The binomial and the Poisson distribution

Useful distributions

Francesco Corona

Department of Computer Science
Federal University of Ceará, Fortaleza

The binomial distribution

Useful distributions

The binomial distribution

A **Bernoulli experiment** is a random experiment

Its outcome can be classified in one of two mutually exclusive ways

- `Success` or `failure`, `defective` and `non-defective`
- `Life` or `death`, `female` or `male`
- ...

The binomial distribution (cont.)

A sequence of **Bernoulli trials**

- ⇝ A Bernoulli experiment performed several independent times
- ⇝ The probability of `success` remains the same

Let p indicate the probability of `success` on each trial

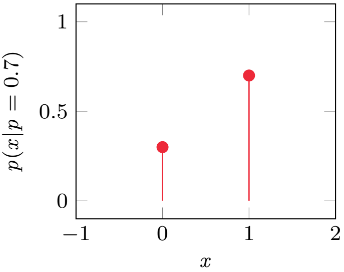
The binomial distribution (cont.)

Let X be a random variable associated with a Bernoulli trial

- $X(\texttt{success}) = 1$
- $X(\texttt{failure}) = 0$

The two outcomes, success/failure, are indicated by 0/1

The PMF of the random variable X



$$p(x) = p^x(1-p)^{1-x} \quad \text{for } x = 0, 1 \tag{1}$$

X is said to have a **Bernoulli distribution**, with parameter p

The binomial distribution (cont.)

The expected value of X

$$\begin{aligned}\mu = E(X) &= \sum_{x \in \{0,1\}} x p(x) = \sum_{x \in \{0,1\}} x p^x (1-p)^{(1-x)} \\ &= (0)(1-p) + (1)(p) \\ &= p\end{aligned}$$

The variance of X

$$\begin{aligned}\sigma^2 = \text{var}(X) &= p^2(1-p) + (1-p)^2 p \\ &= p(1-p)\end{aligned}$$

The standard deviation of X

$$\sigma = \big[p(1-p)\big]^{1/2}$$

The binomial distribution (cont.)

A sequence of n Bernoulli trials

Suppose that n independent Bernoulli trails are performed

- Each with the same success probability

An observed sequence of n trials is a n-tuple of 0s and 1s

$$\underbrace{0, 0, 1, 0, 1, 1, 1, 0, \cdots}_{n \text{ times}}$$
$$\underbrace{1, 0, 1, 0, 0, 0, 1, 0, \cdots}_{n \text{ times}}$$
$$\underbrace{1, 1, 0, 1, 0, 1, 0, 0, \cdots}_{n \text{ times}}$$

There are a total of 2^n possible such sequences[1]

[1] Among them, those without repetitions (permutations) are $n!/(n-n)! = n!$.

The binomial distribution (cont.)

We are interested in the distribution of the number of successes in n trials

- The order of their occurrence is not of our interest

The number of sequences with a single 1 (success) out of n trials

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$$

The number of sequences with two 1s (successes) out of n trials

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = n(n-1)/2$$

The number of sequences with x 1s (successes) out of n trials

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

If we observe x successes ($x = 0, 1, 2 \dots, n$), then also $(n-x)$ failures occur

The binomial distribution (cont.)

Consider a sequence of n Bernoulli trials

Let the number of observed successes in n trials be the random variable X

- The possible values of X are $x = 0, 1, 2, \dots, n$

The binomial distribution (cont.)

Let X_i denote the Bernoulli random variable associated with the i-th trial

- The probabilities of success and failure on each trial are p and $1 - p$

$$p^x(1-p)^{1-x}, \text{ for } x = \{0, 1\}$$

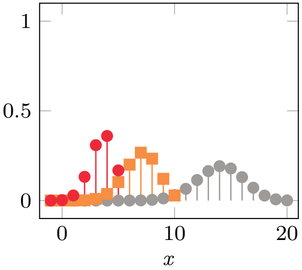
The probability of any of the possible ways to get x successes in n trails

⇝ The Bernoulli random variables are assumed independent

$$\sum_{x=0}^{n} p^x(1-p)^{1-x} = p^x(1-p)^{n-x}$$

The binomial distribution (cont.)

