CK0255/TIP8421: First exam (AP01) September 29, 2017

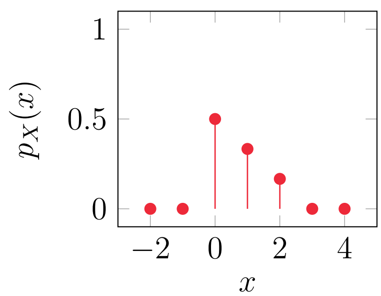
Q 01 (15%). Let $p_X(x)$ be the PMF of a random variable of the discrete type

$$p_X(x) = \begin{cases} 1/2, & x = 0 \\ 1/3, & x = 1 \\ 1/6, & x = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{1}$$

1. Find the $\mathcal{D}_X$, the range of the RV.
2. Find $P(X \geq 1.5)$.
3. Find $P(0 < X < 2)$.
4. Find $P(X = 0|X < 2)$.
5. Find the CDF of X, $F_X(x)$.

Solution: 1.) The range can be found from the PMF, it consists of all possible values of X.

$$\mathcal{D}_X = \{0, 1, 2\}$$



2.) Event $X \geq 1.5$ only happens if $X = 2$, thus

$$\begin{aligned} P(X \geq 1.5) &= P(X = 2) \\ &= p_X(x = 2) = 1/6 \end{aligned}$$

3.) Similarly, we have

$$\begin{aligned} P(0 < X < 2) &= P(X = 1) \\ &= p_X(x = 1) = 1/3 \end{aligned}$$

4.) To determine this conditional probability, we use $P(A|B) = P(A, B)/P(B)$. We have

$$\begin{aligned} P(X = 0|X < 2) &= \frac{P(X = 0, X < 2)}{P(X < 2)} = \frac{P(X = 0)}{P(X < 2)} \\ &= \frac{p_X(x = 0)}{p_X(X = 0) + p_X(X = 1)} = \frac{1/2}{[1/2 + 1/3]} = 3/5 \end{aligned}$$

Q 02 (15%). Let X and Y be two random variables such that $Y = -2X + 3$. We know that $E(Y) = 1$ and that $E(Y^2) = 9$,

1. Find $E(X)$.
2. Find $\mathrm{Var}(X)$.

Solution: We have $Y = -2X + 3$. By the linearity of expectation, we have

$$\begin{aligned} E(Y) &= -2E(X) + 3 = 1 \\ &\rightsquigarrow 1 = -2E(X) + 3 \\ &\rightsquigarrow E(X) = 1 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(Y) = 4\mathrm{Var}(X) &= E(Y^2) - [E(Y)]^2 = 9 - 1 = 8 \\ &\rightsquigarrow \mathrm{Var}(X) = 2 \end{aligned}$$

Q 03 (30%). Let X be a random variable of the discrete type with the PMF

$$p_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1, 2, 3, \ldots \\ 0, & \text{elsewhere} \end{cases}, \quad \text{for some } p \in (0, 1). \tag{2}$$

1. Find $E(1/2^X)$.

[*Hint*] Use the geometric series: $\sum_{k=0}^{\infty} a^k = \sum_{k=1}^{\infty} a^{k-1} = 1/(1-a)$, for some $a \in [0, 1)$.

Solution: Let $q = 1 - p$. By definition, we have

$$\begin{aligned} E(1/2^X) &= \sum_{x=1}^{\infty} \frac{1}{2^x} p_X(x) = \sum_{x=1}^{\infty} \frac{1}{2^x} p q^{x-1} \\ &= \frac{p}{2} \sum_{x=1}^{\infty} \left(\frac{q}{2}\right)^{x-1} = \frac{p}{2} \frac{1}{1 - q/2} \\ &= \frac{p}{1+p} \end{aligned}$$

Q 04 (40%). Let X be a continuous random variable uniform in $(0, 1)$ and let $Y = e^X$.

1. Find the CDF of Y.
2. Find the PDF of Y.
3. Find $E(Y)$ using the PDF of Y and compare the result with $E(e^X)$.

Solution: The PDF and the CDF of X are known objects. In particular, we have

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

1.) Let us first determine the range $\mathcal{D}_Y$ of Y. Since e^x is an increasing function of x and $\mathcal{D}_X = [0, 1]$, we have that $\mathcal{D}_Y = [1, e]$.

We conclude that

$$\begin{aligned} F_Y(y) = P(Y \le y) = 0, &\quad \text{for } y < 1 \\ F_Y(y) = P(Y \le y) = 1, &\quad \text{for } y \ge e \end{aligned}$$

As for $y \in [1, e]$, we can write

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(e^X \le y) \\ &= P\big[X \le \ln(y)\big] = F_X\big[\ln(y)\big] = \ln(y) \end{aligned}$$

Summarising,

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ \ln(y), & 1 \le y < e \\ 1, & y \ge e \end{cases}.$$

2.) The CDF above is a continuous function, the corresponding PDF can be determined by taking its derivative. We have,

$$f_Y(y) = F_Y'(y) = \begin{cases} 1/y, & 1 \le y \le e \\ 0, & \text{elsewhere} \end{cases}$$

Strictly speaking, the CDF is not differentiable about 1 and e, but this is okay as for continuous RV changing the PDF at a finite number of points does not change probabilities.

3.) To find $E(Y)$, we can use

$$\begin{aligned} E(Y) = E(e^X) &= \int_{-\infty}^{\infty} e^x f_X(x) \mathrm{d}x \\ &= \int_0^1 e^x \mathrm{d}x = e - 1 \\ E(Y) &= \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}y \\ &= \int_1^e y(1/y) \mathrm{d}y = e - 1 \end{aligned}$$