Two random variables (A)

Multiple random variables

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Distributions

Two random variables

Distributions

A coin is tossed three times, we are interested in the ordered number pair

(number of Hs on the first two tosses, number of Hs on all three tosses)

- H for heads
- T for tails

Let $\mathcal{C} = \{$TTT, TTH, THT, HTT, THH, HTH, HHT, HHH$\}$ be the sample space

- Let X_1 denote the number of Hs on the first two flips
- Let X_2 denote the number of Hs on all three flips

Distributions

Our interest might be represented by the pair of variables (X_1, X_2)

- $[X_1(\texttt{HTH}), X_2(\texttt{HTH})]$ represents the outcome $(1, 2)$
- $[X_1(\texttt{THH}), X_2(\texttt{THH})]$ represents the outcome $(1, 2)$
- $[X_1(\texttt{TTH}), X_2(\texttt{TTH})]$ represents the outcome $(0, 1)$
- $\cdots$

Distributions (cont.)

X_1 and X_2 are real-valued functions defined on the sample space $\mathcal{C}$

- ⇝ To the space of ordered number pairs

$$\mathcal{D} = \{(0,0),(0,1),(1,1),(1,2),(2,2),(2,3)\}$$

X_1 and X_2 are two random variables defined on $\mathcal{C}$ and with space $\mathcal{D}$

- $\mathcal{D}$ is a two-dimensional set
- Set $\mathcal{D}$ is a subset of $\mathcal{R}^2$

Hence, (X_1, X_2) is a vector function from $\mathcal{C}$ to $\mathcal{D}$

Distributions (cont.)

Definition

Random vector

Consider a random experiment with sample space $\mathcal{C}$

Consider two random variables X_1 *and* X_2 *that assign to each element* $c \in \mathcal{C}$ *one and only one ordered pair of numbers* $X_1(c) = x_1,\ X_2(c) = x_2$

(X_1, X_2) *is called a* ***random vector***

The ***space/range*** *of* (X_1, X_2) *is the set of ordered pairs*

$$\mathcal{D} = \big\{(x_1, x_2) : x_1 = X_1(c), x_2 = X_2(c), c \in \mathcal{C}\big\}$$

Distributions (cont.)

Let the sample space associated with the random vector (X_1, X_2) be $\mathcal{D}$

- Let A be a subset of $\mathcal{D}$ (an event)

Consider an event A, we want to define its probability $P_{X_1,X_2}(A)$

Define P_{X_1,X_2} from the **cumulative distribution function** (**CDF**)

$$F_{X_1,X_2}(x_1, x_2) = P\big[\{X_1 \leq x_1\} \cap \{X_2 \leq x_2\}\big], \quad \forall (x_1, x_2) \in \mathcal{R}^2 \qquad (1)$$

As X_1 and X_2 are random variables, each of the events of the intersection and the intersection of the events are events in the original sample space $\mathcal{C}$

As with random variables we can write

$$P\big[\{X_1 \leq x_1\} \cap \{X_2 \leq x_2\}\big] = P\big[X_1 \leq x_1, X_2 \leq x_2\big]$$

Distributions (cont.)

$$P(a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2) = F_{X_1,X_2}(b_1, b_2) - F_{X_1,X_2}(a_1, b_2) - F_{X_1,X_2}(b_1, a_2) + F_{X_1,X_2}(a_1, a_2) \quad (2)$$

Consider all induced probabilities of sets of the form $(a_1, b_1] \times (a_2, b_2]$

- They can be formulated in terms of the CDF

This CDF is the **joint cumulative distribution function** of (X_1, X_2)

Distributions (cont.)

As with RVs, we are mainly interested in two types of random vectors

- Random vectors of the discrete type
- Random vectors of the continuous type

Distributions (cont.)

Consider a random vector (X_1, X_2) whose space $\mathcal{D}$ is finite or countable

Such a random vector is said to be a **discrete random vector**

- ⇝ Hence, X_1 and X_2 are also both discrete

The **joint probability mass function** (**PMF**) of (X_1, X_2) is defined by

$$p_{X_1,X_2}(x_1, x_2) = P[X_1 = x_1, X_2 = x_2], \quad \forall (x_1, x_2) \in \mathcal{D} \tag{3}$$

The PMF uniquely defines the CDF

Distributions (cont.)

Two properties characterise the PMF

$$\begin{aligned} p_{X_1,X_2}(x_1,x_2) &\in [0,1] \\ \sum_{\mathcal{D}}\sum p_{X_1,X_2}(x_1,x_2) &= 1 \end{aligned} \tag{4}$$

For an event $B \in \mathcal{D}$, we have

$$P\big[(X_1,X_2) \in B\big] = \sum_B p_{X_1,X_2}(x_1,x_2)$$

Distributions (cont.)

Example

Consider the discrete random vector (X_1, X_2)

x_1/x_2	0	1	2	3
0	1/8	1/8	0	0
1	0	2/8	2/8	0
2	0	0	1/8	1/8

This is the tabulated PMF

Distributions (cont.)

We define the **support** of discrete random vectors (X_1, X_2)

⇝ All the points (x_1, x_2) in the range of (X_1, X_2) such that $p(x_1, x_2) > 0$

$$\mathcal{S} = \big\{(x_1, x_1) : p(x_1, x_2) > 0, x_1, x_2 \in \mathcal{D}\big\}$$

Distributions (cont.)

Consider a random vector (X_1, X_2) with range $\mathcal{D}$

- Assume its CDF $F_{X_1,X_2}(x_1, x_2)$ is continuous

Such a random vector is said to be a **continuous random vector**

Usually continuous random vectors have the CDF that can be represented as an integral of a non-negative function

$$F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(w_1, w_2) \mathrm{d}w_1 \mathrm{d}w_2, \quad \forall (x_1, x_2) \in \mathcal{R}^2 \quad (5)$$

We call the integrand the **joint probability density** (**CDF**)

Distributions (cont.)

Then,

$$\frac{\partial F_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2} = f_{X_1,X_2}(x_1,x_2) \tag{6}$$

(Except, possibly, on events with probability zero)

Distributions (cont.)

A PDF is characterised by two properties

$$f_{X_1,X_2}(x_1, x_2) \geq 0 \tag{7}$$

$$\int_{\mathcal{D}} f_{X_1,X_2}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = 1 \tag{8}$$

For an event $A \in \mathcal{D}$, we have

$$P[(X_1, X_2) \in A] = \int_A f_{X_1,X_2}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$$

$P[(X_1, X_2)] \in A$ is the volume under surface $z = f_{X_1,X_2}(x_1, x_2)$ over set A

Distributions (cont.)

Remark

When clear, we drop the subscript X_1, X_2 from joint CDFs/PDFs/PMFs

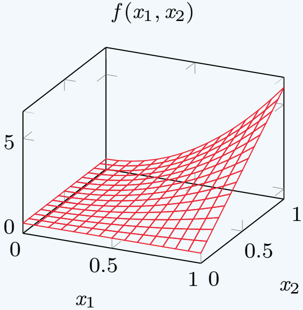
Also, we may use the notation f_{12} to denote f_{X_1,X_2}

Besides X_1, X_2, we sometimes also use X, Y

Distributions (cont.)

