The normal distribution

Useful distributions

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Useful distributions

The normal distribution

Motivation for the normal distribution is found in the central limit theorem

Normal distributions provide an important family of distributions

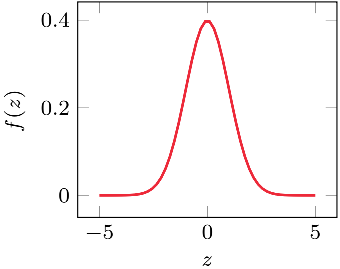
$\rightsquigarrow$ Applications and inference

We first introduce the standard normal distribution

- Then, the general normal distribution

The normal distribution

Consider the integral



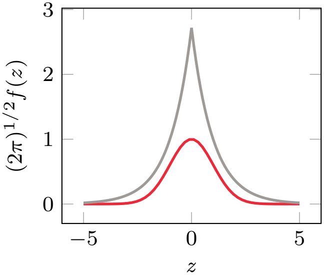
$$I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-z^2}{2}\Big) \mathrm{d}z \quad (1)$$

This integral exists

The normal distribution (cont.)

Consider the relevant part of the integrand

It is a positive continuous function bounded by an integrable function



$$\exp\Big(\frac{-z^2}{2}\Big)$$

That is,

$$0 < \exp\Big(\frac{-z^2}{2}\Big) < \exp\big(-|z|+1\big), \quad -\infty < z < \infty$$

and

$$\int_{-\infty}^{\infty} \exp\big(-|z|+1\big)\mathrm{d}z = 2e$$

The normal distribution (cont.)

To evaluate the integral, we note that $I > 0$

We have that I^2

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{z^2}{2}\Big) \mathrm{d}x \cdot \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{w^2}{2}\Big) \mathrm{d}w \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\Big(-\frac{z^2+w^2}{2}\Big) \mathrm{d}z \mathrm{d}w \end{aligned}$$

The integral can be computed by changing to polar coordinates

The normal distribution (cont.)

Set $z = r\cos\theta$ and $w = r\sin\theta$

Then,

$$
\begin{aligned}
I^2 &= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\int_0^{\infty} e^{-r^2/2} r\mathrm{d}r}_{\int x e^{cx^2}\mathrm{d}x = 1/(2c)e^{cx^2} = 1} \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \\
&= 1
\end{aligned}
$$

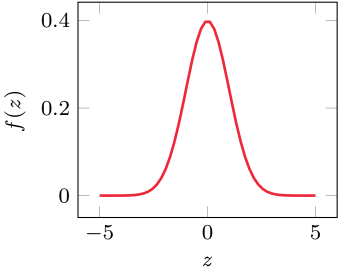
The integrand of Equation (1) is positive on $\mathcal{R}$ and integrates to 1 over $\mathcal{R}$

The normal distribution (cont.)

The integrand is a PDF of a continuous random variable with support $\mathcal{R}$

Denote this random variable Z, Z has the PDF

$$f(z) = \frac{1}{\sqrt{(2\pi)}} \exp\left(\frac{-z^2}{2}\right), \text{ for } -\infty < x < \infty \tag{2}$$



The normal distribution (cont.)

For $t \in \mathcal{R}$, the MGF of Z can be derived by a completion of a square

$$
\begin{aligned}
E[\exp(tZ)] &= \int_{-\infty}^{\infty} \exp(tz) \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}z^2\Big) \mathrm{d}z \\
&= \exp\Big(\frac{1}{2}t^2\Big) \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\Big[-\frac{1}{2}(z-t)^2\Big] \mathrm{d}z \qquad (3) \\
&= \exp\Big(\frac{1}{2}t^2\Big) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big[-\frac{1}{2}w^2\Big] \mathrm{d}w
\end{aligned}
$$

For the last integral, we made a 1-to-1 change of variable $w = z - t$

The integral in Equation (3) has value 1, by the identity

$$
f(z) = \frac{1}{\sqrt{(2\pi)}} \exp\Big(\frac{-z^2}{2}\Big), \text{ for } -\infty < x < \infty,
$$

The normal distribution (cont.)

Thus, the MGF of Z

$$M_Z(t) = e^{(1/2t^2)}, \text{ for } -\infty < x < \infty \tag{4}$$

The first two derivative of $M_Z(t)$ are

$$M'_Z(t) = te^{(1/2t^2)}$$
$$M''_Z(t) = e^{(1/2t^2)} + t^2 e^{(1/2t^2)}$$

Upon evaluating these derivatives at $t = 0$, the mean and variance of Z

$$\begin{aligned} \rightsquigarrow \quad E(Z) &= 0 \\ \rightsquigarrow \quad \text{Var}(Z) &= 1 \end{aligned} \tag{5}$$

The normal distribution (cont.)

Define the continuous random variable $X = bZ + a$, for $b > 0$

- This is a 1-to-1 transformation

We want to derive the PDF of X

- The inverse of the transformation is $z = b^{-1}(x - a)$
- The Jacobian is $J = b^{-1}$

Because $b > 0$, the PDF of X

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{b} \exp\Big[-\frac{1}{2} \Big(\frac{x-a}{b} \Big)^2 \Big], \text{ for } -\infty < x < \infty$$

The normal distribution (cont.)

By Equation (5), we have

$$\rightsquigarrow \quad E(X) = a$$
$$\rightsquigarrow \quad \mathrm{Var}(X) = b^2$$

The PDF of X can be written using those quantities

- $\rightsquigarrow$ a can be replaced by $\mu = E(X)$
- $\rightsquigarrow$ b^2 can be replaced by $\sigma^2 = \mathrm{Var}(X)$

The normal distribution (cont.)

Definition 1.1

Normal distribution

*A random variable X has a **normal distribution** if its PDF is*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)\Big], \quad x \in (-\infty, +\infty) \tag{6}$$

The parameters μ and σ^2 are the mean and the variance of X

We often write that X has a $N(\mu, \sigma^2)$ distribution

The normal distribution (cont.)

