Some important inequalities

Probability and distributions

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We present some important (famous?) inequalities involving expectations

Some important inequalities (cont.)

Theorem

Let X be a random variable and let m be a positive integer

Suppose $E(X^m)$ exists

If k is a positive integer such that $k \leq m$, then $E(X^k)$ exists

Proof: We prove it for the continuous case

Let $f(x)$ be the PDF of X, then

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^k f(x)\mathrm{d}x &= \int_{|x|\leq 1} |x|^k f(x)\mathrm{d}x + \int_{|x|>1} |x|^k f(x)\mathrm{d}x \\ &\leq \int_{|x|\leq 1} f(x)\mathrm{d}x + \int_{|x|>1} |x|^m f(x)\mathrm{d}x \\ &\leq \int_{-\infty}^{\infty} f(x)\mathrm{d}x + \int_{-\infty}^{\infty} |x|^m f(x)\mathrm{d}x \\ &\leq 1 + E(|X|^m) < \infty\end{aligned} \tag{1}$$

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Some important inequalities (cont.)

Theorem

Markov's inequality

Let $u(X)$ be a non-negative function of the random variable X

If $E\big[u(X)\big]$ exists, then for every positive constant c,

$$P\big[u(X) \geq c\big] \leq \frac{E\big[u(X)\big]}{c}$$

Proof

We prove it for the continuous case

Let $A = \{x : u(x) \leq c\}$ and let $f(x)$ denote the PDF of X, then

$$E\big[u(X)\big] = \int_{-\infty}^{\infty} u(x) f(x) = \int_A u(x) f(x) \mathrm{d}x + \int_{A^c} u(x) f(x) \mathrm{d}x$$

Each of the integrals in the very RHS is non-negative

- The LHS is greater than or equal to either of them
- Specifically,

$$E\big[u(X)\big] \geq \int_A u(x) f(x) \mathrm{d}x$$

Some important inequalities (cont.)

$$E\big[u(X)\big] \geq \int_A u(x)f(x)\mathrm{d}x$$

If $x \in A$, then $u(x) \geq c$

Accordingly, the RHS does not increase if we replace $u(c)$ by c

$$E\big[u(X)\big] \geq c\int_A f(x)\mathrm{d}x$$

Since,

$$\int_A f(x)\mathrm{d}x = P(X \in A) = P\big[u(X) \geq c\big],$$

it follows that

$$E\big[u(X)\big] \geq cP\big[u(X) \geq c\big]$$

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Some important inequalities (cont.)

The previous theorem generalises Chebyshev's inequality

Theorem

Chebyshev's inequality

Let the random variable X have a distribution of probability

Assume a finite variance $Var(X)$[1]

Then, for every $k > 0$

$$P\left[|X - \mu| \geq k\sqrt{Var(X)}\right] \leq 1/k^2 \quad (2)$$

or, equivalently

$$P\left[|X_{\mu}| < k\sqrt{Var(X)}\right] \geq 1 - 1/k^2$$

[1] This implies that the mean $\mu = E(X)$ exists (earlier theorem)

Some important inequalities (cont.)

Proof

In Markov's inequality, we can take $u(X) = (X - u)^2$ and $c = k^2 \text{Var}(X)$

$$P\big[(X-\mu)^2 \geq k^2 \text{Var}(X)\big] \leq \frac{E\big[(X-\mu)^2\big]}{k^2 \text{Var}(X)}$$

Since the numerator on the RHS is $\text{Var}(X)$, the inequality becomes

$$P\big[|X-\mu| \geq k\sqrt{\text{Var}(X)}\big] \leq 1/k^2$$

This is the desired result for a non-negative number k greater than one

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Some important inequalities (cont.)

Example

Let X have the PDF

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{3}}, & x \in (-\sqrt{3}, \sqrt{3}) \\ 0, & \text{elsewhere} \end{cases}$$

Let $\mu = 0$ and $\sigma^2 = 1$

If $k = 3/2$, then we have the exact probability

$$P\big(|X - \mu| \geq k\sigma\big) = P\Big(|X| \geq \frac{3}{2}\Big) = 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}} \mathrm{d}x = 1 - \sqrt{3}/2$$

Some important inequalities (cont.)

By Chebyshev's inequality, this probability is bounded from above

$$1/k^2 = 4/9$$

As $1 - \sqrt{3}/2 \approx 0.314$, the exact probability is smaller than the upper bound

Some important inequalities (cont.)

If $k = 2$, we have the exact probability

$$P\big(|X - \mu| \geq 2\sigma\big) = P\big(|X| \geq 2\big) = 0,$$

Again. much smaller than the upper bound from Chebyshev's inequality

$$1/k^2 = 1/4$$

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Some important inequalities (cont.)