Example

Let $f(x_1, x_2)$ be the PDF of two RVs X_1 and X_2 of the continuous type



$$f(x_1, x_2) = \begin{cases} 6x_1^2 x_2, & 0 < x_1, x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Distributions (cont.)

We have,

$$P\left(0 < X_1 < \frac{3}{4}, \frac{1}{3} < X_2 < 2\right) = \int_{1/3}^{2}\int_{0}^{3/4} f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2$$
$$= \int_{1/3}^{1}\int_{0}^{3/4} 6x_1^2 x_2 \mathrm{d}x_1 \mathrm{d}x_2 + \int_{1}^{2}\int_{0}^{3/4} 0 \mathrm{d}x_1 \mathrm{d}x_2$$
$$= 3/8 + 0 = 3/8$$

This probability is the volume under the surface $f(x_1, x_2)$

- above rectangular set $\big\{(x_1, x_2) : 0 < x_1 < 3/4, 1/3 < x_2 < 1\big\} \in \mathcal{R}^2$

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Distributions (cont.)

Consider a continuous random vector (X_1, X_2)

The **support** of (X_1, X_2) contains all points (x_1, x_2) for which $f(x_1, x_2) > 0$

- We denote the support of a random vector by $\mathcal{S}$, with $\mathcal{S} \subset \mathcal{D}$

Distributions (cont.)

Definition of a PDF $f_{X_1,X_2}(x_1, x_2)$ over $\mathcal{R}^2$ extended by using zero elsewhere

- We do this consistently to avoid an incessant reference to $\mathcal{D}$

Once done, we can replace

$$\int_{\mathcal{D}}\int f_{X_1,X_2}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Distributions (cont.)

We extend the PMF $p_{X_1,X_2}(x_1, x_2)$ over a convenient set with zero elsewhere

Hence, we can replace

$$\sum\sum_{\mathcal{D}} p_{X_1,X_2}(x_1, x_2)$$

by

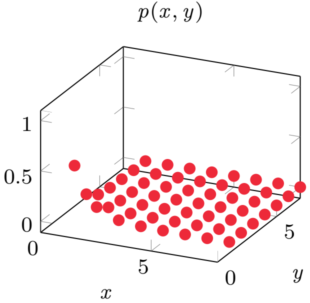
$$\sum_{x_1}\sum_{x_2} p(x_1, x_2)$$

Distributions (cont.)

If a PMF or a PDF in one/more variables is defined explicitly, we can observe by inspection whether the RVs are of the continuous or of the discrete type

Distributions (cont.)

A PMF of two discrete-type variables X and Y

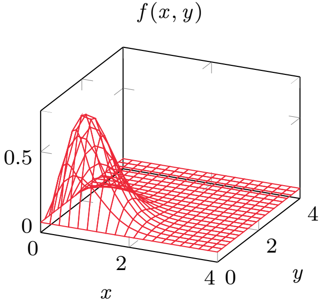


$$p(x, y) = \begin{cases} \dfrac{8}{4^{x+y}}, & x, y = 1, 2, 3, \dots \\ 0, & \text{elsewhere} \end{cases}$$

x/y	1	2	3	4	5	6	7	8
1	0.5000	0.1250	0.0312	0.0078	0.0020	0.0005	0.0001	0.0000
2	0.1250	0.0312	0.0078	0.0020	0.0005	0.0001	0.0000	0.0000
3	0.0312	0.0078	0.0020	0.0005	0.0001	0.0000	0.0000	0.0000
4	0.0078	0.0020	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000
5	0.0020	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Distributions (cont.)

A PDF of two continuous-type variables X and Y



$$f(x,y) = \begin{cases} 4xye^{-(x^2+y^2)}, & x, y \in (0, \infty) \\ 0, & \text{elsewhere} \end{cases}$$

Distributions (cont.)

Let (X_1, X_2) be a random vector, then both X_1 and X_2 are random variables

- We get their distribution in terms of the joint distribution of (X_1, X_2)

Recall that the event which defined the CDF of X_1 at x_1 is $\{X_1 \leq x_1\}$

$$\begin{aligned}\{X_1 \leq x_1\} &= \{X_1 \leq x_1\} \cap \{-\infty < X_2 < +\infty\} \\ &= \{X_1 \leq x_1, -\infty < X_2 < +\infty\}\end{aligned}$$

Taking probabilities, we have

$$F_{X_1}(x_1) = P\big[X_1 \leq x_1, -\infty < X_2 < +\infty\big], \text{ for all } x_1 \in \mathcal{R} \tag{9}$$

Distributions (cont.)

$$F_{X_1}(x_1) = P\big[X_1 \leq x_1, -\infty < X_2 < +\infty\big], \text{ for all } x_1 \in \mathcal{R}$$

We can write the equation as $F_{X_1}(x_1) = \lim_{x_2 \uparrow \infty} F(x_1, x_2)$

- We have a relation between CDFs

This can be extended to either the PMF or the PDF, depending on (X_1, X_2)

Distributions (cont.)

Discrete case

Let $\mathcal{D}_{X_1}$ be the support of X_1

For $x_1 \in \mathcal{D}_{X_1}$, we have

$$F_{X_1}(x_1) = \sum_{w_1 \leq x_1} \sum_{-\infty < x_2 < \infty} p_{X_1,X_2}(w_1, x_2)$$
$$= \sum_{w_1 \leq x_1} \Big\{ \sum_{x_2 < \infty} p_{X_1,X_2}(w_1, x_2) \Big\}$$

By uniqueness of CDFs, quantity in brackets must be the PMF of X_1 at w_1

$$p_{X_1}(x_1) = \sum_{x_2 < \infty} p_{X_1,X_2}(x_1, x_2), \text{ for all } x_1 \in \mathcal{D}_{X_1} \tag{10}$$

To determine the probability that X_1 is x_1

- ⇝ Fix x_1 and sum p_{X_1,X_2} over all of x_2

Distributions (cont.)

Theorem

Let (X_1, X_2) be a discrete random vector with joint PMF $p_{X_1,X_2}(x_1, x_2)$

Then, the marginal PMFs of X_1 and X_2 are

$$p_{X_1}(x_1) = \sum_{x_2} p_{X_1,X_2}(x_1, x_2)$$

$$p_{X_2}(x_2) = \sum_{x_1} p_{X_1,X_2}(x_1, x_2)$$

Proof

For any x_1, let $A_x = \big\{(x_1, x_2) : -\infty < x_2 < \infty\big\}$ be a line in the plane

- First coordinate equal x_1

Then, for any x_1,

$$\begin{aligned} p_{X_1}(x_1) &= P(X_1 = x_1) = P(X_1 = x_1, -\infty < X_2 < \infty) = P\big[(X_1, X_2) \in A_x\big] \\ &= \sum_{(x_1,x_2) \in A_x} p_{X_1,X_2}(x_1, x_2) = \sum_{x_2} p_{X_1,X_2}(x_1, x_2) \end{aligned}$$

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Distributions (cont.)