$p(x|p = 0.7, n = 5|10|20)$



$$p(x) = \begin{cases} \binom{n}{x} \dfrac{p^x}{(1-p)^{x-n}}, & x = 0, 1, \dots, n \\ 0, & \text{elsewhere} \end{cases} \quad (2)$$

The binomial distribution (cont.)

Remark

If n is a positive integer

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}$$

Thus,

$$\begin{aligned}\sum_x p(x) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \big[(1-p)+p\big]^n \\ &= 1\end{aligned}$$

It is clear that $p(x) \geq 0$

$p(x)$ satisfies the conditions of being a PMF of a RV X of the discrete type

The binomial distribution (cont.)

A RV X with the PMF of the form of $p(x)$ has a **binomial distribution**

- Any such $p(x)$ is called a **binomial PMF**, with parameters n and p

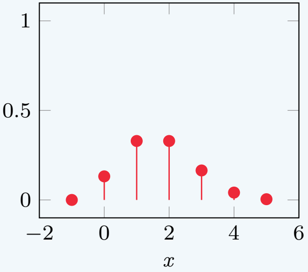
A binomial PMF is denoted by the symbol $b(n, p)$

The binomial distribution (cont.)

Example

If we say that $X \sim b(5, 1/3)$, we mean that X has the binomial PMF

$p(x|p = 1/3, n = 5)$



$$p(x) = \begin{cases} \binom{5}{x} \dfrac{(1/3)^x}{(2/3)^{x-5}}, & x = 0, 1, \ldots, 5 \\ 0, & \text{elsewhere} \end{cases} \quad (3)$$

The binomial distribution (cont.)

The MGF of a binomial distribution

$$
\begin{aligned}
M(t) = \sum_{x} e^{(tx)} p(x) &= \sum_{x=0}^{n} e^{(tx)} \binom{n}{x} p^{x} (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} \left[pe^{(t)} \right]^{x} (1-p)^{n-x} \\
&= \left[(1-p) + e^{t} \right]^{n}
\end{aligned}
$$

for all real values of t

The binomial distribution (cont.)

The mean μ and the variance σ^2 of X may be computed from $M(t)$

$$
\begin{aligned}
M'(t) = & n\big[(1-p) + pe^{(t)}\big]^{n-1} pe^{(t)} \\
M''(t) = & n\big[(1-p) + pe^{(t)}\big]^{n-1} pe^{(t)} + \\
& n(n-1)\big[(1-p) + pe^{(t)}\big]^{n-2} \big[pe^{(t)}\big]^2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mu &= M'(0) = np \\
\sigma^2 &= M''(0) = np + n(n-1)p^2 - (np)^2 = np(1-p)
\end{aligned}
$$

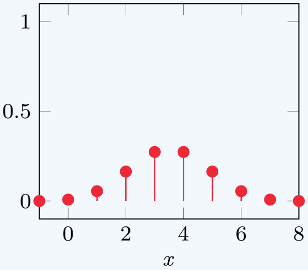
The binomial distribution (cont.)

Example

Let X be the number of successes in $n = 7$ independent fair experiments

The PMF of X

$p(x|p = 0.5, n = 7)$



$$p(x) = \begin{cases} \binom{7}{x} \dfrac{(1/2)^x}{(1/2)^{x-7}}, & x = 0, 1, \ldots, 7 \\ 0, & \text{elsewhere} \end{cases} \quad (4)$$

The binomial distribution (cont.)

The MGF of X

$$M(t) = \big(1/2 + 1/2e^t\big)^7$$

The mean of X

$$\rightsquigarrow \quad \mu = np = 7/2$$

The variance of X

$$\rightsquigarrow \quad \sigma^2 = np(1-p) = 7/4$$

The binomial distribution (cont.)

Furthermore,

$$P(0 \leq X \leq 1) = \sum_{x=0}^{1} p(x) = \frac{1}{128} + \frac{7}{128} = 8/128$$

$$P(X = 5) = p(5) = \frac{7!}{5!2!}(1/2)^5(1/2)^2 = 21/128$$

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The binomial distribution (cont.)

The commands to obtain binomial probabilities for $X \sim b(n, p)$

- `dbinom(k,n,p)`, $P(X = k)$
- `pbinom(k,n,p)`, $P(X \leq k)$

The binomial distribution (cont.)