Remark

Consider the random variable Z with the PDF

$$f(z) = \frac{1}{\sqrt{(2\pi)}} \exp\Big(\frac{-z^2}{2}\Big), \text{ for } -\infty < x < \infty$$

The RV Z is said to have a $N(0,1)$ distribution

- We call Z a **standard normal** RV

The normal distribution (cont)

Consider the calculation of the MGF of X

We use the relationship

$$X = \underbrace{\sigma}_{b} Z + \underbrace{\mu}_{a}$$

Remember that the MGF of Z

$$M_Z(t) = e^{(1/2t^2)}, \text{ for } -\infty < x < \infty$$

The normal distribution (cont)

We obtain,

$$E\Big[\exp(tX)\Big] = E\Big\{\exp\Big[t(\sigma Z + \mu)\Big]\Big\} = \exp(\mu t) E\Big[\exp(t\sigma Z)\Big]$$
$$= \exp(\mu t)\exp\Big(\frac{1}{2}\sigma^2 t^2\Big) = \exp\Big(\mu t + \frac{1}{2}\sigma^2 t^2\Big)$$

for $-\infty < t < +\infty$

The normal distribution (cont)

Remark

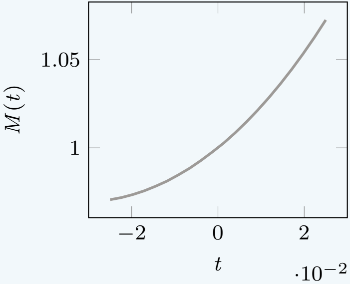
Consider the relationship between Z and X

$\rightsquigarrow$ $X \sim N(\mu, \sigma^2)$ if and only if $Z = (X - \mu)/\sigma \sim N(0, 1)$

The normal distribution (cont.)

Example

Consider the random variable X with the MGF



$$M(t) = e^{2t+32t^2}$$

$$M(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

X has a normal distribution with $\mu = 2$ and $\sigma^2 = 64$, $N(2, 54)$

The random variable $Z = (X - 2)/8$ has a $N(0, 1)$ distribution

The normal distribution (cont.)

Example

We can derive all the moments of the standard normal RV using its MGF

- We can do the same for a general normal RV

Let $X \sim N(\mu, \sigma^2)$

Hence, for all nonnegative integers k, by the binomial theorem[1]

$$E(X^k) = E\big[(\sigma Z + \mu)^k\big] = \sum_{j=0}^{k} \binom{k}{j} \sigma^j E(Z^j) \mu^{k-j} \qquad (7)$$

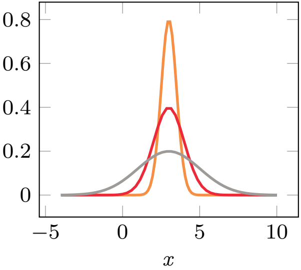
- All the odd moments of Z are zero
- All even moments can be calculated

■

[1] $(x+y)^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

The normal distribution (cont.)

$f(x|\mu = 3, \sigma^2 = 1/4|1|4)$



$$f_X(x) = \frac{1}{\sqrt{2\pi}b} \exp\Big[-\frac{1}{2}\Big(\frac{x-a}{b}\Big)^2\Big]$$

- The graph is symmetric about a vertical axis at $x = \mu$
- The graph has its maximum of $1/(\sigma\sqrt{2\pi})$ at $x = \mu$
- The graph has the x-axis as horizontal asymptote
- The points of inflection are at $x = \mu \pm \sigma$

The normal distribution (cont)

There are many practical applications that involve normal distributions

$\rightsquigarrow$ We need to be able to readily calculate probabilities

Normal PDFs contain a factor of the form $\exp(-s^2)$

- Antiderivatives are not in closed-form
- Numerical integration must be used

The normal distribution (cont)

We can use the relation between normal and standard normal variables

- $\rightsquigarrow$ $X \sim N(\mu, \sigma^2)$ if and only if $Z = (X - \mu)/\sigma \sim N(0, 1)$

Thus, we need only calculate probabilities for standard normal RVs

The normal distribution (cont)

Consider the CDF of a standard normal random variables Z

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\Big(\frac{-w^2}{2}\Big) \mathrm{d}w \tag{8}$$

Let $X \sim N(\mu, \sigma^2)$

Suppose that we want to compute $F_X(x) = P(X \leq x)$ for some x

Then, for $Z = (X - \mu)/\sigma$

$$F_X(x) = P(X \leq x) = P\Big(Z \leq \frac{x-\mu}{\sigma}\Big) = \Phi\Big(\frac{x-u}{\sigma}\Big)$$

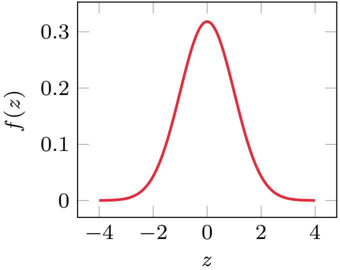
Thus, only integration for $\Phi(z)$ is needed

The normal distribution

Suppose we are interested in the value x_p such that $p = F_X(x_p)$ for some p

Take $z_p = \Phi^{-1}(p)$, then $x_p = \sigma z_p + \mu$

Consider the standard normal density



$$\Phi(z) = \frac{1}{\sqrt{(2\pi)}} \exp\left(\frac{-z^2}{2}\right)$$

The area to the left of z_p is p

- $\Phi(z_p) = p$

We can use an abbreviated table of probability

The normal distribution

Suppose we are interested in $\Phi(-z)$, for some $z > 0$

Since the PDF is symmetric,

$$\rightsquigarrow \quad \Phi(-z) = 1 - \Phi(z) \tag{9}$$

The normal distribution (cont.)

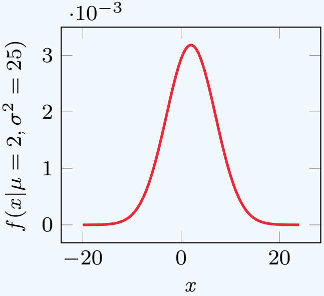
Consider a random variable $X \sim N(a, b)$

- **`pnorm(x,a,b)`**, $P(X \leq x)$
- **`dnorm(x,a,b)`**, the PDF of X at x

The normal distribution (cont.)

Example

Let the random variable $X \sim N(2, 25)$



Using abbreviated tables of probabilities

$$
\begin{aligned}
P(0 < X < 10) &= \Phi\Big(\frac{10-2}{5}\Big) - \Phi\Big(\frac{0-2}{5}\Big) \\
&= \Phi(1.6) - \Phi(-0.4) \\
&= 0.945 - (1 - 0.655) = 0.600
\end{aligned}
$$

The normal distribution (cont.)

$$
\begin{aligned}
P(-8 < X < 1) &= \Phi\Big(\frac{1-2}{5}\Big) - \Phi\Big(\frac{-8-2}{5}\Big) \\
&= \Phi(-0.2) - \Phi(-2) \\
&= (1-0.579) - (1-0.977) = 0.398
\end{aligned}
$$

■

The normal distribution (cont.)

Example

Let the random variable $X \sim N(\mu, \sigma^2)$

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= \Phi\Big(\frac{\mu + 2\sigma - \mu}{\sigma}\Big) - \Phi\Big(\frac{\mu - 2\sigma - \mu}{\sigma}\Big) \\ &= \Phi(2) - \Phi(-2) \\ &= 0.977 - (1 - 0.977) = 0.954 \end{aligned} \quad (10)$$

■

The normal distribution (cont.)

