CK0255/TIP8421: Second exam (AP02) November 24, 2017

Q 01 (20%). Complete the table of the joint probability mass function and the marginal mass functions of the discrete random vector (X, Y) [70%]

$p(X=x, Y=y)$	$y=1$	$y=2$	$y=3$	$p(X=x)$
$x=1$	$-$	0.2	$-$	0.4
$x=2$	$-$	$-$	0.3	$-$
$p(Y=y)$	$-$	0.3	0.4	

Calculate $P(X+Y=3)$ [30%]

Solution:

$p(X=x, Y=y)$	$y=1$	$y=2$	$y=3$	$p(X=x)$
$x=1$	$-$	0.2	0.1	0.4
$x=2$	$-$	0.1	0.3	$-$
$p(Y=y)$	$-$	0.3	0.4	

$p(X=x, Y=y)$	$y=1$	$y=2$	$y=3$	$p(X=x)$
$x=1$	0.1	0.2	0.1	0.4
$x=2$	$-$	0.1	0.3	$-$
$p(Y=y)$	$-$	0.3	0.4	

$p(X=x, Y=y)$	$y=1$	$y=2$	$y=3$	$p(X=x)$
$x=1$	0.1	0.2	0.1	0.4
$x=2$	$-$	0.1	0.3	0.6
$p(Y=y)$	0.3	0.3	0.4	

$p(X=x, Y=y)$	$y=1$	$y=2$	$y=3$	$p(X=x)$
$x=1$	0.1	0.2	0.1	0.4
$x=2$	0.2	0.1	0.3	0.6
$p(Y=y)$	0.3	0.3	0.4	

$P(X+Y=3) = 0.4.$

Q 02 (40%). Consider the function $f(\mathbf{x}) = 2e^{(-x_1-x_2)}$ for $0 \leq x_2 \leq x_1 < \infty$ and $f(\mathbf{x}) = 0$ elsewhere.

1. [60%] Show that $f(\mathbf{x})$ is the probability density function of a random vector $\mathbf{X}$

2. [40%] Calculate the marginal probability density functions

Solution: Function $f(x,y)$ is continuous and positive over the support $\mathcal{S}$. It suffices to check the normalisation condition.

$$\begin{aligned}\int_{\mathcal{S}}\int f(\mathbf{x})\mathrm{d}\mathbf{x} &= \int_0^\infty \int_0^{x_1} f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = \int_0^\infty \int_0^{x_1} 2e^{(-x_1-x_2)}\mathrm{d}x_2\mathrm{d}x_1 \\ &= \int_0^\infty \int_0^{x_1} 2e^{(-x_1)}e^{(-x_2)}\mathrm{d}x_2\mathrm{d}x_1 = \int_0^\infty 2e^{(-x_1)}\Big[\int_0^{x_1} e^{(-x_2)}\mathrm{d}x_2\Big]\mathrm{d}x_1 \\ &= \int_0^\infty 2e^{(-x_1)}(1-e^{-x_1})\mathrm{d}x_1 = \Big[e^{(-2x_1)} - 2e^{(-x_1)}\Big]_{x_1=0}^{x_1\to\infty} = 1\end{aligned} \tag{1}$$

The marginals

$$\begin{aligned} f(x_1|x_1<0) &= \int_{-\infty}^{\infty} f(\mathbf{x})\mathrm{d}x_2 = 0 \\ f(x_1|x_1\geq 0) &= \int_{-\infty}^{\infty} f(\mathbf{x})\mathrm{d}x_2 = \int_0^{x_1} 2e^{(-x_1-x_2)}\mathrm{d}x_2 = \int_0^{x_1} 2e^{(-x_1)}e^{(-x_2)}\mathrm{d}x_2 \\ &= 2e^{(-x_1)}\int_0^{x_1} e^{(-x_2)}\mathrm{d}x_2 = 2e^{(-x_1)}\big[1-e^{(-x_1)}\big]\end{aligned} \tag{2}$$

$$\begin{aligned} f(x_2|x_2<0) &= \int_{-\infty}^{\infty} f(\mathbf{x})\mathrm{d}x_1 = 0 \\ f(x_2|x_2\geq 0) &= \int_{-\infty}^{\infty} f(\mathbf{x})\mathrm{d}x_2 = \int_{x_2}^{\infty} 2e^{(-x_1-x_2)}\mathrm{d}x_1 = \int_{x_2}^{\infty} 2e^{(-x_1)}e^{(-x_2)}\mathrm{d}x_1 \\ &= 2e^{(-x_2)}\int_0^{x_1} e^{(-x_2)}\mathrm{d}x_2 = 2e^{(-2x_2)}\end{aligned} \tag{3}$$

Q 03 (40%). The probability density function of the continuous random vector (X,Y) with support the triangle $\mathcal{S} = \{(x,y) : 0 \leq y \leq x \leq 2\}$ is $f(x,y) = Kxy$.

1. [20%] Calculate the value of K
2. [40%] Calculate the marginal probability density functions
3. [40%] Calculate the covariance of X and Y

Solution: By the normalisation condition

$$\begin{aligned} 1 &= \int_{\mathcal{S}}\int f(x,y)\mathrm{d}x\mathrm{d}y = K\int_0^2 x\mathrm{d}x \int_0^x y\mathrm{d}y = K/2\int_0^2 x^3\mathrm{d}x = 2K \\ &\rightsquigarrow K = 1/2\end{aligned} \tag{4}$$

The marginals

$$\begin{aligned} f(x|x \in [0,2]) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y = \frac{x}{2}\int_0^x y\mathrm{d}y = x^3/4 \\ f(y|y \in [0,2]) &= \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x = \frac{y}{2}\int_y^2 x\mathrm{d}x = y(4-y^2)/4 \end{aligned} \tag{5}$$

The expectations

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)\mathrm{d}x = \int_0^2 x^4/4\mathrm{d}x = 8/5 \\ E(Y) &= \int_{-\infty}^{\infty} yf(y)\mathrm{d}y = \int_0^2 y^2(4-y^2)/4\mathrm{d}y = 16/15 \end{aligned} \tag{6}$$

$$\begin{aligned} E(XY) &= \int_{\mathcal{T}}\int xyf(x,y)\mathrm{d}x\mathrm{d}y = \int_0^2\int_0^2 xyf(x,y)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\int_0^2 x^2\mathrm{d}x\int_0^x y^2\mathrm{d}y = \frac{1}{2}\frac{1}{3}\int_0^2 x^5\mathrm{d}x \\ &= 16/9 \quad (\cancel{32/9}) \end{aligned}$$

The covariance

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 16/9 \quad (\cancel{32/9}) - (8/5)(16/5) = \cancel{416/225}$$