}{p(A = \texttt{tr}, B = \texttt{fa})} = 1 \quad (47)$$

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## Probabilistic reasoning (cont.)

**Remark**

We are not required to make a full probabilistic model

We do not need to specify $p(A, B)$

⇝ The situation is common in limiting situations of probabilities

⇝ Probabilities being either 0 or 1

Probabilistic reasoning corresponds to traditional logic systems

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## Probabilistic reasoning (cont.)

### Example

**Aristotles - Modus Ponens**

Consider the following logic statements '*All apples are fruit*' and '*All fruits grow on trees*', they lead to '*All apples grow on trees*'

From statements $A \Rightarrow F$ and $F \Rightarrow T$, we infer (transititivity) $A \Rightarrow T$

This may be reduced to probabilistic reasoning

- '*All apples are fruits*' corresponds to $p(F = \texttt{tr}|A = \texttt{tr}) = 1$
- '*All fruits grow on trees*' corresponds to $p(T = \texttt{tr}|F = \texttt{tr}) = 1$

We want to show that this implies one of the two

- $p(T = \texttt{tr}|A = \texttt{tr}) = 1$, '*All apples grow on trees*'
- $p(T = \texttt{fa}|A = \texttt{tr}) = 0$, '*All apples do not grow on non-trees*'

$$p(T = \texttt{fa}|A = \texttt{tr}) = \frac{p(T = \texttt{fa}, A = \texttt{tr})}{p(A = \texttt{tr})}$$



## Probabilistic reasoning (cont.)

$$p(T = \texttt{fa}|A = \texttt{tr}) = \frac{p(T = \texttt{fa}, A = \texttt{tr})}{p(A = \texttt{tr})}$$

Assuming that $p(A = \texttt{tr}) > 0$, these equal to $p(T = \texttt{fa}, A = \texttt{tr}) = 0$

$$p(T = \texttt{fa}, A = \texttt{tr}) = p(T = \texttt{fa}, A = \texttt{tr}, F = \texttt{tr}) + p(T = \texttt{fa}, A = \texttt{tr}, F = \texttt{fa}) \quad (48)$$

We need to show that both terms on the right-hand side are zero



## Probabilistic reasoning (cont.)

- Since $p(T = \texttt{fa}|F = \texttt{tr}) = 1 - p(T = \texttt{tr}|F = \texttt{tr}) = 1 - 1 = 0$,

$$p(T = \texttt{fa}, A = \texttt{tr}, F = \texttt{tr}) \leq p(T = \texttt{fa}, F = \texttt{tr}) = p(T = \texttt{fa}|F = \texttt{tr})p(F = \texttt{tr}) = 0, \quad (49)$$

- By assumption $p(F = \texttt{fa}|A = \texttt{tr}) = 0$,

$$p(T = \texttt{fa}, A = \texttt{tr}, F = \texttt{fa}) \leq p(A = \texttt{tr}, F = \texttt{fa}) = p(F = \texttt{fa}|A = \texttt{tr})p(A = \texttt{tr})) = 0, \quad (50)$$

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## Probabilistic reasoning (cont.)

### Example

**Aristotles - Inverse Modus Ponens**

Consider the following logic statement'*If A is true then B is true*' leads to deduce that '*If B is false then A is false*'

We show how this can be represented by using probabilistic reasoning

'*If A is true then B is true*' corresponds to

$$p(B = \texttt{tr}|A = \texttt{tr}) = 1$$

We may infer

$$p(A = \texttt{fa}|B = \texttt{fa}) = 1 - p(A = \texttt{tr}|B = \texttt{fa}) = 1 - \frac{p(B = \texttt{fa}|A = \texttt{tr})p(A = \texttt{tr})}{p(B = \texttt{fa}|A = \texttt{tr})p(A = \texttt{tr}) + p(B = \texttt{fa}|A = \texttt{fa})p(A = \texttt{fa})} = 1 \quad (51)$$

It follows since $p(B = \texttt{fa}|A = \texttt{tr}) = 1 - p(B = \texttt{br}|A = \texttt{tr}) = 1 - 1 = 0$

■

## Probabilistic reasoning (cont.)

### Example

**Soft XOR gate**

What about inputs $A$ and $B$, knowing the output is 0?