Theorem 1.1

Consider a random variable $X \sim N(\mu, \sigma^2)$ with $\sigma^2 > 0$

The random variable $V = (X - \mu)^2/\sigma^2$ is $\chi^2(1)$

Proof

The random variable $V = W^2$, with $W = (X - \mu)/\sigma \sim N(0, 1)$

The CDF $G(v)$ for V

$$G(v) = P(W^2 \leq v) = P(-\sqrt{v} \leq W \leq \sqrt{v}), \quad \text{for } v \geq 0$$

The normal distribution (cont.)

That is,

$$G(v) = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{(-w^2/2)} \mathrm{d}w, \text{ for } 0 \leq v$$

Moreover,

$$G(v) = 0, \text{ for } v < 0$$

Consider a change of the integration variable, $w = \sqrt{y}$

$$\rightsquigarrow \quad G(v) = \int_0^{v} \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{(-y/2)} \mathrm{d}y, \text{ for } 0 \leq v$$

The normal distribution (cont.)

Hence, the PDF $g(v) = G'(v)$ of random variable V of the continuous type

$$g(v) = \begin{cases} \dfrac{1}{\sqrt{\pi}\sqrt{2}} v^{(1/2-1)} e^{(-v/2)}, & 0 < v < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Since $g(v)$ is a PDF,

$$\int_0^{\infty} g(v)\mathrm{d}v = 1,$$

Then, it must be $\Gamma(1/2) = \sqrt{\pi}$ and therefore $V \sim \chi^2(1)$

■

The normal distribution (cont.)

A main property of the normal distribution is additivity under independence

Theorem 1.2

Let $X_1, \dots, X_n$ be independent random variables

Suppose that $X_i \sim N(\mu_i, \sigma_i^2)$, for $i = 1, \dots, n$,

Let $Y = \sum_{i=1}^n a_i X_i$, for some constants $a_1, \dots, a_n$

Then, the distribution of Y is $N(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$

Proof

For $t \in \mathcal{R}$, the MGF of Y

$$
\begin{aligned}
M_Y(t) &= \prod_{i=1}^n \exp\left[t a_i \mu_i + (1/2) t^2 a_i^2 \sigma_i^2\right] \\
&= \exp\left[t \sum_{i=1}^n a_i \mu_i + (1/2) t^2 \sum_{i=1}^n a_i^2 \sigma_i^2\right]
\end{aligned}
$$

This is the MGF of a $N(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2)$ distribution $\blacksquare$

The normal distribution (cont.)

Let $X_1, X_2, \ldots, X_n$ be random sample from a $N(\mu, \sigma^2)$

A corollary gives the distribution of the sample mean

$$\rightsquigarrow \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Corollary

Let $X_1, \ldots, X_n$ be IID random variables with common $N(\mu, \sigma^2)$ distribution

Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

Then, $\overline{X}$ *has a* $N(\mu, \sigma^2/n)$ *distribution*

Proof

In Theorem 1.2, take $a_i = (1/n)$, $\mu_i = \mu$ and $\sigma_i^2 = \sigma^2$, for $i = 1, 2, \ldots, n$

$\blacksquare$

The multivariate normal distribution

Useful distributions

The multivariate normal distribution

The multivariate normal distribution for a n-dimensional random vector

- Examples for the bivariate case, $n = 2$

The derivation is simplified by first discussing the standard case

- Then, we proceed with the general cases

The multivariate normal distribution (cont.)

Consider the random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)'$

- $Z_1, \ldots, Z_n$ are IID $N(0, 1)$ random variables

Then, the density of $\mathbf{Z}$

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{1}{2} z_i^2 \Big\} = \Big(\frac{1}{2\pi} \Big)^{n/2} \exp\Big\{ -\frac{1}{2} \sum_{i=1}^{n} z_i^2 \Big\} \\ &= \Big(\frac{1}{2\pi} \Big)^{n/2} \exp\Big\{ -\frac{1}{2} \mathbf{z}'\mathbf{z} \Big\}, \text{ for } \mathbf{z} \in \mathcal{R}^n \end{aligned} \quad (11)$$

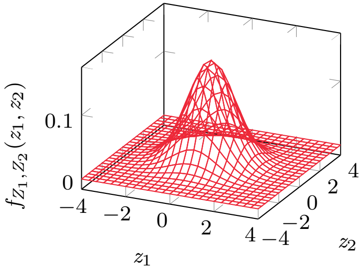
The multivariate normal distribution (cont.)

Consider the random vector $\mathbf{Z} = (Z_1, Z_2)'$

Since $n = 2$, we have

$$f_{\mathbf{Z}}(\mathbf{z}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{1}{2} z_i^2 \Big\} = \Big(\frac{1}{2\pi}\Big)^{n/2} \exp\Big\{ -\frac{1}{2} \sum_{i=1}^{n} z_i^2 \Big\},$$

$$= 1/(2\pi) \exp\Big\{ -1/2 \sum_{i=1}^{2} z_i^2 \Big\} = 1/(2\pi) \exp\Big\{ -1/2 (z_1^2 + z_2^2) \Big\}$$



The multivariate normal distribution (cont.)

The Z_is have mean zero, unit variance and they are uncorrelated

Then, the mean and covariance matrix of $\mathbf{Z}$

$$\begin{aligned} E(\mathbf{Z}) &= \mathbf{0} \\ \text{Cov}(\mathbf{Z}) &= \mathbf{I}_n \end{aligned} \tag{12}$$

- $\mathbf{I}_n$ indicates an identity matrix of order n

The multivariate normal distribution

The MGF of each of the Z_is evaluated at t_i is $e^{(t_i^2/2)}$

As the Z_is are independent, the MGF of $\mathbf{Z}$

$$
\begin{aligned}
M_{\mathbf{Z}}(\mathbf{t}) &= E\Big[\prod_{i=1}^{n} \exp\{t_i Z_i\}\Big] = \prod_{i=1}^{n} E[\exp\{t_i Z_i\}] \\
&= \exp\Big\{\frac{1}{2}\sum_{i=1}^{n} t_i^2\Big\} = \exp\Big\{\frac{1}{2}\mathbf{t}'\mathbf{t}\Big\}, \text{for all } \mathbf{t} \in \mathcal{R}^n
\end{aligned} \tag{13}
$$

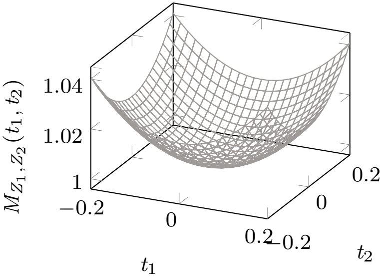
The multivariate normal distribution

Consider the random vector $\mathbf{Z} = (Z_1, Z_2)'$

Since $n = 2$, we have

$$M_{\mathbf{Z}}(\mathbf{t}) = E\Big[\prod_{i=1}^{n} \exp\{t_i Z_i\}\Big] = \prod_{i=1}^{n} E[\exp\{t_i Z_i\}]$$

$$= \exp\Big\{\frac{1}{2}\sum_{i=1}^{2} t_i^2\Big\} = \exp\{1/2(t_1^2 + t_2^2)\}, \text{ for all } (t_1, t_2) \in \mathcal{R}^2$$



The multivariate normal distribution (cont.)