Example

Let X be a random variable with the MGF

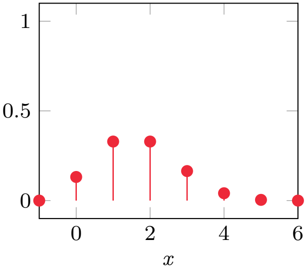
$$M(t) = \left(2/3 + 1/3e^t\right)^5$$

Then, X has a binomial distribution with $n = 5$ and $p = 1/3$

The binomial distribution (cont.)

The PMF of X

$p(x|p = 1/3, n = 5)$



$$p(x) = \begin{cases} \binom{5}{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{5-x}, & x = 0, 1, \ldots, 5 \\ 0, & \text{elsewhere} \end{cases}$$

The binomial distribution (cont.)

The mean of X

$$\mu = np = 5/3$$

The variance of X

$$\sigma^2 = np(1-p) = 10/9$$

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The binomial distribution (cont.)

Example

Let the random variable Y be equal to the number of successes out of n independent random experiments with probability of success equal p

The ratio Y/n is called the **relative frequency of success**

Recalling Chebyshev's inequality, for all $\varepsilon > 0$, we have

$$P\Big(\Big|\frac{Y}{n} - p\Big| \geq \varepsilon\Big) \leq \frac{\text{Var}(Y/n)}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2}$$

For any fixed ε, the RHS goes to zero with n

$$\lim_{n\uparrow\infty} P\Big(\Big|\frac{Y}{n} - p\Big| \geq \varepsilon\Big) = 0$$

The Binomial and related distributions (cont.)

Thus,

$$\lim_{n\uparrow\infty} P\Big(\Big|\frac{Y}{n} - p\Big| < \varepsilon\Big) = 1$$

The relative frequency of success is close to the probability of success p

- For sufficiently large n
- (And any $\varepsilon > 0$)

The result is a form of the **Weak Law of Large Numbers** (**WLLN**)

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The binomial distribution (cont.)

Theorem

Let $X_1, X_2, \dots, X_m$ be independent random variables

Let each X_i have the binomial distribution $b(n_i, p)$, for $i = 1, 2, \dots, m$

Let $Y = \sum_{i=1}^{m} X_i$

Then, Y has a binomial distribution $b(\sum_{i=1}^{m} n_i, p)$

Proof

The MGF of X_i

$$M_X(t) = \big[(1-p) + pe^{(t)}\big]^{n_i}$$

By independence,

$$M_Y(t) = \prod_{i=1}^{m} \big[(1-p) - pe^{(t)}\big]^{n_i} = \big[(1-p) + pe^{(t)}\big]^{\sum_{i=1}^{m} n_i}$$

Hence, Y has the binomial distribution $b(\sum_{i=1}^{m} n_i, p)$

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The Poisson distribution

Useful distributions

The Poisson distribution

Recall[2] the series

$$1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

The series converges absolutely, for *all* values of m, to $\exp(m)$

Consider the function $p(x|m)$, for some $m > 0$

$$p(x|m) = \begin{cases} \frac{m^x}{x!} e^{(-m)}, & x = 0, 1, 2, \dots \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

[2] Walter Rudin. *Real and complex analysis*, page 1.

The Poisson distribution (cont.)

Since $m > 0$, then $p(x) \geq 0$

Moreover,

$$\sum_x p(x) = \sum_{x=0}^{\infty} \frac{m^x e^{(-m)}}{x!} = e^{(-m)} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$
$$= e^{(-m)} e^{(m)} = 1$$

$p(x)$ satisfies the condition of being a PMF of a discrete type of RV

The Poisson distribution (cont.)

A RV X with the PMF of the form $p(x)$ has a **Poisson distribution**

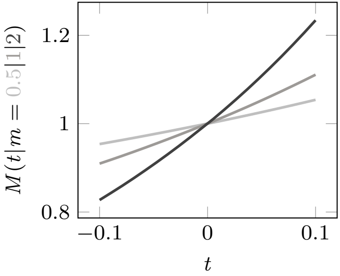
- Any such $p(x)$ is called a **Poisson PMF** with parameter m

The Poisson distribution (cont.)