| $A$ | $B$ | $A$ xor $B$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

The 'standard' XOR gate

1. $A$ and $B$ were both 0
2. $A$ and $B$ were both 1

We do not know which state $A$ is in, it could equally likely be 0 or 1

A 'soft' XOR gate stochastically outputs $C = 1$ depending on its inputs

| $A$ | $B$ | $p(C = 1 \vert A, B)$ |
|---|---|---|
| 0 | 0 | 0.10 |
| 0 | 1 | 0.99 |
| 1 | 0 | 0.80 |
| 1 | 1 | 0.25 |

Additionally, let $A \perp\!\!\!\perp B$ and

- $p(A = 1) = 0.65$
- $p(B = 1) = 0.77$



## Probabilistic reasoning (cont.)

What's up with $p(A = 1 | C = 0)$?

$$\begin{aligned} p(A = 1, C = 0) &= \sum_B p(A = 1, B, C = 0) \\ &= \sum_B p(C = 0 | A = 1, B) p(A = 1) p(B) \\ &= p(A = 1) p(C = 0 | A = 1, B = 0) p(B = 0) + \\ &\quad p(A = 1) p(C = 0 | A = 1, B = 1) p(B = 1) \\ &= 0.65 \times (0.2 \times 0.23 + 0.75 \times 0.77) = 0.405275 \end{aligned} \quad (52)$$



## Probabilistic reasoning (cont.)

$$\begin{aligned} p(A = 0, C = 0) &= \sum_B p(A = 0, B, C = 0) \\ &= \sum_B p(C = 0 | A = 0, B) p(A = 0) p(B) \\ &= p(A = 0) p(C = 0 | A = 0, B = 0) p(B = 0) + \\ &\quad p(A = 1) p(C = 0 | A = 0, B = 1) p(B = 1) \\ &= 0.35 \times (0.9 \times 0.23 + 0.01 \times 0.77) = 0.075145 \end{aligned} \quad (53)$$

$$\begin{aligned} p(A = 1 | C = 0) &= \frac{p(A = 1, C = 0)}{p(A = 1, C = 0) + p(A = 0, C = 0)} \\ &= \frac{0.405275}{0.405275 + 0.075145} \\ &= 0.8436 \end{aligned} \quad (54)$$

■



## Probabilistic reasoning (cont.)

### Example

**Larry the lair**

Larry is typically late for school

When his mum asks whether or not he was late, never admits to being late

- We denote Larry being late with $L = \texttt{late}$, otherwise $L = \texttt{not late}$

The response Larry gives is denoted by $R_L$

- $p(R_L = \texttt{not late} | L = \texttt{not late}) = 1$
- $p(R_L = \texttt{late} | L = \texttt{late}) = 0$

The remaining two values are determined by normalisation

- $p(R_L = \texttt{late} | L = \texttt{not late}) = 0$
- $p(R_L = \texttt{not late} | L = \texttt{late}) = 1$

Given that $R_L = \texttt{not late}$, what is the probability that Larry was late?

$$p(L = \texttt{late} | R_L = \texttt{not late})$$

## Probabilistic reasoning (cont.)

Using Bayes' rule,

$$p(L = \texttt{late}|R_L = \texttt{not late}) = \frac{p(L = \texttt{late}, R_L = \texttt{not late})}{p(R_L = \texttt{not late})}$$
$$= \frac{p(L = \texttt{late}, R_L = \texttt{not late})}{p(L = \texttt{late}, R_L = \texttt{not late}) + p(L = \texttt{not late}, R_L = \texttt{not late})} \quad (55)$$

---

We recognise

$$p(L = \texttt{late}, R_L = \texttt{not late}) = \underbrace{p(R_L = \texttt{not late}|L = \texttt{late})}_{1} p(L = \texttt{late}) \quad (56)$$

$$p(L = \texttt{not late}, R_L = \texttt{not late}) = \underbrace{p(R_L = \texttt{not late}|L = \texttt{not late})}_{1} p(L = \texttt{not late}) \quad (57)$$

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## Probabilistic reasoning (cont.)

$$p(L = \texttt{late}|R_L = \texttt{not late}) = \frac{p(L = \texttt{late})}{p(L = \texttt{late}) + p(L = \texttt{not late})} \quad (58)$$
$$= p(L = \texttt{late})$$

Larry's mother knows that he never admits to being late

- Her belief about whether or not he was late is unchanged
- (regardless of what Larry actually says)

In the last step we used normalisation, $p(L = \texttt{late}) + p(L = \texttt{not late}) = 1$

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## Probabilistic reasoning (cont.)