$\mathbf{Z}$ is said to have a **multivariate normal distribution**

$\mathbf{Z}$ has a $N_n(\mathbf{0}, \mathbf{I}_n)$ distribution, mean $\mathbf{0}$, covariance matrix $\mathbf{I}_n$

The multivariate normal distribution (cont.)

Let $\mathbf{\Sigma}$ be a $n \times n$, symmetric and positive-definite matrix $(\mathbf{z}'\mathbf{\Sigma}\mathbf{z} > 0, \forall \mathbf{z})$

From linear algebra, we can always decompose $\mathbf{\Sigma}$

$$\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma} \tag{14}$$

$\mathbf{\Lambda}$ is the diagonal matrix $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\rightsquigarrow$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues of $\mathbf{\Sigma}$

The columns of $\mathbf{\Gamma}'$, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, are the corresponding eigenvectors

$$\mathbf{\Gamma}' = \begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{n,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{n,2} \\ \vdots & \vdots & & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{n,n} \end{bmatrix}$$

The multivariate normal distribution (cont.)

$$\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma}$$

The factorisation is called the **spectral decomposition** of $\mathbf{\Sigma}$

- Matrix $\mathbf{\Gamma}$ is orthogonal ($\mathbf{\Gamma}^{-1} = \mathbf{\Gamma}'$, thus $\mathbf{\Gamma}\mathbf{\Gamma}' = \mathbf{I}$)

The multivariate normal distribution (cont.)

Remark

Eigenvalues and eigenvectors

A $(n \times n)$ matrix $\mathbf{A}$ can be used to transform a n-vector $\mathbf{x}$ to another one $\mathbf{y}$

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Consider a scalar variable λ_i and a particular value of $\mathbf{x}$ $(= \mathbf{e}_i)$ such that

$$\mathbf{y}_{\mathbf{e}_i} = \mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

- λ_i is often called an **eigenvalue** of $\mathbf{A}$
- $\mathbf{e}_i$ is the corresponding **eigenvector**

The multivariate normal distribution (cont.)

$$\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Every $(n \times n)$ matrix has n eigenvalues and up to n eigenvectors

They can be jointly specified

$$\begin{aligned}
(\lambda_1 \mathbf{I}_n - \mathbf{A})\mathbf{e}_1 &= \mathbf{0} \\
(\lambda_2 \mathbf{I}_n - \mathbf{A})\mathbf{e}_2 &= \mathbf{0} \\
&\cdots \\
(\lambda_n \mathbf{I}_n - \mathbf{A})\mathbf{e}_n &= \mathbf{0}
\end{aligned}$$

If $(\lambda_i \mathbf{I}_n - \mathbf{A})\mathbf{e}_i = \mathbf{0}$ is satisfied, then so is $(\lambda_i \mathbf{I}_n - \mathbf{A})\alpha\mathbf{e}_i = \mathbf{0} \quad (i = 1, \dots, n)$

$\rightsquigarrow$ Eigenvectors are specified up to any multiplicative constant α

The multivariate normal distribution (cont.)

The set of n vector equations can have non-trivial solutions

The n values of λ_i need be solutions to the scalar equation

$$|\lambda \mathbf{I}_n - \mathbf{A}| := \Delta(\lambda) = 0$$

⇝ $(\lambda \mathbf{I}_n - \mathbf{A})$ is called the **characteristic matrix** of $\mathbf{A}$

⇝ $\Delta(\lambda)$ is the **characteristic polynomial** of $\mathbf{A}$

The determinant gives a n-degree polynomial in λ

- It can be factored as a product of n binomials

$$\begin{aligned}\Delta(\lambda) &= \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0\\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)\\ &= 0\end{aligned}$$

- Each of the binomial roots is an eigenvalue of $\mathbf{A}$

The multivariate normal distribution (cont.)

If the roots are all distinct, there will be n independent eigenvectors

$\rightsquigarrow$ If the roots are repeated, there may be fewer eigenvectors

For a distinct eigenvalue λ_i, the eigenvector $\mathbf{e}_i$ is contained in $\text{Adj}(\lambda_i \mathbf{I}_n - \mathbf{A})$

$$\text{Adj}(\lambda_i \mathbf{I}_n - \mathbf{A}) = \begin{bmatrix} \alpha_1 \mathbf{e}_i & \alpha_2 \mathbf{e}_i & \cdots & \alpha_n \mathbf{e}_i \end{bmatrix}$$

Since α_j are arbitrary constants, any non-zero column represents $\mathbf{e}_i$

$\text{Adj}(\lambda_i \mathbf{I}_n - \mathbf{A})$ is computed for each root $(i = 1, \dots, n)$

- A single eigenvector is chosen from each evaluation

The multivariate normal distribution (cont.)

Together, the n eigenvectors form the columns of the **modal matrix** $\mathbf{E}$

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$$

Here, the eigenvectors are scaled so that $|\mathbf{e}_i| = 1$ $(i = 1, \ldots, n)$

The multivariate normal distribution (cont.)

The modal matrix can be used to diagonalise $\mathbf{A}$

To transform $\mathbf{A}$ into a diagonal matrix (of eigenvalues)

$$\begin{bmatrix} \mathbf{A}\mathbf{e}_1 & = & \lambda_1 \mathbf{e}_1 \\ \mathbf{A}\mathbf{e}_2 & = & \lambda_2 \mathbf{e}_2 \\ \vdots & & \vdots \\ \mathbf{A}\mathbf{e}_n & = & \lambda_n \mathbf{e}_n \end{bmatrix}, \text{ or } \mathbf{AE} = \mathbf{E}\mathbf{\Lambda}$$

$\mathbf{\Lambda}$ is a diagonal matrix of eigenvalues

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

The multivariate normal distribution (cont.)