Example

Let the RV X of the discrete type have probabilities

- $\frac{1}{8}, \frac{6}{8}, \frac{1}{8}$, at the points $x = -1, 0, +1$, respectively

Let $\mu = 0$ and $\sigma^2 = 1/4$

If $k = 2$, then $1/k^2 = 1/4$ and $P(|X - \mu| \geq k\sigma) = P(|X| > 1) = 1/4$

The probability $P(|X - \mu| \geq k\sigma)$ reaches the upper bound $1/k^2 = 1/4$

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- The inequality can not be improved
- Further assumptions about the distribution of X are needed

Some important inequalities (cont.)

Definition

A function ϕ defined on an interval (a, b) such that $-\infty \leq a < b \leq \infty$

Function ϕ is said to be ***convex*** *if for all x, y in (a, b) and for all $\gamma \in (0, 1)$*

$$\phi\big[\gamma x + (1-\gamma) y\big] \leq \gamma \phi(x) + (1-\gamma) y$$

We say ϕ is ***strictly convex*** *if the inequality is strict*

Some important inequalities (cont.)

Theorem

If ϕ is differentiable on (a, b), then

- *ϕ is convex iff $\phi'(x) \leq \phi'(y)$, for all $a < x < y < b$*
- *ϕ is strictly convex iff $\phi'(x) < \phi'(y)$, for all $a < x < y < b$*

If ϕ is twice-differentiable on (a, b), then

- *ϕ is convex iff $\phi''(x) \geq 0$, for all $a < x < y < b$*
- *ϕ is strictly convex iff $\phi''(x) > 0$, for all $a < x < y < b$*

A very useful probability inequality follows from convexity

Some important inequalities (cont.)

Theorem

Jenses's inequality

Let ϕ be convex on an open interval I and X be a RV with support in I

Assume that X has a finite expectation

Then,

$$\phi\big[E(X)\big] \leq E\big[\phi(X)\big] \tag{3}$$

If ϕ is strictly convex, the inequality is strict, unless X is a constant RV

Some important inequalities (cont.)

Proof

We assume that ϕ has a second derivative (only convexity is needed)

Expand ϕ into a order 2 Taylor series about $\mu = E[X]$

$$\phi(x) = \phi(\mu) + \phi'(\mu)(x - \mu) + \frac{\phi''(\zeta)(x - \mu)^2}{2}$$

ζ is between x and μ

On the RHS, the last term on the RHS is nonnegative

$$\phi(x) \geq \phi(\mu) + \phi'(\mu)(x - \mu)$$

Taking expectations of both sides yields

Provided X is not a constant, the inequality is strict if $\phi''(x) > 0, \forall x \in (a, b)$

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Some important inequalities (cont.)

Example

Let X be a non-degenerate RV with mean μ and a finite second moment

- $\rightsquigarrow$ Then, $\mu^2 < E(X^2)$

Obtained by Jensen's inequality using a strictly convex function $\phi(t) = t^2$

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Some important inequalities (cont.)

Example

Harmonic and geometric means

Let $\{a_1, \cdots, a_n\}$ be a set of positive numbers

Set a distribution for RV X by placing mass $1/n$ on each of the numbers a_i

Then, the mean of X is the **arithmetic mean** (**AM**)

$$E(X) = n^{-1} \sum_{i=1}^{n} a_i$$

Since $-\log(x)$ is a convex function, by Jensen's inequality, we have

$$-\log\Big(\frac{1}{n}\sum_{i=1}^{n} a_i\Big) \leq E[-\log(X)] = -n^{-1}\sum_{i=1}^{n}\log(a_i) = -\log(a_1 a_2 \cdots a_n)^{1/n}$$

or, equivalently

$$\Big(\frac{1}{n}\sum_{i=1} n\, a_i\Big) \geq \log(a_1 a_2 \cdots a_n)^{1/n}$$

Some important inequalities (cont.)

Hence,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i \tag{4}$$

The quantity on the LHS is the **geometric mean** (**GM**)

- For any finite set of positive numbers, GM $\leq$ AM

Some important inequalities (cont.)

By replacing a_i by its reciprocal $1/a_i$ (also positive),

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i} \geq \Big(\frac{1}{a_1}\frac{1}{a_2}\cdots\frac{1}{a_n}\Big)^{1/n}$$

or equivalently

$$\frac{1}{1/n\sum_{i=1}^{n}1/a_i} \leq (a_1\,a_2\cdots a_n)^{1/n}$$

The quantity on the LHS is the **harmonic mean** (**HM**)

Remark

Putting things together

- For any finite set of positive numbers, HM $\leq$ GM $\leq$ AM