Consider a tabulated joint PMF

- Rows comprised of X_1 support values
- Columns comprised of X_2 support values

The distribution of X_1 can be obtained by marginal sums of the columns

Distributions (cont.)

Example

Consider the discrete random vector (X_1, X_2) with the tabulated PMF

x_1/x_2	0	1	2	3	$p_{X_1}(x_1)$
0	1/8	1/8	0	0	2/8
1	0	2/8	2/8	0	4/8
2	0	0	1/8	1/8	2/8
$p_{X_2}(x_2)$	1/8	3/8	3/8	1/8	

Joint probabilities have been summed in each row and each column

The sums are added to the margins of table

- Last column is the PMF of X_1
- Last row is the PMF of X_2

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Distributions (cont.)

Such distributions are often referred to as **marginal PMFs**

Distributions (cont.)

Continuous case

Let $\mathcal{D}_{X_1}$ be the support of X_1 and $x_1 \in \mathcal{D}_{X_1}$

For $x_1 \in \mathcal{R}$, equation $F_{X_1}(x_1) = P\big[X_1 \leq x_1, -\infty < X_2 < \infty\big]$ equals to

$$F_{X_1}(x_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{\infty} f_{X_1,X_2}(w_1, x_2) \mathrm{d}x_2 \mathrm{d}w_1$$
$$= \int_{-\infty}^{x_1} \Big\{ \int_{-\infty}^{\infty} f_{X_1,X_2}(w_1, x_2) \mathrm{d}x_2 \Big\} \mathrm{d}w_1$$

By uniqueness of CDFs, quantity between brackets must be the PDF of X_1

- evaluated at w_1

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) \mathrm{d}x_2, \quad \forall x \in \mathcal{D}_X$$

Distributions (cont.)

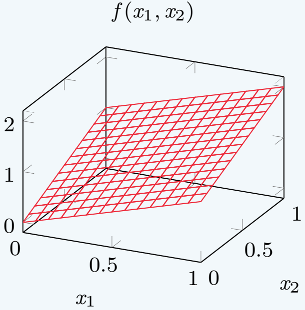
In the continuous case

- The marginal PDF of X_1 is found by integrating out x_2
- The marginal PDF of X_2 is found by integrating out x_1

Distributions (cont.)

Example

Let X_1 and X_2 have joint PDF



$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1, x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Distributions (cont.)

The marginal PDF of X_1

$$f_1(x_1) = \int_0^1 (x_1 + x_2) \mathrm{d}x_2 = x_1 + 1/2, \quad 0 < x_1 < 1$$

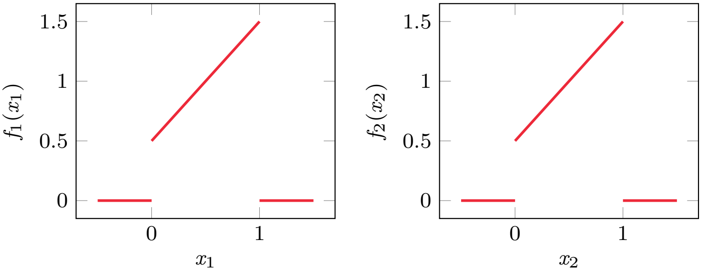
and zero elsewhere

The marginal PDF of X_2

$$f_2(x_2) = \int_0^1 (x_1 + x_2) \mathrm{d}x_1 = 1/2 + x_2, \quad 0 < x_2 < 1$$

and zero elsewhere

Distributions (cont.)



A probability as $P(X_1 \leq 1/2)$ is computed from either $f_1(x_1)$ or $f(x_1, x_2)$

$$\int_0^{1/2} \int_0^1 f(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1 = \int_0^{1/2} f_1(x_1) \mathrm{d}x_1 = 3/8$$

Distributions (cont.)

A probability as $P(X_1 + X_2 \leq 1)$ must be from the joint PDF $f(x_1, x_2)$

$$\int_0^1 \int_0^{1-x_1} (x_1 + x_2) \mathrm{d}x_2 \mathrm{d}x_1 = \int_0^1 \Big[x_1(1 - x_1) + \frac{(1 - x_1)^2}{2} \Big] \mathrm{d}x_1$$
$$= \int_0^1 (1/2 - 1/2 x_1^2) \mathrm{d}x_1 = 1/3$$

This is the volume under surface $f(x_1, x_2) = x_1 + x_2$

- Above set $\big\{(x_1, x_2) : 0 < x_1, x_1 + x_2 \leq 1\big\}$

Expectation

Distributions of two random variables

Expectation

Let (X_1, X_2) be a random vector and let $Y = g(X_1, X_2)$

- for some real-valued function $g : \mathcal{R}^2 \to \mathcal{R}$

Y is a random vector, we can determine its expectation

$\rightsquigarrow$ By getting its distribution

Expectation (cont.)

Suppose (X_1, X_2) is of the continuous type

Then $E(Y)$ exists if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big|g(x_1, x_2)\big| f_{X_1,X_2}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 < \infty$$

$$\rightsquigarrow \quad E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Expectation (cont.)

Likewise, suppose (X_1, X_2) is of the discrete type

Then $E(Y)$ exists if

$$\sum_{x_1}\sum_{x_2} \big|g(x_1, x_2)\big| p_{X_1,X_2}(x_1, x_2) < \infty$$

$$\rightsquigarrow \quad E(Y) = \sum_{x_1}\sum_{x_2} g(x_1, x_2) p_{X_1,X_2}(x_1, x_2)$$

Expectation (cont.)

Theorem

E is a linear operator

Let (X_1, X_2) be a random vector

Let $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g(X_1, X_2)$ be RVs whose expectations exist

For all real numbers k_1 and k_2,

$$E(k_1 Y_1 + k_2 Y_2) = k_1 E(Y_1) + k_2 E(Y_2)$$

Proof

For the continuous case

Existence of the expected value of $k_1 Y_1 + k_2 Y_2$ follows

- Assumptions
- Triangle inequality
- Linearity of integrals

Expectation (cont.)

By using the triangle inequality and the linearity of the integrals

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big|k_1 g_1(x_1,x_2)+k_2 g_2(x_1,x_2)\big| f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\ \leq \big|k_1\big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big|g_1(x_1,x_2)\big| f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\ +\big|k_2\big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big|g_2(x_1,x_2)\big| f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 < \infty\end{aligned}$$

Expectation (cont.)

By using the linearity of the integral

$$
\begin{aligned}
& E(k_1 Y_1 + k_2 Y_2) \\
& \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \big[k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)\big] f_{X_1, X_2}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\
& \qquad = k_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\
& + k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = k_1 E(Y_1) + k_2 E(Y_2)
\end{aligned}
$$

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Expectation (cont.)