$\mathbf{E}$ is non-singular because the $\mathbf{e}_i$ are linearly independent

$$\mathbf{A} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^{-1}$$

From the diagonal matrix of eigenvalues to the original square matrix

The inverse transformation

$$\boldsymbol{\Lambda} = \mathbf{E}^{-1}\mathbf{A}\mathbf{E}$$

The multivariate normal distribution (cont.)

If $\mathbf{A}$ is a real, symmetric matrix, its eigenvalues are real

- The eigenvectors are orthogonal to each other

There are n eigenvectors, even with repeated roots

The eigenvectors corresponding to distinct eigenvalues

$$\text{Adj}(\lambda_i \mathbf{I}_n - \mathbf{A}) = \begin{bmatrix} \alpha_1 \mathbf{e}_i & \alpha_2 \mathbf{e}_i & \cdots & \alpha_n \mathbf{e}_i \end{bmatrix}$$

The eigenvectors of eigenvalues λ_i with multiplicity m

$$\frac{d^{m-1}}{d\lambda^{m-1}} \big[\text{Adj}(\lambda_i \mathbf{I}_n - \mathbf{A})\big]\Big|_{\lambda=\lambda_i} = \begin{bmatrix} \cdots & \mathbf{e}_{i_1} & \mathbf{e}_{i_2} & \cdots & \mathbf{e}_{i_m} & \cdots \end{bmatrix}$$

The multivariate normal distribution (cont.)

Remark

An alternative way to write the spectral decomposition

$$\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i' \tag{15}$$

$$\underbrace{\begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{n,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{n,2} \\ \vdots & \vdots & & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{n,n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\lambda_2 \mathbf{v}_2} \underbrace{\begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,n} \\ \vdots & \vdots & & \vdots \\ v_{n,1} & v_{n,2} & \cdots & v_{n,n} \end{bmatrix}}_{\mathbf{v}_2'}$$

The multivariate normal distribution (cont.)

Because the $\lambda_i s$ are non-negative, we can define the diagonal matrix

$$\mathbf{\Lambda}^{1/2} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n})$$

$$= \begin{bmatrix} {\lambda_1}^{1/2} & 0 & \cdots & 0 \\ 0 & {\lambda_2}^{1/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {\lambda_n}^{1/2} \end{bmatrix}$$

Then, the orthogonality of $\mathbf{\Gamma}$ implies

$$\mathbf{\Sigma} = \mathbf{\Gamma}' \underbrace{\mathbf{\Lambda}^{1/2} \underbrace{\mathbf{\Gamma}\mathbf{\Gamma}'}_{\mathbf{I}} \mathbf{\Lambda}^{1/2}}_{\mathbf{\Lambda}} \mathbf{\Gamma}$$

The multivariate normal distribution (cont.)

$$\mathbf{\Sigma} = \underbrace{\mathbf{\Gamma}' \, \mathbf{\Lambda}^{1/2} \, \underbrace{\mathbf{\Gamma}\mathbf{\Gamma}'}_{\mathbf{I}} \, \mathbf{\Lambda}^{1/2}}_{\mathbf{\Lambda}} \, \mathbf{\Gamma}$$

Define the square root of the positive semi-definite matrix $\mathbf{\Sigma}$

$$\mathbf{\Sigma}^{1/2} = \mathbf{\Gamma}' \mathbf{\Lambda}^{1/2} \mathbf{\Gamma} \tag{16}$$

- $\mathbf{\Sigma}^{1/2}$ is symmetric and positive semi-definite

Suppose $\mathbf{\Sigma}$ is positive definite and all its eigenvalues are strictly positive

Then, we have

$$(\mathbf{\Sigma}^{1/2})^{-1} = \mathbf{\Gamma}' \mathbf{\Lambda}^{-1} \mathbf{\Gamma} = \mathbf{\Sigma}^{-1/2} \tag{17}$$

The multivariate normal distribution (cont.)

Let $\mathbf{Z}$ have a $N_n(\mathbf{0}, \mathbf{I}_n)$ distribution

Let $\mathbf{\Sigma}$ be a positive semi-definite symmetric matrix

Let $\boldsymbol{\mu}$ be a $n \times 1$ vector of constants

Define the random vector $\mathbf{X}$

$$\mathbf{X} = \mathbf{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu} \tag{18}$$

By Equation (12), we have

$$\begin{aligned} E[\mathbf{X}] &= \boldsymbol{\mu} \\ \text{Cov}[\mathbf{X}] &= \mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma} \end{aligned} \tag{19}$$

The multivariate normal distribution (cont.)

The MGF of $\mathbf{X}$

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) = E\Big[\exp\big\{\mathbf{t}'\mathbf{X}\big\}\Big] &= E\Big[\exp\big\{\mathbf{t}'\boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \mathbf{t}'\boldsymbol{\mu}\big\}\Big] \\ &= \exp\big(\mathbf{t}'\boldsymbol{\mu}\big)E\Big\{\exp\Big[\big(\boldsymbol{\Sigma}^{1/2}\mathbf{t}\big)'\mathbf{Z}\Big]\Big\} \\ &= \exp\big(\mathbf{t}'\boldsymbol{\mu}\big)\exp\Big[(1/2)\big(\boldsymbol{\Sigma}^{1/2}\mathbf{t}\big)'\boldsymbol{\Sigma}^{1/2}\mathbf{t}\Big] \\ &= \exp\big(\mathbf{t}'\boldsymbol{\mu}\big)\exp\Big[(1/2)\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\Big] \end{aligned} \tag{20}$$

The multivariate normal distribution (cont.)

Definition

Multivariate normal

A n-dimensional random vector $\mathbf{X}$ has a ***multivariate normal distribution*** *if its MGF is*

$$M_{\mathbf{X}}(\mathbf{t}) = \exp\left[\mathbf{t}'\boldsymbol{\mu} + (1/2)\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right] \qquad (21)$$

for all $\mathbf{t} \in \mathcal{R}^n$

- $\boldsymbol{\Sigma}$ *is a symmetric, positive semi-definite matrix*
- $\boldsymbol{\mu} \in \mathcal{R}^n$

We say that $\mathbf{X}$ has a $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution

The multivariate normal distribution (cont.)