**Example**

**Larry the lair and his sister Sue**

Unlike Larry, his sister Sue always tells the truth to her mother

- (As to whether or not Larry is late for school)

$$p(R_S = \texttt{not late}|L = \texttt{not late}) = 1$$
$$\implies p(R_S = \texttt{late}|L = \texttt{not late}) = 0$$
$$p(R_S = \texttt{late}|L = \texttt{late}) = 1$$
$$\implies p(R_S = \texttt{not late}|L = \texttt{late}) = 0$$

We also assume that $p(R_S, R_L|L) = p(R_S|L)p(R_L|L)$

Then, we write

$$p(R_S, R_L, L) = p(R_L|L)p(R_S|L)p(L) \quad (59)$$

Given $R_S = \texttt{late}$ and $R_L = \texttt{not late}$, what the probability that he late?

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## Probabilistic reasoning (cont.)

Using Bayes' rule,

$$p(L = \texttt{late}|R_L = \texttt{nlate}, R_S = \texttt{late}) =$$
$$\frac{1}{Z} p(R_S = \texttt{late}|L = \texttt{late}) p(R_L = \texttt{nlate}|L = \texttt{late}) p(L = \texttt{late}) \quad (60)$$

The normalisation term $1/Z$,

$$\frac{1}{Z} = p(R_S = \texttt{late}|L = \texttt{late}) p(R_L = \texttt{nlate}|L = \texttt{late}) p(L = \texttt{late})$$
$$+ p(R_S = \texttt{late}|L = \texttt{nlate}) p(R_L = \texttt{nlate}|L = \texttt{nlate}) p(L = \texttt{nlate}) \quad (61)$$

Hence,

$$p(L = \texttt{late}|R_L = \texttt{not late}, R_S = \texttt{late}) =$$
$$\frac{1 \times 1 \times p(L = \texttt{late})}{1 \times 1 \times p(L = \texttt{late}) + 0 \times 1 \times p(L = \texttt{not late})} = 1 \quad (62)$$

Larry's mother knows that Sue tells the truth, no matter what Larry says

■

## Probabilistic reasoning (cont.)

**Example**

**Luke**

Luke has been told he is lucky and has won a prize in the lottery

5 prizes available

- 10 $\rightsquigarrow (p_1)$
- 100 $\rightsquigarrow (p_2)$
- $1K$ $\rightsquigarrow (p_3)$
- $10K$ $\rightsquigarrow (p_4)$
- $1M$ $\rightsquigarrow (p_5)$

$p_0$ is the prior probability of winning no prize

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

- Luke asks '*Did I win 1M?!*', '*I'm afraid not sir*' the lottery guy
- '*Did I win 10K?!*' asks Luke, '*Again, I'm afraid not sir*'

What is the probability that Luke has won 1K?



## Probabilistic reasoning (cont.)

We denote

- $W = 1$ for the first prize (10)
- $W = 2, \dots, 5$ for the remaining prices (100, 1K, 10K, 1M)
- $W = 0$ for no prize (0)

$$\begin{aligned} p(W = 3 | W \neq 5, W \neq 4, W \neq 0) &= \frac{p(W = 3, W \neq 5, W \neq 4, W \neq 0)}{p(W \neq 5, W \neq 4, W \neq 0)} \\ &= \frac{p(W = 3)}{\underbrace{p(W = 1 \text{ or } W = 2 \text{ or } W = 3)}_{\text{events are mutually exclusive}}} \\ &= \frac{p_3}{p_1 + p_2 + p_3} \end{aligned} \tag{63}$$



## Probabilistic reasoning (cont.)

The results makes intuitive sense

We remove the impossible states of $W$

The probability to win 1K is proportional to its prior probability ($p_3$)

- normalisation is the total set of possible probability left

■



# Prior, likelihood and posterior

## Reasoning under uncertainty

# Prior, likelihood and posterior

*Tell me something about variable $\Theta$, given that*

i) *I have observed data $\mathcal{D}$*
ii) *I have some knowledge of the data generating mechanism*

The quantity of interest

$$p(\Theta|\mathcal{D}) = \frac{p(\mathcal{D}|\Theta)p(\Theta)}{p(\mathcal{D})} = \frac{p(\mathcal{D}|\Theta)p(\Theta)}{\int_{\Theta} p(\mathcal{D}|\Theta)p(\Theta)} \quad (64)$$

A **generative model** $p(\mathcal{D}|\Theta)$ of the data

A **prior belief** $p(\Theta)$ about which variable values are appropriate

- We infer the **posterior distribution** $p(\Theta|\mathcal{D})$ of the variables
- (In the light of the observed data)



