Signals and distributions

Linear systems and ATML

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Canonical signals

Signals and distributions

Canonical signals

We describe some **signals** or **functions** in the real variable t, time

$$f : \mathcal{R} \rightarrow \mathcal{C}$$

In our studies, such functions are often discontinuous

- We introduce a generalisation of function
- The **distribution**

Unit step

Canonical signals

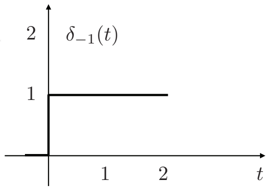
Unit step

Definition

Unit step

The unit step, denoted as $\delta_{-1}(t)$, is a function

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



The function is continuous over the domain, except for the origin

- Discontinuity, size 1

Signals and distributions
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Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Unit step (cont.)

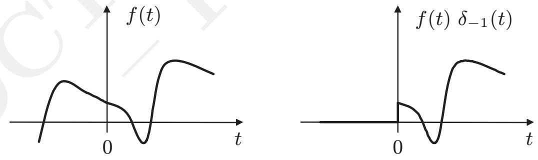
We can use the unit step to define new functions

Consider some function $f(t) : \mathcal{R} \rightarrow \mathcal{R}$, we have

$$f(t)\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

Values of $f(t)$ for $t < 0$ have been set to zero

Graphically,



Signals and distributions
UFC/DC CK0255|TIP8244 2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Ramps

Canonical signals

Signals and distributions
UFC/DC CK0255|TIP8244 2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

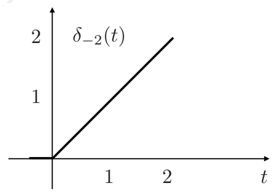
Ramps

Definition

Unit ramp

The integral of the unit step is called the unit ramp, $\delta_{-2}(t)$

$$\begin{aligned} \delta_{-2}(t) &= \int_{-\infty}^{t} \delta_{-1}(\tau)\mathrm{d}\tau \\ &= \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases} \\ &= t\delta_{-1}(t) \end{aligned}$$



Ramps (cont.)

Definition

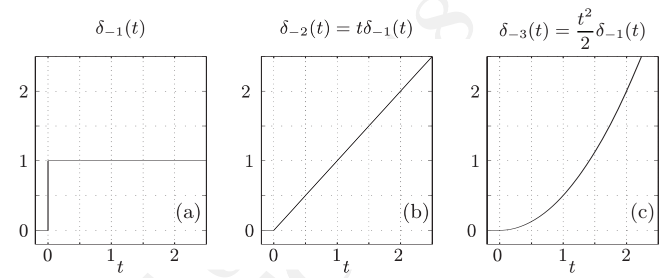
Ramp functions

The family of ramp functions $\delta_{-k}(t)$ can be be recursively defined for $k > 2$

$$\delta_{-k}(t) = \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{k-1 \text{ times}} \delta_{-1}(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \dfrac{t^{k-1}}{(k-1)!}, & t \geq 0 \end{cases}$$
$$= \frac{t^{k-1}}{(k-1)!} \delta_{-1}(t)$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Ramps (cont.)



- **Quadratic ramp**, $k = 3$

$$\rightsquigarrow \quad \delta_{-3}(t) = \frac{t^2}{2!} \delta_{-1}(t)$$

- **Cubic ramp**, $k = 4$

$$\rightsquigarrow \quad \delta_{-4}(t) = \frac{t^3}{3!} \delta_{-1}(t)$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Ramps (cont.)

Definition

Exponential ramp

*A generalisation of the ramp function is the **exponential ramp**, or **cisoid***

$$\frac{t^k}{k!} e^{at} \delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^k}{k!} e^{at}, & t \geq 0 \end{cases}$$

It is defined in terms of two parameters

$\rightsquigarrow$ $k \in \mathcal{N}$

$\rightsquigarrow$ $a \in \mathcal{C}$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Ramps (cont.)

$$\frac{t^k}{k!} e^{at} \delta_{-1}(t)$$

Particular cases that can be generated from the exponential ramp

$\rightsquigarrow$ $a = 0$ and $k = 1, 2, \cdots$, the family of ramp functions

$$\frac{t^k}{k!} \cdot 1 \cdot \delta_{-1}(t)$$

$\rightsquigarrow$ $a = 0$ and $k = 0$, the unit ramp

$$(1/1) \cdot 1 \cdot \delta_{-1}(t)$$

$\rightsquigarrow$ $k = 0$ and $a \in \mathcal{R}$, exponential function e^{at}

$\rightsquigarrow$ $k = 0$ and $a = j\omega \in \mathcal{I}$, a linear combinations of the resulting exponential ramps can be used to describe sinusoidal functions

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$
$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Linear combinations of ramps can be used for polynomial functions

$$c_2 t^2 + c_1 t + c_0$$

Impulse

Canonical signals

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse

We can extend the family of canonical signals

We consider the derivatives of the unit step (so far, we only integrated it)

- The results of classical calculus cannot be used for the purpose
- The derivative of a discontinuous function is not defined

We need to generalise the concept of function

$\rightsquigarrow$ The distribution

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse (cont.)