Remark

The expected value of any function $g(X_2)$ of X_2 can be found in two ways

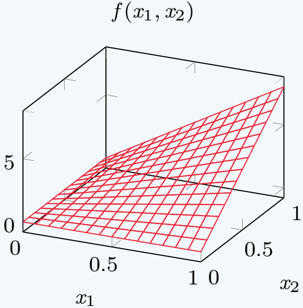
$$E\big[g(X_2)\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_2) f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = \int_{-\infty}^{\infty} g(x_2) f_{X_2}(x_2)\mathrm{d}x_2$$

The single integral is obtained from the double, by integrating on x_1 first

Expectation (cont.)

Example

Let X_1 and X_2 have the PDF

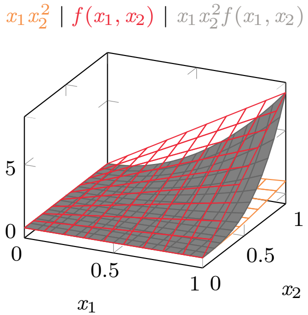


$$f(x_1, x_2) = \begin{cases} 8x_1x_2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Expectation (cont.)

Then,

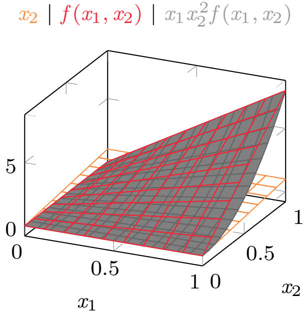
$$\begin{aligned} E(X_1 X_2^2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 x_2^2) f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 x_2^2)(8 x_1 x_2) \mathrm{d}x_1 \mathrm{d}x_2 = \int_0^1 \int_0^{x_2} 8 x_1^2 x_2^3 \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \int_0^1 8/3 x_2^6 \mathrm{d}x_2 = 8/21 \end{aligned}$$



Expectation (cont.)

In addition,

$$E(X_2) = \int_0^1 \int_0^{x_2} x_2(8x_1 x_2)\mathrm{d}x_1 \mathrm{d}x_2 = 4/5$$



X_2 has the PDF $f_2(x_2) = 4x_2^3$ for $0 < x_2 < 1$ and zero elsewhere

- Thus, the expectation can be obtained also from

$$E(X_2) = \int_0^1 x_2(4x_2^3)\mathrm{d}x_2 = 4/5$$

Expectation (cont.)

$$
\begin{aligned}
E(7X_1X_2^2 + 5X_2) &= 7E(X_1X_2^2) + 5E(X_2) \\
&= (7)(8/21) + (5)(4/5) = 20/3
\end{aligned}
$$

■

Expectation (cont.)

Example

Let X_1 and X_2 have the PDF

$$f(x_1, x_2) = \begin{cases} 8x_1 x_2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Suppose the random variable Y is defined by $Y = X_1/X_2$

- We are interested in $E(Y)$

We can determine it in two ways

Expectation (cont.)

The first way is by definition (find the distro then determine expectation)

- The CDF of Y, for $0 < y \leq 1$

$$F_Y(y) = P(Y \leq y) = P(X_1 \leq yX_2) = \int_0^1 \int_0^{yx_2} 8x_1x_2 \mathrm{d}x_1 \mathrm{d}x_2$$
$$= \int_0^1 4y^2x_2^3 \mathrm{d}x_2 = y^2$$

- Hence, the PDF of Y

$$f_Y(y) = F'_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

This yields

$$E(Y) = \int_0^1 y(2y)\mathrm{d}y = 2/3$$

Expectation (cont.)

For the second way, we use $E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) p_{X_1,X_2}(x_1, x_2)$

- We find $E(Y)$ directly

$$E(Y) = E\big(X_1/X_2\big) = \int_0^1 \Big\{ \int_0^{x_2} (x_1/x_2)(8x_1x_2)\mathrm{d}x_1 \Big\}\mathrm{d}x_2$$
$$= \int_0^1 8/3x_2^3\mathrm{d}x_2 = 2/3$$

■

Expectation (cont.)

Definition

Moment generating function

Let $\mathbf{X} = (X_1, X_2)'$ *be a random vector*

Suppose $E\big[\exp(t_1 X_1 + t_2 X_2)\big]$ *exists for* $|t_1| < h_1$ *and* $|t_2| < h_2$

- h_1 *and* h_2 *are positive numbers*

Then, this quantity is indicated by $M_{X_1,X_2}(t_1, t_2)$

- *The* ***moment generating function*** *(**MGF**) of* $\mathbf{X}$

The MGF of a random vector uniquely determines its distribution

- If it exists

As in the single-variable case

Expectation (cont.)

Let $\mathbf{t} = (t_1, t_2)'$

Similarly to the MGF of a RV, we can write the MGF of $\mathbf{X}$ as

$$M_{X_1,X_2}(\mathbf{t}) = E\left[e^{\mathbf{t}'\mathbf{X}}\right]$$

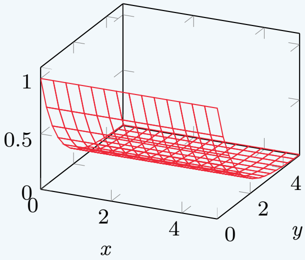
The MGFs of X_1 and X_2 are $M_{X_1,X_2}(t_1, 0)$ and $M_{X_1,X_2}(0, t_2)$, respectively

Expectation (cont.)

Example

Let the continuous-type random variables X and Y have the joint PDF

$f(x,y)$ with $(x,y) \in \mathcal{R}^2$

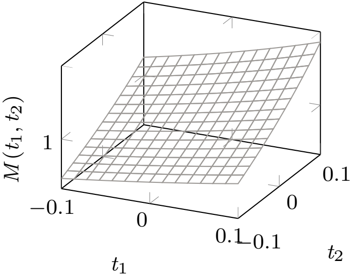


$$f(x,y) = \begin{cases} \exp(-y), & 0 < x < y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Expectation (cont.)

The MGF of this joint distribution

$$M(t_1, t_2) = \int_0^{\infty} \int_x^{\infty} \exp(t_1 x + t_2 y - y) \mathrm{d}x \mathrm{d}y$$
$$= \frac{1}{(1 - t_1 - t_2)(1 - t_2)}, \text{ for } t_1 + t_2 < 1 \text{ and } t_2 < 1$$

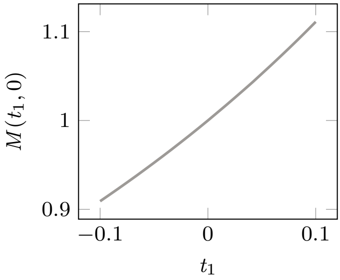


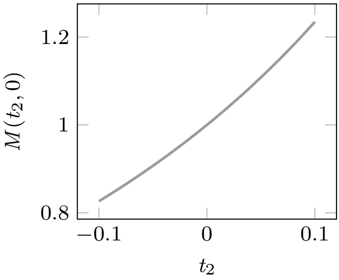
Expectation (cont.)