# Prior, likelihood and posterior (cont.)

The **most probable a posteriori** (**MAP**) setting maximises the posterior

$$\Theta_* = \arg\max_{\Theta} \big[p(\Theta|\mathcal{D})\big]$$

Consider a flat prior, $p(\Theta)$ being a constant (with $\Theta$)

The MAP solution is equivalent to the **maximum likelihood** solution

- The $\Theta$ that maximises the **likelihood** $p(\mathcal{D}|\Theta)$

---

The use of the generative model suits well with physical modelling

- We typically postulate how to generate observed phenomena
- (Assuming we know the model)



# Prior, likelihood and posterior (cont.)

One might postulate a model of how to generate a time-series

Consider the displacements for a swinging pendulum

- Unknown mass, length and dumping constant

We infer the unknown physical properties of the pendulum

- Using the generative model, given the displacements



# Prior, likelihood and posterior (cont.)

**Example**

**Pendulum**

Consider a pendulum and let $x_t$ be the angular displacement at time $t$

We measure the displacement and the measurements are independent

- The likelihood of a sequence $x_1, \dots, x_T$

$$p(x_1, \dots, x_T|\Theta) = \prod_{t=1}^{T} p(x_t|\Theta) \quad (65)$$

It depends on the knowledge of the problem parameter $\Theta$

Assume first that the model is correct and that measurements $x$ are perfect

$\rightsquigarrow$ We can express the physical model of the oscillations

$$\rightsquigarrow \quad x_t = \sin(\Theta t), \quad (66)$$

$\Theta$ is the unknown constants of the pendulum ($\sqrt{g/L}$)

- $g$ is the gravitational attraction
- $L$ the pendulum length

## Prior, likelihood and posterior (cont.)

Now assume that we have a poor instrument to measure the displacements

- Suppose that measurements have a Gaussian distribution
- Suppose that the variance $\sigma^2$ is known

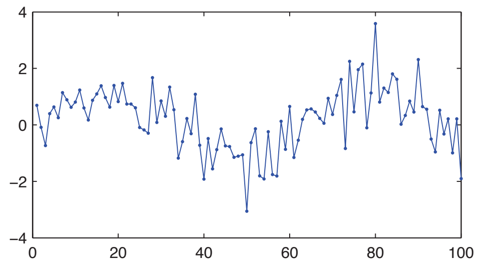
Then, we can re-express the physical model of the oscillations

$$\rightsquigarrow \quad x_t = \sin(\Theta t) + \varepsilon_t \tag{67}$$

$\varepsilon_t$ is a zero mean Gaussian noise with variance $\sigma^2$

---

We have noisy observations (data) of the displacements $x_1, \dots, x_{100}$

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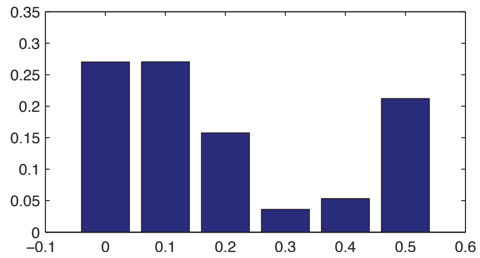
## Prior, likelihood and posterior (cont.)

Consider a set of possible parameters $\Theta$, we can place a prior $p(\Theta)$ over it

- Express our prior belief (before even seeing the measurements)
- Our trust in the appropriateness of different values of $\Theta$

---

Suppose that we define a prior belief $p(\Theta)$ on 5 possible values of $\Theta$

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## Prior, likelihood and posterior (cont.)

The posterior distribution

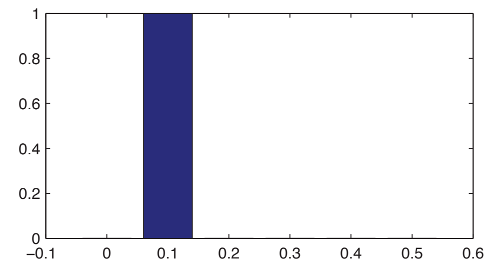
$$p(\Theta|x_1, \dots, x_N) \propto p(\Theta) \prod_{t=1}^{T} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{1}{2\sigma^2}\Big[x_t - \sin(\Theta t)\Big]^2\Big\}}_{p(x_t|\theta)} \tag{68}$$

The posterior belief over the assumed values of $\Theta$ becomes strongly peaked

- For a large number of measurements, despite noisy measurements

---

The posterior belief on $\Theta$, $p(\Theta|x_1, \dots, x_N)$

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## Two dice: Individual scores

**Example**

Two fair dice are rolled and someone tells that the sum of scores is 9

$\rightsquigarrow$ What is the posterior distribution of the dice scores?