The definition is for positive semi-definite positive matrices $\mathbf{\Sigma}$

- $\mathbf{\Sigma}$ is usually positive definite

$\rightsquigarrow$ We can get the density of $\mathbf{X}$

If $\mathbf{\Sigma}$ is positive definite, then so is $\mathbf{\Sigma}^{1/2}$

- Its inverse is $(\mathbf{\Sigma}^{1/2})^{-1} = \mathbf{\Gamma}'\mathbf{\Lambda}^{-1}\mathbf{\Gamma} = \mathbf{\Sigma}^{-1/2}$

The multivariate normal distribution (cont.)

$$\mathbf{X} = \mathbf{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$$

The transformation between $\mathbf{X}$ and $\mathbf{Z}$ is 1-to-1

The inverse transformation

$$\mathbf{Z} = \mathbf{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$$

The Jacobian

$$\left|\mathbf{\Sigma}^{-1/2}\right| = \left|\mathbf{\Sigma}\right|^{-1/2}$$

The multivariate normal distribution (cont.)

Hence, upon simplification, the PDF of $\mathbf{X}$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\Big[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\Big], \text{ for } \mathbf{x} \in \mathcal{R}^n \quad (22)$$

The multivariate normal distribution (cont.)

Theorem 2.1

Suppose $\mathbf{X}$ *has a* $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *distribution*

Let $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, *where* $\mathbf{A}$ *is a* $m \times n$ *matrix and* $\mathbf{b} \in \mathcal{R}^m$

Then, $\mathbf{Y}$ *has a* $N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$ *distribution*

The multivariate normal distribution (cont.)

Proof

From $M_{\mathbf{X}}(\mathbf{t}) \exp \big[\mathbf{t}'\boldsymbol{\mu} + (1/2)\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\big]$, for all $\mathbf{t} \in \mathcal{R}^m$

The MGF of $\mathbf{Y}$

$$
\begin{aligned}
M_{\mathbf{Y}}(\mathbf{t}) &= E\big[\exp\big(\mathbf{t}'\mathbf{Y}\big)\big] \\
&= E\big\{\exp\big[\mathbf{t}'(\mathbf{A}\mathbf{X} + \mathbf{b})\big]\big\} \\
&= \exp(\mathbf{t}'\mathbf{b}) E\big\{\exp\big[\big(\mathbf{A}'\mathbf{t}\big)'\mathbf{X}\big]\big\} \\
&= \exp(\mathbf{t}'\mathbf{b}) \exp\big[\big(\mathbf{A}'\mathbf{t}\big)'\boldsymbol{\mu} + \big(1/2\big)\big(\mathbf{A}'\mathbf{t}\big)'\boldsymbol{\Sigma}\big(\mathbf{A}'\mathbf{t}\big)\big] \\
&= \exp\big[\mathbf{t}'\big(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}\big) + \big(1/2\big)\mathbf{t}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\mathbf{t}\big]
\end{aligned}
$$

This is the MGF of a $N_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$ distribution

■

The multivariate normal distribution (cont.)

A corollary gives us the marginal distributions of a multivariate normal RV

Let $\mathbf{X}_1$ be any sub-vector of $\mathbf{X}$ of dimension $m < n$

We can always rearrange means and correlations

There is no loss in writing

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \tag{23}$$

$\mathbf{X}_2$ is of dimension $p = n - m$

The multivariate normal distribution (cont.)

In the same way, we partition mean and covariance matrix of $\mathbf{X}$

$$\begin{aligned} \boldsymbol{\mu} &= \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \\ \boldsymbol{\Sigma} &= \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \end{aligned} \tag{24}$$

- $\boldsymbol{\Sigma}_{11}$ is the covariance matrix of $\mathbf{X}_1$
- $\boldsymbol{\Sigma}_{12}$ contains all covariances between the components of $\mathbf{X}_1$ and $\mathbf{X}_2$

The multivariate normal distribution (cont.)

We define $\mathbf{A}$ to be the matrix

$$\mathbf{A} = \Big[[\mathbf{I}_m] \Big| [\mathbf{0}_{mp}] \Big] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

- $\mathbf{I}_m$ indicates a $m \times m$ identity matrix
- $\mathbf{0}_{mp}$ indicates a $m \times p$ matrix of zeros

Then, $\mathbf{X}_1 = \mathbf{A}\mathbf{X}$

The multivariate normal distribution (cont.)

We apply Theorem 2.1 to the transformation $\mathbf{X}_1 = \mathbf{A}\mathbf{X}$

We have the following corollary

Corollary 2.1

Suppose $\mathbf{X}$ *has a* $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *distribution, partitioned as in Equation (23-24)*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$$
$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Then, $\mathbf{X}_1$ *has a* $N_m(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ *distribution*

This is a useful result

The corollary shows that any marginal distribution of $\mathbf{X}$ is also normal

- $\leadsto$ Mean and covariance matrix are those from the partial vector

The multivariate normal distribution (cont.)

Example

Consider the multivariate normal in the case when $n = 2$

- $\rightsquigarrow$ The bivariate normal

We use common notation (X, Y) rather than (X_1, X_2)

Suppose that $(X, Y) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$
$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_2^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} \quad (25)$$

- $\rightsquigarrow$ μ_1 and σ_1^2 are the mean and variance of X
- $\rightsquigarrow$ μ_2 and σ_2^2 are the mean and variance of Y

$\sigma_{12} = \rho(\sigma_1 \sigma_2)$ is the covariance between X and Y

- $\rightsquigarrow$ ρ is the correlation coefficient between X and Y

The multivariate normal distribution (cont.)

Substituting $\rho\sigma_1\sigma_2$ for σ_{12} in $\mathbf{\Sigma}$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_2^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{bmatrix}$$

The determinant of $\mathbf{\Sigma}$ is $\sigma_1^2\sigma_2^2(1-\rho^2)$ (remember that $\rho^2 \leq 1$)

- Suppose that $\rho^2 < 1$, so that $\mathbf{\Sigma}$ is invertible

The inverse of $\mathbf{\Sigma}$

$$\mathbf{\Sigma}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_2\sigma_1 & \sigma_1^2 \end{bmatrix} \qquad (26)$$

The multivariate normal distribution (cont.)

From this expression, the PDF of (X, Y)

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{(-q/2)}, \quad -\infty < x, y < \infty \tag{27}$$

with

$$q = \frac{1}{1-\rho^2}\Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{x-\mu_2}{\sigma_2}\Big) + \Big(\frac{y-\mu_2}{\sigma^2}\Big)^2\Big] \tag{28}$$

The multivariate normal distribution (cont.)

In general, if X and Y are independent RVs, correlation coefficient is zero

If they are normal, then $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$

- (by Corollary 2.1)

The multivariate normal distribution (cont.)

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{(-q/2)}, \quad -\infty < x, y < \infty$$

$$q = \Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 + \Big(\frac{y-\mu_2}{\sigma^2}\Big)^2\Big]$$

From the joint PDF of (X, Y), let the correlation coefficient be zero

- X and Y are independent

For the bivariate normal, independence corresponds to $\rho = 0$

The generalisation to the multivariate case also holds true

The multivariate normal distribution (cont.)