Moment-generating functions of the marginal distributions of X and Y

$$M(t_1, 0) = \frac{1}{1 - t_1}, \quad t_1 < 1$$

$$M(t_2, 0) = \frac{1}{(1 - t_2)^2}, \quad t_2 < 1$$



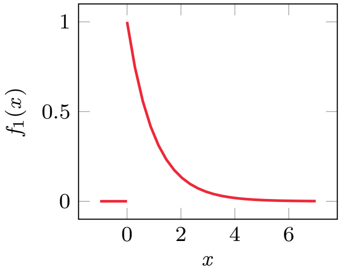


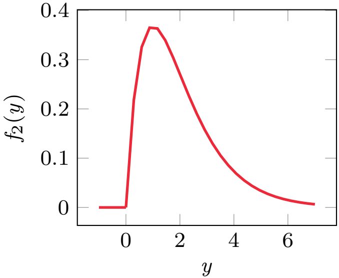
Expectation (cont.)

These MGFs are those of the marginal probability density functions

$$f_1(x) = \int_x^{\infty} e^{-y} \mathrm{d}y = e^{-x}, \quad 0 < x < \infty \quad \text{(zero elsewhere)}$$

$$f_2(y) = e^{-y} \int_0^{y} \mathrm{d}x = y e^{-y}, \quad 0 < y < \infty \quad \text{(zero elsewhere)}$$





Expectation (cont.)

We also need to define the expected value of a random vector itself

- It is a not new concept

It is defined from component-wise expectations

Definition

Expected value of a random vector

Let $\mathbf{X} = (X_1, X_2)'$ be a random vector

*The **expected value** of $\mathbf{X}$ exists if the expectations of X_1 and X_2 exist*

*If it exists, then the **expected value** has the form*

$$E(\mathbf{X}) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} \tag{11}$$

Transformations

Two random variables

Transformations

Let (X_1, X_2) be a random vector, suppose we know the joint distribution

We are interested in the distribution of a transformation of (X_1, X_2)

$\rightsquigarrow$ $Y = g(X_1, X_2)$

We could try to obtain the CDF of Y or we could use a transformation

We extend transformation theory for random variables to random vectors

- We present discrete and continuous cases separately

Transformations (cont.)

Discrete case

Let $p_{X_1,X_2}(x_1, x_2)$ be the joint PMF of two discrete RVs X_1 and X_2

Let $\mathcal{S}$ be the bi-dimensional set of points where $p_{X_1,X_2}(x_1, x_2) > 0$

- $\mathcal{S}$ is the support of (X_1, X_2)

Let $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a 1-to-1 map from $\mathcal{S}$ onto $\mathcal{T}$

The joint PMF of the new RVs $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$

$$p_{Y_1,Y_2}(y_1, y_2) = \begin{cases} p_{X_1,X_2}\big[w_1(y_1, y_2), w_2(y_1, y_2)\big], & (y_1, y_2) \in \mathcal{T} \\ 0, & \text{elsewhere} \end{cases}$$

$x_1 = w_1(y_1, y_2)$, $x_2 = w_2(y_1, y_2)$, inverses of $y_1 = u_1(x_1, x_2)$, $y_2 = u_2(x_1, x_2)$

Transformations (cont.)

From the joint PMF $p_{Y_1,Y_2}(y_1, y_2)$, we can determine both marginals

- The marginal PMF of Y_1, by summing on y_2
- The marginal PMF of Y_2, by summing on y_1

Transformations (cont.)

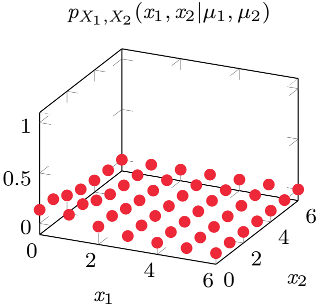
Example

Let X_1 and X_2 have the joint PMF

$$p_{X_1,X_2}(x_1,x_2) = \begin{cases} \dfrac{\mu_1^{x_1}\mu_2^{x_2}e^{-\mu_1}e^{-\mu_2}}{x_1!x_2!}, & \begin{cases} x_1 = 0,1,2,3,\cdots \\ x_2 = 0,1,2,3,\cdots \end{cases} \\ 0, & \text{elsewhere} \end{cases}$$

μ_1, μ_2 are positive real numbers (in the plots, $\mu_1 = 1, \mu_2 = 1$)

Space $\mathcal{S}$ is the set of points (x_1, x_2) with x_1 and x_2 non-negative integers



Transformations (cont.)

We wish to find the PMF of $Y_1 = X_1 + X_2$

If we use the change of variable method, we need to define a second RV Y_2

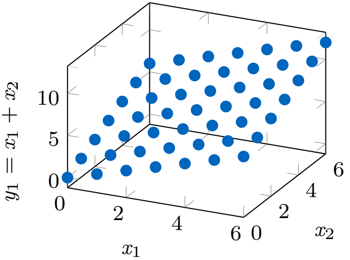
- This RV is of no interest to us

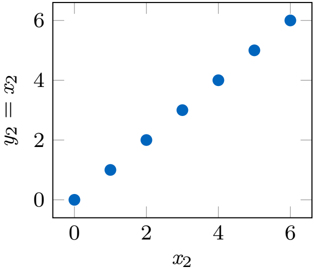
Let us choose it in such a way that we have a simple 1-to-1 transformation

Let us take $Y_2 = X_2$

Transformations (cont.)

Then $y_1 = x_1 + x_2$ and $y_2 = x_2$ represent a 1-to-1 transformation

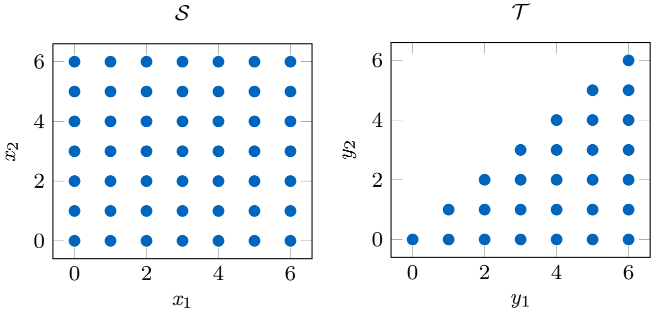




- It maps $\mathcal{S}$ onto $\mathcal{T}$

$$\mathcal{T} = \{(y_1, y_2) : y_2 = 0, 1, \ldots, y_1 \text{ and } y_1 = 0, 1, 2, \ldots\}$$

Transformations (cont.)



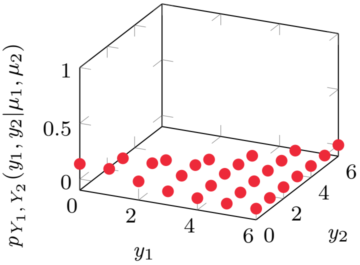
Note that if $(y_1, y_2) \in \mathcal{T}$, then $0 \leq y_2 \leq y_1$

Transformations (cont.)

The inverse functions are $x_1 = y_1 - y_2$ and $x_2 = y_2$

Thus, the joint PMF of Y_1 and Y_2 is

$$p_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \dfrac{\mu_1^{y_1 - y_2} \mu_2^{y_2} e^{-\mu_1 - \mu_2}}{(y_1 - y_2)! y_2!}, & (y_1, y_2) \in \mathcal{T} \\ 0, & \text{elsewhere} \end{cases}$$



Transformations (cont.)