The MGF of a Poisson distribution

$$\begin{aligned} M(t) = \sum_x e^{tx} p(x) &= \sum_{x=0}^{\infty} e^{tx} \frac{m^x e^{-m}}{x!} \\ &= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!} \\ &= e^{-m} e^{me^t} = e^{m(e^t - 1)} \end{aligned}$$

for all real values of t



The Poisson distribution (cont.)

We have,

$$M'(t) = e^{m(e^t - 1)}(me^t)$$
$$M''(t) = e^{m(e^t - 1)}(me^t) + e^{m(e^t - 1)}(me^t)^2$$

Thus,

$$\rightsquigarrow \mu = M'(0) = m$$
$$\rightsquigarrow \sigma^2 = M''(0) - \mu^2 = m + m^2 - m^2 = m$$

The Poisson distribution has $\mu = \sigma^2 = m > 0$

The Poisson distribution (cont.)

An alternative form of the Poisson PMF

$$p(x) = \begin{cases} \dfrac{\mu^x e^{-\mu}}{x!}, & x = 0, 1, 2, \dots \\ 0, & \text{elsewhere} \end{cases}$$

Thus, the parameter μ in a Poisson PMF is the mean μ

The Poisson distribution (cont.)

Let X be a RV with the Poisson distribution with parameter $m = \mu$

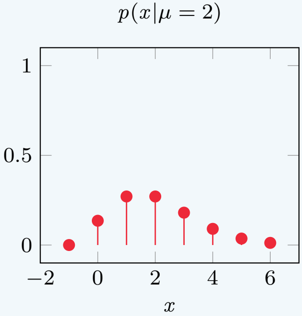
Then,

- `dpois(k,m)`, $P(X = k)$
- `ppois(k,m)`, $P(X \leq k)$

The Poisson distribution (cont.)

Example

Suppose that X has a Poisson distribution with $\mu = 2$, then the PMF of X



$$p(x) = \begin{cases} \dfrac{2^x e^{-2}}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{elsewhere} \end{cases}$$

The Poisson distribution (cont.)

The variance of this distribution is $\sigma^2 = \mu = 2$

We can compute $P(1 \leq X)$ or use the table of distributions

$$\begin{aligned} P(1 \leq X) &= 1 - P(X = 0) \\ &= 1 - p(0) = 1 - e^{-2} \\ &= 0.865 \end{aligned}$$

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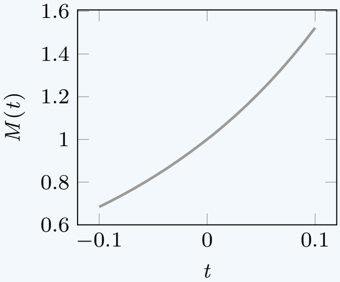
The Poisson distribution (cont.)

Example

Let the random variable X have the MGF



$$M(t) = e^{4(e^t - 1)}$$

Then, X has a Poisson distribution

Accordingly,

$$P(X = 3) = \frac{4^3 e^{-4}}{3!} = \frac{32}{3} e^{-4}$$

The Poisson distribution (cont.)

By the table of distributions,

$$P(X = 3) = P(X \leq 3) - P(X \leq 2) = 0.433 - 0.238 = 0.195$$

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The Poisson distribution (cont.)

The Poisson distribution satisfies an important additive property

Theorem

Suppose $X_1, \dots, X_n$ are independent random variables

Suppose that each X_i has a Poisson distribution with parameter m

Then, $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with parameter $\sum_{i=1}^{n} m_i$

Proof

We obtain the result by determining the MGF of Y

$$M_Y(t) = E\big[\exp(tY)\big] = \prod_{i=1}^{n} \exp\Big\{m_i\big[\exp(t) - 1\big]\Big\}$$

$$= \exp\Big\{\sum_{i=1}^{n} m_i\big[\exp(t) - 1\big]\Big\}$$

The Poisson distribution (cont.)

By the uniqueness of MGFs, Y has a Poisson distribution with parameter

$$\rightsquigarrow \quad \mu = \sum_{i=1}^{n} m_i$$

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