If two random variables are independent their covariance is zero

- The converse is not necessarily true

Yet, it can be shown that this is true for the multivariate normal

Theorem 2.2

Suppose that $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Suppose the partitioning

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Then, $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent if and only if* $\boldsymbol{\Sigma}_{12} = \mathbf{0}$

The multivariate normal distribution (cont.)

Proof

Note that $\mathbf{\Sigma}_{12} = \mathbf{\Sigma}'_{21}$

The joint MGF of $\mathbf{X}_1$ and $\mathbf{X}_2$

$$M_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{t}_1, \mathbf{t}_2) =$$

$$\exp\Big\{\mathbf{t}'_1\boldsymbol{\mu}_1 + \mathbf{t}'_2\boldsymbol{\mu}_2 + \frac{1}{2}\Big(\mathbf{t}'_1\mathbf{\Sigma}_{11}\mathbf{t}_1 + \mathbf{t}'_2\mathbf{\Sigma}_{22}\mathbf{t}_2 + \mathbf{t}'_2\mathbf{\Sigma}_{21}\mathbf{t}_1 + \mathbf{t}'_1\mathbf{\Sigma}_{12}\mathbf{t}_2\Big)\Big\} \qquad (29)$$

We used

$$\mathbf{t}' = (\mathbf{t}'_1, \mathbf{t}'_2)$$

The multivariate normal distribution (cont.)

By Corollary (2.1),

- $\mathbf{X}_1 \sim N_m(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$
- $\mathbf{X}_2 \sim N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$

The product of their marginal MGFs

$$M_{\mathbf{x}_1}(\mathbf{t}_1)M_{\mathbf{x}_2}(\mathbf{t}_2) = \exp\left\{\mathbf{t}_1'\boldsymbol{\mu}_1 + \mathbf{t}_2'\boldsymbol{\mu}_2 + \frac{1}{2}\Big(\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1 + \mathbf{t}_2'\boldsymbol{\Sigma}_{22}\mathbf{t}_2\Big)\right\} \quad (30)$$

For $\mathbf{X}_1$ and $\mathbf{X}_2$ be independent, Equation (29) and (30) must be identical

The multivariate normal distribution (cont.)

$$\exp\Big\{\mathbf{t}_1'\boldsymbol{\mu}_1 + \mathbf{t}_2'\boldsymbol{\mu}_2 + \frac{1}{2}\Big(\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1 + \mathbf{t}_2'\boldsymbol{\Sigma}_{22}\mathbf{t}_2 + \mathbf{t}_2'\boldsymbol{\Sigma}_{21}\mathbf{t}_1 + \mathbf{t}_1'\boldsymbol{\Sigma}_{12}\mathbf{t}_2\Big)\Big\}$$

$$\exp\Big\{\mathbf{t}_1'\boldsymbol{\mu}_1 + \mathbf{t}_2'\boldsymbol{\mu}_2 + \frac{1}{2}\Big(\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1 + \mathbf{t}_2'\boldsymbol{\Sigma}_{22}\mathbf{t}_2\Big)\Big\}$$

Independence of $\mathbf{X}_1$ and $\mathbf{X}_2$ is verified when $\boldsymbol{\Sigma}_{12} = \mathbf{0}'$ and hence $\boldsymbol{\Sigma}_{21} = \mathbf{0}$

- $\rightsquigarrow$ The covariances between their components are all 0

■

The multivariate normal distribution (cont.)

Theorem

Suppose $\mathbf{X}$ *has a* $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *distribution, partitioned as in Equation (23-24)*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

Assume that $\boldsymbol{\Sigma}$ *is positive definite*

Then, the conditional distribution of $\mathbf{X}_1 | \mathbf{X}_2$

$$N_m \Big[\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} (\mathbf{X}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} \Big]$$

The multivariate normal distribution (cont.)

Proof

Consider the joint distribution of $\mathbf{W} = \mathbf{X}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2$ and $\mathbf{X}_2$

The joint distribution can be obtained from the transformation

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{X}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_m & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$$

This is a linear transformation

The joint distribution is multivariate normal (Theorem 2.1)

The multivariate normal distribution (cont.)

The means

- $E(\mathbf{W}) = \boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2$
- $E(\mathbf{X}_2) = \boldsymbol{\mu}_2$

The covariance matrix

$$\underbrace{\begin{bmatrix} \mathbf{I}_m & -\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}}_{\boldsymbol{\Sigma}} \underbrace{\begin{bmatrix} \mathbf{I}_m & \mathbf{0}' \\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{I}_p \end{bmatrix}}_{\mathbf{A}'}$$

$$= \begin{bmatrix} \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} & \mathbf{0}' \\ \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

The multivariate normal distribution (cont.)

By Theorem 2.2, the random vectors $\mathbf{W}$ and $\mathbf{X}_2$ are independent

- The conditional of $\mathbf{W}_2|\mathbf{X}_2$ equals the marginal of $\mathbf{W}$

$$\mathbf{W}|\mathbf{X}_2 \sim N_m\big(\boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\big)$$

Because of this independence,

$$\begin{aligned} &\mathbf{W} + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2|\mathbf{X}_2 \\ &\qquad \sim N_m\big(\boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\big) \end{aligned}$$

■

The multivariate normal distribution (cont.)

Example

Suppose that $(X, Y) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_2^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$$

- $\rightsquigarrow$ μ_1 and σ_1^2 are the mean and variance of Y $(= X_1)$
- $\rightsquigarrow$ μ_2 and σ_2^2 are the mean and variance of X $(= X_2)$

$$N_m\big(\boldsymbol{\mu}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\big)$$

The expression shows the conditional distribution of Y given $X = x$

$$N\Big[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2)\Big]$$

The multivariate normal distribution (cont.)

The coefficient of x in the conditional mean of $E(Y|x)$ is $\rho\sigma_2/\sigma_1$

$$E(Y|x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}$$

The conditional mean of Y, given that $X = x$ is linear in x

The σ_1 and σ_2 are the respective standard deviations

ρ is the correlation coefficient of X and Y

Remark

This follows from the fact that the coefficient of x in a linear conditional mean $E(Y|x)$ is the product of correlation coefficient and σ_2/σ_1

The multivariate normal distribution (cont.)

The mean of the conditional distribution of Y given $X = x$

- It depends upon x (unless $\rho = 0$)

The variance $\sigma_2^2(1 - \rho^2)$ is the same for all real values of x

Thus, given that $X = x$, the conditional probability that Y is within $(\approx 2.57)\sigma_2\sqrt{(1 - \rho^2)}$ units of the conditional mean is 0.99, whatever is x

Most of the probability for the distribution of (X, Y) lies within the band

$$\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1) \pm (\approx 2.57)\sigma_2\sqrt{1 - \rho^2}$$

about the plot of the linear conditional mean

■

The multivariate normal distribution (cont.)