The marginal PMF of Y_1

$$
\begin{aligned}
p_{Y_1}(y_1) &= \sum_{y_2=0}^{y_1} p_{Y_1,Y_2}(y_1, y_2) \\
&= \frac{e^{-\mu_1-\mu_2}}{y_1!} \sum_{y_2=0}^{y_1} \frac{y_1!}{(y_1-y_2)!y_2!} \mu_1^{y_1-y_2} \mu_2^{y_2} \\
&= \frac{(\mu_1+\mu_2)^{y_1} e^{-\mu_1-\mu_2}}{y_1!}, \quad y_1 = 0, 1, 2, \dots
\end{aligned}
$$

and zero elsewhere

(*) The third equality holds from the binomial expansion

Transformations (cont.)

Continuous case

We consider an example which is illustrative of the CDF technique

Transformations (cont.)

Example

Choose at random a point (X, Y) from $\mathcal{S} = \big\{(x, y) : 0 < x, y < 1\big\}$

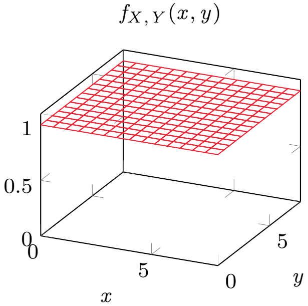
The interest is not in X or in Y, but in $Z = X + Y$

We need a suitable probability model

- $\rightsquigarrow$ Then, we can find the PDF of Z

Transformations (cont.)

Assume that the probability distribution over the unit square is uniform



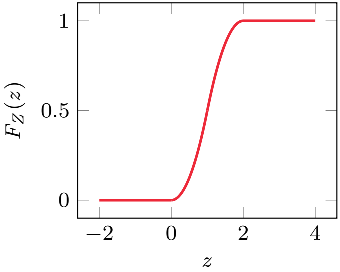
$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

This describes the probability model

Transformations (cont.)

Let the CDF of Z be denoted by $F_Z(z) = P(X + Y \leq z)$, then

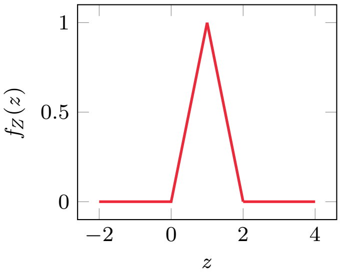
$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \int_0^z \int_0^{z-x} \mathrm{d}y\mathrm{d}x = z^2/2, & 0 \leq z < 1 \\ 1 - \int_{z-1}^1 \int_{z-x}^1 \mathrm{d}y\mathrm{d}x = 1 - \dfrac{(2-z)^2}{2}, & 1 \leq z < 2 \\ 1, & 2 \leq z \end{cases}$$



Transformations (cont.)

The $F'_Z(z)$ exists for all values of z

Then, the PMF of Z



$$f_Z(z) = \begin{cases} z, & 0 < z < 1 \\ 2 - z, & 1 \leq z < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Transformations (cont.)

We consider in general the transformation technique for continuous variables

Let (X_1, X_2) have the jointly continuous distribution with PDF $f_{X_1,X_2}(x_1, x_2)$

- The support set is of (X_1, X_2) is $\mathcal{S}$

Suppose RVs Y_1 and Y_2 are given by $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$

Functions $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a 1-to-1 transformation

- A map from the set $\mathcal{S} \in \mathcal{R}^2$ onto a set $\mathcal{T} \in \mathcal{R}^2$
- $\mathcal{T}$ is the support of (Y_1, Y_2)

Transformations (cont.)

If we express each of x_1 and x_2 in terms of y_1 and y_2, we can write

- $x_1 = w_1(y_1, y_2)$
- $x_2 = w_2(y_1, y_2)$

The **Jacobian** of the transformation is the determinant of order 2

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

It is assumed that

- These first-order derivatives are continuous
- The Jacobian is not identically zero in $\mathcal{T}$

Transformations (cont.)

By use of a theorem in analysis[1], we can find the joint PDF of (Y_1, Y_2)

Let A be a subset of $\mathcal{S}$

Let B denote a mapping of A under the one-to-one transformation

Map is 1-to-1, events $\{(X_1, X_2) \in A\}$ and $\{(Y_1, Y_2) \in B\}$ are equivalent

$$\begin{aligned} P\big[(Y_1, Y_2) \in B\big] &= P\big[(X_1, X_2) \in A\big] \\ &= \int\int_A f_{X_1,X_2}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \end{aligned}$$

[1] R. Creighton Buck, *Advanced calculus*.

Transformations (cont.)

We wish now to change variables of integration

- $y_1 = u_1(x_1, x_2),\ y_2 = u_2(x_1, x_2)$

or

- $x_1 = w_1(y_1, y_2),\ x_2 = w_2(y_1, y_2)$

It has been proved in analysis

$$\int\int_A f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = \int\int_B f_{X_1,X_2}\big[w_1(y_1,y_2), w_2(y_1,y_2)\big]\big|J\big|\mathrm{d}y_1\mathrm{d}y_2$$

Thus, for every set $B \in \mathcal{T}$

$$P\big[(Y_1, Y_2) \in B\big] = \int\int_B f_{X_1,X_2}\big[w_1(y_1,y_2), w_2(y_1,y_2)\big]\big|J\big|\mathrm{d}y_1\mathrm{d}y_2$$

Transformations (cont.)

This implies that the joint PDF $f_{Y_1,Y_2}(y_1, y_2)$ of Y_1 and Y_2

$$f_{Y_1,Y_2}(x_1, x_2) = \begin{cases} f_{X_1,X_2}\big[w_1(y_1, y_2), w_2(y_1, y_2)\big]\big|J\big|, & (y_1, y_2) \in \mathcal{T} \\ 0, & \text{elsewhere} \end{cases}$$

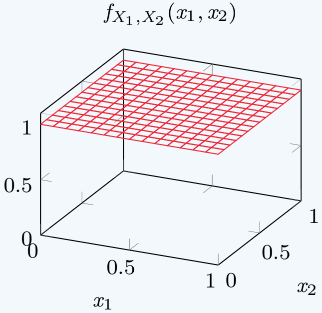
The marginal PDF $f_{Y_1}(y_1)$ of Y_1, from the joint PDF $f_{Y_1,Y_2}(y_1, y_2)$

- In the usual manner, by integrating on y_2

Transformations (cont.)