$\rightsquigarrow$ The score of die $a$ is denoted by $s_a$, with $\text{dom}(s_a) = \{1, 2, 3, 4, 5, 6\}$

$\rightsquigarrow$ The score of die $b$ is denoted by $s_b$, with $\text{dom}(s_b) = \{1, 2, 3, 4, 5, 6\}$

The three variables involved are then $s_a$, $s_b$ and $t = s_a + s_b$

- We jointly model them

$$p(t, s_a, s_b) = \underbrace{p(t|s_a, s_b)}_{\text{likelihood}} \underbrace{p(s_a, s_b)}_{\text{prior}} \tag{69}$$

## Two dice: Individual scores (cont.)

The prior $p(s_a, s_b)$ is the joint probability of scores $s_a$ and $s_b$

- Without knowing anything else
- Assuming no dependency in the rolling

$$p(s_a, s_b) = p(s_a)p(s_b) \qquad (70)$$

Since dice are fair, both $p(s_a)$ and $p(s_b)$ are uniform distributions

$$p(s_a) = p(s_b) = 1/6$$

The likelihood $p(t|s_a, s_b)$ states the total score $t = s_a + s_b$

$$p(t|s_a, s_b) = \mathbb{I}[t = s_a + s_b] \qquad (71)$$

Function $\mathbb{I}[A]$ is such that $\mathbb{I}[A] = 1$ if statement $A$ is true, 0 otherwise



## Two dice: Individual scores (cont.)

| $p(s_a)p(s_b)$ | $s_a = 1$ | $s_a = 2$ | $s_a = 3$ | $s_a = 4$ | $s_a = 5$ | $s_a = 6$ |
|---|---|---|---|---|---|---|
| $s_b = 1$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| $s_b = 2$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| $s_b = 3$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| $s_b = 4$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| $s_b = 5$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| $s_b = 6$ | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |

| $p(t = 9 \vert s_a, s_b)$ | $s_a = 1$ | $s_a = 2$ | $s_a = 3$ | $s_a = 4$ | $s_a = 5$ | $s_a = 6$ |
|---|---|---|---|---|---|---|
| $s_b = 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 3$ | 0 | 0 | 0 | 0 | 0 | 1 |
| $s_b = 4$ | 0 | 0 | 0 | 0 | 1 | 0 |
| $s_b = 5$ | 0 | 0 | 0 | 1 | 0 | 0 |
| $s_b = 6$ | 0 | 0 | 1 | 0 | 0 | 0 |



## Two dice: Individual scores (cont.)

The complete model is explicitly defined

$$p(t, s_a, s_b) = p(t = 9|s_a, s_b)p(s_a)p(s_b) \qquad (72)$$

| | $s_a = 1$ | $s_a = 2$ | $s_a = 3$ | $s_a = 4$ | $s_a = 5$ | $s_a = 6$ |
|---|---|---|---|---|---|---|
| $s_b = 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 3$ | 0 | 0 | 0 | 0 | 0 | 1/36 |
| $s_b = 4$ | 0 | 0 | 0 | 0 | 1/36 | 0 |
| $s_b = 5$ | 0 | 0 | 0 | 1/36 | 0 | 0 |
| $s_b = 6$ | 0 | 0 | 1/36 | 0 | 0 | 0 |



## Two dice: Individual scores (cont.)

The posterior

$$p(s_a, s_b|t = 9) = \frac{p(t = 9|s_a, s_b)p(s_a)p(s_b)}{p(t = 9)} \qquad (73)$$

| | $s_a = 1$ | $s_a = 2$ | $s_a = 3$ | $s_a = 4$ | $s_a = 5$ | $s_a = 6$ |
|---|---|---|---|---|---|---|
| $s_b = 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $s_b = 3$ | 0 | 0 | 0 | 0 | 0 | 1/4 |
| $s_b = 4$ | 0 | 0 | 0 | 0 | 1/4 | 0 |
| $s_b = 5$ | 0 | 0 | 0 | 1/4 | 0 | 0 |
| $s_b = 6$ | 0 | 0 | 1/4 | 0 | 0 | 0 |

$$p(t = 9) = \sum_{s_a, s_b} p(t = 9|s_a, s_b)p(s_a)p(s_b) = 4 \times 1/36 = 1/9 \qquad (74)$$

The posterior is given by equal mass in only 4 non-zero elements