Theorem

Suppose $\mathbf{X}$ *has a* $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ *distribution*

Let $\boldsymbol{\Sigma}$ *is positive definite*

Then, the RV $W = (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma} (\mathbf{X} - \boldsymbol{\mu})$ *has a* $\chi^2(n)$ *distribution*

Proof

Write $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2} \boldsymbol{\Sigma}^{1/2}$, with $\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Gamma}' \boldsymbol{\Lambda}^{1/2} \boldsymbol{\Gamma}$

Then, $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})$ is $N_n(\mathbf{0}, \mathbf{I}_n)$

Let $W = \mathbf{Z}'\mathbf{Z} = \sum_{i=1}^n Z_i^2$

The multivariate normal distribution (cont.)

Each Z_i has a $N(0,1)$ distribution, for $i = 1, 2, \dots, n$

From Theorem 1.1, it follows that Z_i^2 has a $\chi^2(1)$ distribution

Variables $Z_1, \dots, Z_n$ are independent standard normal RVs

- Thus, $\sum_{i=1}^n Z_i^2 = W$ has a $\chi^2(n)$ distribution
- (by an earlier corollary)

$\blacksquare$

An application

The normal distribution

Principal components analysis

An application

Principal components analysis

We consider applications of the multivariate normal distribution

- Principal components analysis (PCA)

It results in a linear function of a multivariate normal random vector

- The function preserves the 'total variation in the problem'
- The random vector has independent components

Principal components analysis (cont.)

Let the random vector $\mathbf{X}$ has the multivariate normal distribution $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

- $\boldsymbol{\Sigma}$ is positive definite

Consider the spectral decomposition of $\boldsymbol{\Sigma} = \boldsymbol{\Gamma}' \boldsymbol{\Lambda} \boldsymbol{\Gamma}$

⇝ The eigenvalues form the main diagonal of $\boldsymbol{\Lambda}$

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

⇝ The corresponding eigenvectors are the columns of $\boldsymbol{\Gamma}$

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$$

Assume without loss of generality that the eigenvalues are sorted

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

Principal components analysis (cont.)

Define the random vector $\mathbf{Y} = \mathbf{\Gamma}(\mathbf{X} - \boldsymbol{\mu})$

$\mathbf{\Gamma}\mathbf{\Sigma}\mathbf{\Gamma}' = \mathbf{\Lambda}$, thus $\mathbf{Y}$ has a $N_n(\mathbf{0}, \mathbf{\Lambda})$ distribution (by Theorem 2.1)

- The components $Y_1, Y_2, \dots, Y_n$ are independent RVs
- Y_i has a $N(0, \lambda_i)$ distribution, for $i = 1, 2, \dots, n$

The random vector $\mathbf{Y}$ is the vector of **principal components**

Principal components analysis (cont.)

Total variation of a random vector sums the variances of its components

For a random vector $\mathbf{X}$, because $\mathbf{\Gamma}$ is an orthogonal matrix

$$TV(\mathbf{X}) = \sum_{i=1}^{n} \sigma_i^2 = \text{Tr}(\mathbf{\Sigma}) = \text{Tr}(\mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma}) = \text{Tr}(\mathbf{\Lambda}\mathbf{\Gamma}\mathbf{\Gamma}') = \sum_{i=1}^{n} \lambda_i = TV(\mathbf{Y})$$

Hence, $\mathbf{X}$ and $\mathbf{Y}$ share the same total variation

Principal components analysis (cont.)

Consider the first component of $\mathbf{Y}$

$$Y_1 = \mathbf{v}_1'(\mathbf{X} - \boldsymbol{\mu})$$

- This is a linear combination of the components of $\mathbf{X} - \boldsymbol{\mu}$
- Because of orthogonality, $||\mathbf{v}_1||^2 = \sum_{j=1}^{n} v_{1j}^2 = 1$

Principal components analysis (cont.)

Consider any other linear combination of $(\mathbf{X} - \boldsymbol{\mu})$

$$\mathbf{a}'(\mathbf{X} - \boldsymbol{\mu}), \text{ with } ||\mathbf{a}||^2 = 1$$

As $\mathbf{a} \in \mathcal{R}^n$ and because $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ form a basis for $\mathcal{R}^n$

- We must have

$$\mathbf{a} = \sum_{j=1}^{n} a_j \mathbf{v}_j$$

  for some set of scalars $a_1, a_2, \dots, a_n$

Furthermore, the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal

$$\mathbf{a}'\mathbf{v}_i = \Big(\sum_{j=1}^{n} a_j \mathbf{v}_j\Big)'\mathbf{v}_i = \sum_{j=1}^{n} a_j \mathbf{v}_j'\mathbf{v}_i = a_i$$

Principal components analysis (cont.)

We have that $\mathbf{\Sigma} = \mathbf{\Gamma}'\mathbf{\Lambda}\mathbf{\Gamma} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i'$ and that $\lambda_i > 0$

Then, the inequality

$$\begin{aligned} \text{Var}(\mathbf{a}'\mathbf{X}) &= \mathbf{a}'\text{Cov}(\mathbf{X})\mathbf{a} = \mathbf{a}'\mathbf{\Sigma}\mathbf{a} = \sum_{i=1}^{n} \lambda_i (\mathbf{a}'\mathbf{v}_i)^2 \\ &= \sum_{i=1}^{n} \lambda_i a_i^2 \leq \lambda_1 \sum_{i=1}^{n} a_i^2 = \lambda_1 = \text{Var}(Y_1) \end{aligned} \tag{31}$$

Y_1 has the maximum variance of any other linear combination

$\rightsquigarrow$ Y_1 is called the **first principal component** of $\mathbf{X}$

Principal components analysis (cont.)

Other components $Y_2, Y_3, \dots, Y_n$ share a similar property

Relative to the order of their associated eigenvalue

$\rightsquigarrow$ **Second**, **third**, ..., **n-th principal component**

Principal components analysis (cont.)

Theorem

For $j = 2, \dots, n$ *and* $i = 1, 2, \dots, j-1$,

$$Var[\mathbf{a}'\mathbf{X}] \leq \lambda_j = Var(Y_j)$$

for all vectors $\mathbf{a}$ *such that* $\mathbf{a} \perp \mathbf{v}_i$ *and* $||\mathbf{a}|| = 1$

Proof

The proof follows the lines of that for the first principal component ($\star$)

■