Example

Suppose (X_1, X_2) have the joint PDF



$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & 0 < x_1, x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

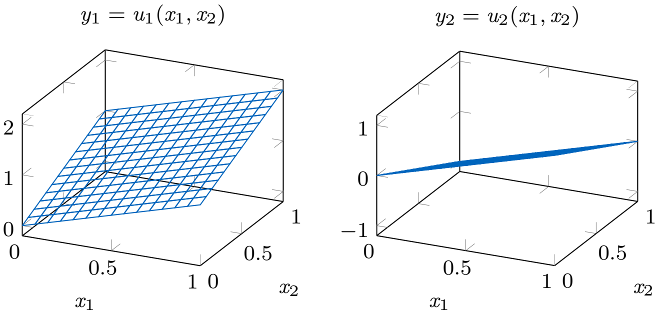
The support of (X_1, X_2) is the set $\mathcal{S} = \{(x_1, x_2) : 0 < x_1, x_2 < 1\}$

Transformations (cont.)

Suppose $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$

The transformation is

$$y_1 = u_1(x_1, x_2) = x_1 + x_2$$
$$y_2 = u_2(x_1, x_2) = x_1 - x_2$$



This transformation is one-to-one

Transformations (cont.)

We determine set $\mathcal{T}$ in the $y_1 - y_2$ plane onto which $\mathcal{S}$ has been mapped

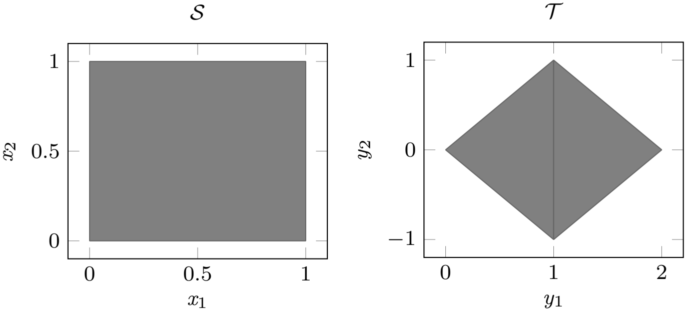
$$x_1 = w_1(y_1, y_2) = 1/2(y_1 + y_2)$$
$$x_2 = w_2(y_1, y_2) = 1/2(y_1 - y_2)$$

The boundaries of $\mathcal{S}$ have been transformed such that

$$x_1 = 0 \text{ into } 0 = 1/2(y_1 + y_2)$$
$$x_1 = 1 \text{ into } 1 = 1/2(y_1 + y_2)$$
$$x_2 = 0 \text{ into } 0 = 1/2(y_1 - y_2)$$
$$x_2 = 1 \text{ into } 1 = 1/2(y_1 - y_2)$$

They define $\mathcal{T}$

Transformations (cont.)



Transformations (cont.)

We could have transformed the boundaries of $\mathcal{S}$

Alternatively, we directly use the inequalities

$$0 < x_1 < 1$$
$$0 < x_2 < 1$$

The four inequalities become

$$0 < 1/2(y_1 + y_2) < 1$$
$$0 < 1/2(y_1 - y_2) < 1$$

They are equivalent to

$$-y_1 < y_2$$
$$y_2 < 2 - y_1$$
$$y_2 < y_1$$
$$y_1 - 2 < y_2$$

They define $\mathcal{T}$

Transformations (cont.)

We need the Jacobian of the inverse transformation

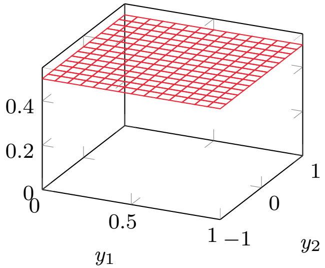
$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

Transformations (cont.)

Hence, the joint PDF of random vector (Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} f_{X_1,X_2}\big[1/2(y_1+y_2), 1/2(y_1-y_2)\big]\big|J\big| = 1/2, & (y_1, y_2 \in \mathcal{T}) \\ 0, & \text{elsewhere} \end{cases}$$

$f_{Y_1,Y_2}(y_1,y_2)$ with $(y_1,y_2) \in \mathcal{R}^2$

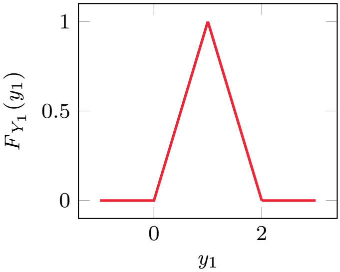


Transformations (cont.)

The marginal PDF of Y_1

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\mathrm{d}y_2$$

$$= \begin{cases} \int_{-y_1}^{y_1} 1/2\mathrm{d}y_2 = y_1, & 0 < y_1 \leq 1 \\ \int_{y_1-2}^{2-y_1} 1/2\mathrm{d}y_2 = 2 - y_1 & 1 < y_1 < 2 \\ 0, & \text{elsewhere} \end{cases}$$

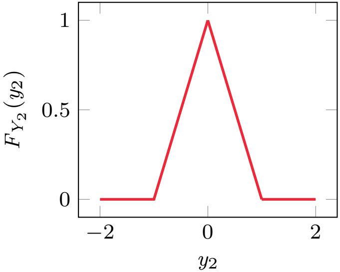


Transformations (cont.)

The marginal PDF of Y_2

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\mathrm{d}y_1$$

$$= \begin{cases} \int_{-y_2}^{y_2+2} 1/2\mathrm{d}y_1 = y_2 + 1, & -1 < y_2 \leq 0 \\ \int_{y_2}^{2-y_2} 1/2\mathrm{d}y_1 = 1 - y_2, & 0 < y_1 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

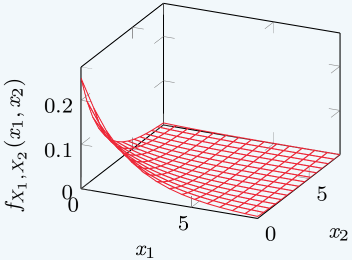


Transformations (cont.)

Example

Let $Y_1 = \frac{1}{2}(X_1 - X_2)$ where X_1 and X_2 have the joint PDF

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1/4 e^{\left(-\frac{x_1 + x_2}{2}\right)}, & 0 < x_1, x_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$



Transformations (cont.)

Let $Y_2 = X_2$

$y_1 = 1/2(x_1 - x_2)$ and $y_2 = x_2$, or equally $x_1 = 2y_1 + y_2$ and $x_2 = y_2$

⇝ Define a 1-to-1 transformation

From

$$\mathcal{S} = \{(x_1, x_2) : 0 < x_1, x_2 < \infty\}$$

onto

$$\mathcal{T} = \{(y_1, y_2) : -2y_1 < y_2 \text{ and } 0 < y_2 < \infty, -\infty < y_1 < \infty\}$$

Transformations (cont.)

The Jacobian of the inverse transformation

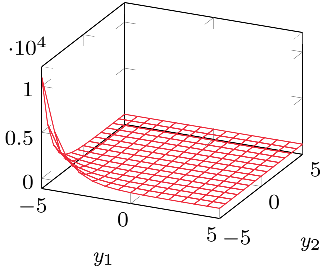
$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

Transformations (cont.)

