CK0255/TIP8421: First exam (AP01-II) October 06, 2017

Q 01 (50%). Let X be a random variable of the continuous type X with the PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & x \geq 0 \\ (1-p)\lambda e^{\lambda x}, & x < 0 \end{cases},$$

with p and λ some scalars with $\lambda > 0$ and $p \in [0, 1]$.

1. Knowing that $f_X(x) \geq 0$ in $\mathcal{R}$, show that $f_X(x)$ is a valid PDF;
2. Find the mean and the variance of X using appropriate expected values.

[*Hint*] Use $\int xe^{cx}\mathrm{d}x = e^{cx}[(cx-1)/c^2]$ and $\int x^2e^{cx}\mathrm{d}x = e^{cx}(x^2/c - 2x/c^2 + 2/c^3)$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{\infty} f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} (1-p)\lambda e^{\lambda x}\mathrm{d}x + \int_{0}^{\infty} p\lambda e^{-\lambda x}\mathrm{d}x \\ &= 1 \end{aligned}$$

$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} f_X(x)\mathrm{d}x + \int_{0}^{\infty} f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} x(1-p)\lambda e^{\lambda x}\mathrm{d}x + \int_{0}^{\infty} xp\lambda e^{-\lambda x}\mathrm{d}x \\ &= -\frac{1-p}{\lambda} + \frac{p}{\lambda} \\ &= \frac{2p-1}{\lambda} \end{aligned}$$

$$\begin{aligned} E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\mathrm{d}x &= \int_{-\infty}^{0} x^2 f_X(x)\mathrm{d}x + \int_{0}^{\infty} x^2 f_X(x)\mathrm{d}x \\ &= \int_{-\infty}^{0} x^2(1-p)\lambda e^{\lambda x}\mathrm{d}x + \int_{0}^{\infty} x^2 p\lambda e^{-\lambda x}\mathrm{d}x \\ &= \frac{2(1-p)}{\lambda^2} + \frac{2p}{\lambda^2} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

$$\mathrm{Var}(X) = \frac{2}{\lambda^2} - \Big(\frac{2p-1}{\lambda}\Big)^2.$$

Q 02 (30%). Let X be a random variable of the continuous type with the PDF

$$f_X(x) = \begin{cases} x^2/9, & 0 < x < 3 \\ 0, & \text{elsewhere} \end{cases}$$

1. Find the PDF $f_Y(y)$ of $Y = X^3$.

Solution: For $y \in (0, 27)$,

$$\begin{aligned} x &= y^{1/3} \\ \mathrm{d}x/\mathrm{d}y &= 1/3y^{-2/3} \\ \rightsquigarrow f_Y(y) &= f_X\left[y^{1/3)}\right]\mathrm{d}x/\mathrm{d}y = \left[(1/3)(y^{-2/3})\right]\left[y^{2/3}/9\right] = 1/27 \quad \text{(zero elsewhere)} \end{aligned}$$

Q 03 (10%). Let X be a random variable of the continuous type with the PDF

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Consider a random rectangle of sides X and $(1 - X)$.

1. What is the expected value of the area of the rectangle?

Solution:

$$E\big[X(1-X)\big] = \int_0^1 \big[x(1-x)\big](3x^2)\mathrm{d}x = 3/20$$

Q 04 (10%). Let X be a random variable such that $E[(X-k)^2]$ exists for all $k \in \mathcal{R}$.

1. Show that $k = E(X)$ is a minimiser of $E[(X-k)^2]$.

Solution:

$$\begin{aligned} f(k) = E[(X-k)^2] &= E(X^2) - 2kE(X) + k^2 \\ \rightsquigarrow f'(k) &= -2E(X) + 2k = 0 \\ \rightsquigarrow k &= E(X) \end{aligned}$$