Matrícula (First-name Last-name) **CK0255/TIP8421: Homework D**

Exercise A.1. Let X and Y be two random variables with the joint PDF $f(x,y)=1$ for $0<x,y<1$ and zero elsewhere. Find the CDF and the PDF of the product $Z=XY$.

Solution:

$$
\begin{aligned}
G(z) &= P(XY \leq z) = 1 - \int_z^1 \int_{z/x}^1 \mathrm{d}y\mathrm{d}x \\
&= 1 - \int_z^1 \Big(1 - z/x\Big)\mathrm{d}x = z - z\log z \\
g(z) = G'(z) &= \begin{cases} -\log(z), & 0<z<1 \\ 0, & \text{elsewhere} \end{cases}
\end{aligned}
$$

$(-\log(z)>0?)$

Exercise A.2. Let X_1 and X_2 be two random variables with the joint PMF $p(x_1,x_2)=1/12(x_1+x_2)$ for $x_1,x_2=1,2$ and zero elsewhere.

Compute

- $E(X_1)$ and $E(X_1^2)$
- $E(X_2)$ and $E(X_2^2)$
- $E(X_1,X_2)$. Is $E(X_1X_2)=E(X_1)E(X_2)$?
- $E(2X_1-6X_2^2+7X_1X_2)$

Exercise A.3. Let X_1 and X_2 be two random variables of the continuous type with the joint PDF $f_{X_1,X_2}(x_1,x_2)$, for $-\infty<x_i<\infty$, $i=1,2$. Let $Y_1=X_1+X_2$ and $Y_2=X_2$.

- Find the joint PDF f_{Y_1,Y_2}

Solution:
The inverse transformation equations,

- $x_1=y_1-y_2$
- $x_2=y_2$

The Jacobian,

$$J = \begin{bmatrix} \partial x_1/\partial y_1 & \partial x_1/\partial y_2 \\ \partial x_2/\partial y_1 & \partial x_2/\partial y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1.$$

The space ,

$$\mathcal{T} = \big\{(y_1,y_2): -\infty < y_1,y_2 < \infty\big\}.$$

The joint PDF,

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1-y_2,y_2).$$

Exercise A.4. Let the random variables X and Y have the joint PDF $f_{XY}(x,y) = 1/\pi$ for $(x-1)^2 + (y+2)^2 < 1$ and zero elsewhere.

- Find $f_X(x)$ and $f_Y(y)$

Are X and Y independent?

Solution:
The marginal PDF of X_1 ,

$$f_{X_1}(x_1) = \int_{-2+\sqrt{1-(x_1-1)^2}}^{-2-(x_1-1)^2} \frac{1}{\pi} \mathrm{d}x_2 = \frac{2}{\pi}\sqrt{1-(x_1-1)^2}, \quad 0 < x_1 < 2$$

The RVs X_1 and X_2 are not independent.

Exercise A.5. Let X_1, X_2 and X_3 be IID random variables with common PDF $f(x) = \exp(-x)$ for $0 < x < \infty$ and zero elsewhere. Let

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2} \\ Y_2 &= \frac{X_1 + X_2}{X_1 + X_2 + X_3} \\ Y_3 &= X_1 + X_2 + X_3 \end{aligned}$$

- Show that Y_1, Y_2 and Y_3 are mutually independent

Solution: The joint PDF of (X_1, X_2, X_3)

$$f(x_1, x_2, x_3) = \begin{cases} e^{(-\sum_{i=1}^{3} x_i)}, & 0 < x_i < \infty, i = 1,2,3 \\ 0, & \text{elsewhere} \end{cases}$$

The transformation equations,

- $x_1 = y_1 y_2 y_3$
- $x_2 = y_2 y_3 - y_1 y_2 y_3$
- $x_3 = y_3 - y_2 y_3$

The space,

$$\mathcal{T} = \{(y_1, y_2, y_3) : 0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < \infty\}$$

The Jacobian,

$$J = \begin{bmatrix} y_2 y_3 & y_2 y_3 & 0 \\ y_1 y_3 & y_3 - y_1 y_3 & y_3 \\ y_1 y_2 & y_2 - y_1 y_2 & 1 - y_2 \end{bmatrix} = y_2 y_3^2$$

The joint PDF,

$$g(y_1, y_2, y_3) = y_2 y_3^2 e^{(-y_3)} = (1)(2y_2)(y_3^2 e^{-y_3}/2) = g_1(y_1) g_2(y_2) g_3(y_3)$$

The marginal PDFs above

$$\begin{aligned}
g(y_1) &= \int_0^1 \int_0^\infty g(y_1, y_2, y_3)\mathrm{d}y_3\mathrm{d}y_2 = \int_0^1 \int_0^\infty y_2 y_3^2 e^{-y_3}\mathrm{d}y_3\mathrm{d}y_2 = \int_0^1 y_2 \int_0^\infty y_3^2 e^{-y_3}\mathrm{d}y_3\mathrm{d}y_2 \\
&= \int_0^1 y_2 \underbrace{\int_0^\infty y_3^2 e^{(-y_3)}\mathrm{d}y_3}_{\int x^2 e^{cx}\mathrm{d}x = e^{cx}(1/cx^2 - 2/c^2 x + 2/c^3)} \mathrm{d}x_2 = \int_0^1 y_2 \Big[e^{-y_3}(-y_3^2 - 2y_3 - 2)\Big]_0^\infty \mathrm{d}x_2 \\
&= \int_0^1 2y_2\mathrm{d}y_2 = 1 \text{ (the result is obvious, given the PDF of } Y)
\end{aligned}$$

$$\begin{aligned}
g(y_2) &= \int_0^1 \int_0^\infty g(y_1, y_2, y_3)\mathrm{d}y_3\mathrm{d}y_1 = \int_0^1 \int_0^\infty y_2 y_3^2 e^{-y_3}\mathrm{d}y_3\mathrm{d}y_1 = y_2 \int_0^1 \int_0^\infty y_3^2 e^{-y_3}\mathrm{d}y_3\mathrm{d}y_1 \\
&= y_2 \int_0^1 2\mathrm{d}y_1 = 2y_2
\end{aligned}$$

$$\begin{aligned}
g(y_3) &= \int_0^1 \int_0^1 g(y_1, y_2, y_3)\mathrm{d}y_2\mathrm{d}y_1 = \int_0^1 \int_0^1 y_2 y_3^2 e^{-y_3}\mathrm{d}y_2\mathrm{d}y_1 = y_3^2 e^{-y_3} \int_0^1 \int_0^1 y_2\mathrm{d}y_2\mathrm{d}y_1 \\
&= y_3^2 e^{-y_3} \int_0^1 \left[\frac{1}{2}y_2^2\right]_0^1 \mathrm{d}y_1 = y_3^2 e^{-y_3} \int_0^1 \frac{1}{2}\mathrm{d}y_1 = y_3^2 e^{-y_3}/2
\end{aligned}$$

Exercise A.6. Let X_1 and X_2 be independent random variables with means μ_1 and μ_2 and with variances σ_1^2 and σ_2^2, respectively. Show that the mean and the variance of the product $Y = X_1 X_2$ are $\mu_1\mu_2$ and $\sigma_1^2\sigma_2^2 + \mu_1\sigma_2^2 + \mu_2\sigma_1^2$, respectively.