The joint PDF of the random vector (Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} |2|/4e^{(-y_1-y_2)}, & (y_1,y_2) \in \mathcal{T} \\ 0, & \text{elsewhere} \end{cases}$$

$f_{Y_1,Y_2}(y_1,y_2)$ with $(y_1,y_2) \in \mathcal{R}^2$

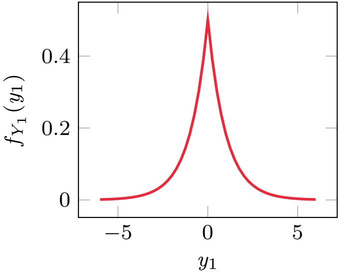


Transformations (cont.)

The marginal PDF of Y_1

$$f_{Y_1}(y_1) = \begin{cases} \int_{-2y_1}^{\infty} 1/2e^{(-y_1-y_2)}\mathrm{d}y_2 = 1/2e^{y_1}, & -\infty < y_1 < 0 \\ \int_{0}^{\infty} 1/2e^{(-y_1-y_2)}\mathrm{d}y_2 = 1/2e^{-y_1}, & 0 \leq y_1 < \infty \end{cases}$$
$$= 1/2e^{-|y_1|}, \text{ for } -\infty < y_1 < \infty$$

This PDF is the **double exponential** or **Laplace** PDF

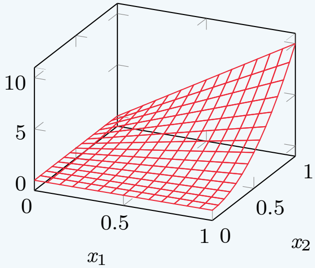


Transformations (cont.)

Example

Let X_1 and X_2 have the joint PDF

$f_{X_1,X_2}(x_1, x_2)$ with $(x_1, x_2) \in \mathcal{R}^2$



$$f_{12}(x_1, x_2) = \begin{cases} 10 x_1 x_2^2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Suppose $Y_1 = X_1/X_2$ and $Y_2 = X_2$

Transformations (cont.)

The inverse transformation is $x_1 = y_1 y_2$ and $x_2 = y_2$, with Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

The inequalities defining the support $\mathcal{S}$ of (X_1, X_2) become

$$\begin{aligned} 0 &< y_1 y_2 \\ y_1 y_2 &< y_2 \\ y_2 &< 1 \end{aligned}$$

The inequalities are equivalent to

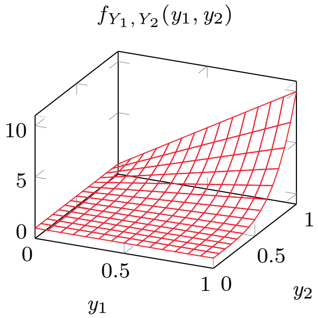
$$\begin{aligned} 0 < y_1 < 1 \\ 0 < y_2 < 1 \end{aligned}$$

They define the support set $\mathcal{T}$ of (Y_1, Y_2)

Transformations (cont.)

Hence, the joint PDF of (Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1,y_2) = 10 y_1 y_2 y_2^2 |y_2| = 10 y_1 y_2^4, \quad (y_1,y_2) \in \mathcal{T}$$



Transformations (cont.)

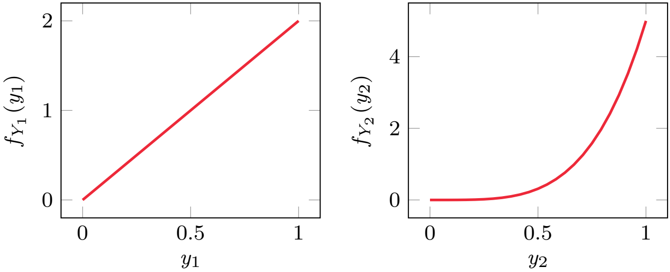
The marginal PDFs

$$\rightsquigarrow f_{Y_1}(y_1) = \int_0^1 10 y_1 y_2^4 \mathrm{d}y_2 = 2y_1, \quad 0 < y_1 < 1$$

and zero elsewhere

$$\rightsquigarrow f_{Y_2}(y_2) = \int_0^1 10 y_2 y_2^4 \mathrm{d}y_1 = 5y_2^4, \quad 0 < y_2 < 1$$

and zero elsewhere



Transformations (cont.)

Variable change and CDF for finding distributions of functions of RVs

There is another model, the moment generating function technique

- It works well for linear functions of RVs

If $Y = g(X_1, X_2)$, then $E(Y)$, if it exists, can be from

$$E(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x_1, x_2) f_{X_1, X_2} \mathrm{d}x_1 \mathrm{d}x_2$$

(Summations replace integrals in discrete case)

Transformations (cont.)

Function $g(X_1, X_2)$ could be chosen to be $e^{\left[tu(X_1, X_2)\right]}$

- ⇝ We would find the MGF of function $Z = u(X_1, X_2)$

If we could then recognise this MGF as belonging to a certain distribution

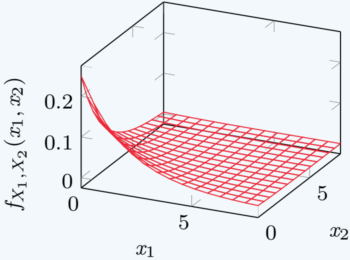
- ⇝ Z would have that distribution

Transformations (cont.)

Example

Let X_1 and X_2 have the joint PDF

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1/4e^{\left(-\dfrac{x_1+x_2}{2}\right)}, & 0 < x_1, x_2 < \infty \\ 0, & \text{elsewhere} \end{cases}$$



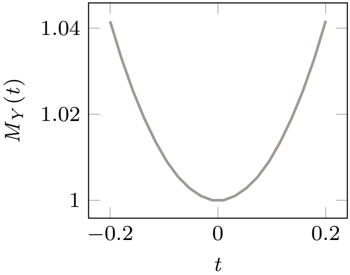
Let $Y = (1/2)(X_1 - X_2)$

Transformations (cont.)

The MGF of Y

$$E(e^{tY}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x_1-x_2)/2}\frac{1}{4}e^{[-(x_1+x_2)/2]}\mathrm{d}x_1\mathrm{d}x_2$$
$$= \Big[\int_0^{\infty}\frac{1}{2}e^{-x_1(1-t)/2}\mathrm{d}x_1\Big]\Big[\int_0^{\infty}\frac{1}{2}e^{-x_2(1+t)/2}\mathrm{d}x_2\Big]$$
$$= \Big[\frac{1}{1-t}\Big]\Big[\frac{1}{1+t}\Big] = \frac{1}{1-t^2}$$

for $1-t>0$ and $1+t>0$, or equivalently $t \in (-1,+1)$



Transformations (cont.)

Consider the MGF of the double exponential distribution

$$\int_{-\infty}^{\infty} e^{tx} \frac{e^{-|x|}}{2} \mathrm{d}x = \int_{-\infty}^{0} \frac{e^{(1+t)x}}{2} \mathrm{d}x + \int_{0}^{\infty} \frac{e^{(1-t)x}}{2} \mathrm{d}x$$
$$= \frac{1}{2(1+t)} + \frac{1}{2(1-t)} = \frac{1}{1-t^2}$$

with $t \in (-1, +1)$

The RV Y does have the double exponential distribution

- Because of the uniqueness of the MGF

■