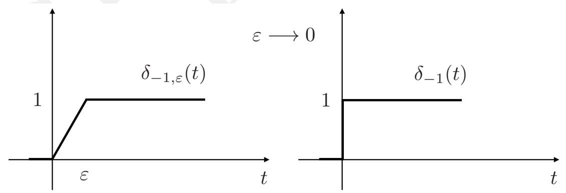
Let $\varepsilon > 0$ be some positive scalar

Define the function $\delta_{-1,\varepsilon}(t)$

$$\delta_{-1,\varepsilon}(t) = \begin{cases} 0, & t < 0 \\ t/\varepsilon, & t \in [0, \varepsilon) \\ 1, & t \geq \varepsilon \end{cases}$$

This function is understood as a continuous approximation of the unit step

$$\rightsquigarrow \lim_{\varepsilon \rightarrow 0} \delta_{-1,\varepsilon}(t) = \delta_{-1}(t)$$



Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse (cont.)

Definition

Finite impulse

Function $\delta_{-1,\varepsilon}(t) = \begin{cases} 0, & t < 0 \\ t/\varepsilon, & t \in [0, \varepsilon) \\ 1, & t \geq \varepsilon \end{cases}$ *is continuous*

Thus, it possesses a derivative

$$\delta_{\varepsilon}(t) = \frac{d}{dt}\delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & t \in [0, \varepsilon) \\ 0, & \textit{elsewhere} \end{cases}$$

Function $\delta_{\varepsilon}(t)$ *is denoted as* ***finite impulse*** *of base* ε

Function $\delta_{\varepsilon}(t)$ is a rectangle, with base ε and with height $1/\varepsilon$

- Area equal to 1, whatever the value of ε

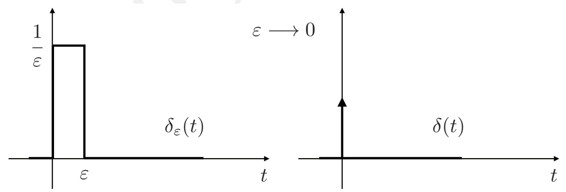
Impulse (cont.)

Definition

Unit impulse *or* ***Dirac function***

We define the derivative of the unit step $\delta_{-1}(t0$

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t) = \frac{d}{dt}\lim_{\varepsilon\to 0}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\frac{d}{dt}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\delta_{\varepsilon}(t)$$

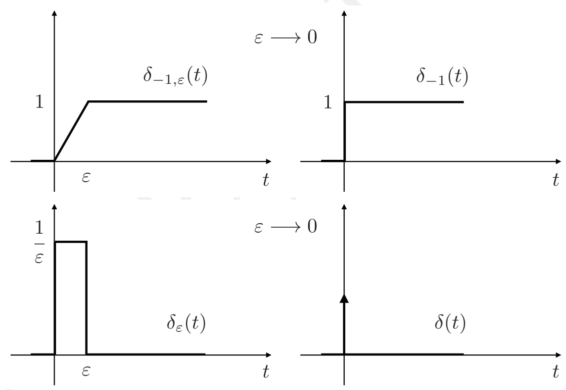


Such a definition is not formally correct in the sense of the classical calculus

- It is valid only if we accept the generalisation of a function
- The impulse $\delta(t)$ is not a function, it is a distribution

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse (cont.)



Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse (cont.)

The following properties hold

⇝ $\delta(t)$ is equal to zero everywhere, except the origin

$$\delta(t) = 0, \quad \text{if } t \neq 0$$

⇝ $\delta(t)$ at the origin is equal to infinity

$$\delta(t) = \infty, \quad \text{if } t = 0$$

⇝ The area under $\delta(t)$ is equal to 1

$$\int_{-\infty}^{+\infty}\delta(t)\mathrm{d}t = \int_{0^-}^{0^+}\delta(t)\mathrm{d}t = 1$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Impulse (cont.)

Theorem

Let $f(t)$ be some continuous function in $t = 0$

- *The product of $f(t)$ and the impulse $\delta(t)$*

$$\rightsquigarrow \quad f(t)\delta(t) = f(0)\delta(t)$$

Let $f(t)$ be some continuous function in $t = T$

- *The product of $f(t)$ and $\delta(t-T)$*

$$\rightsquigarrow \quad f(t)\delta(t-T) = f(T)\delta(t-T)$$

We used function $\delta(t-T)$ to denote the impulse centred in T

Proof

We have that $\delta(t) = 0$, for $t \neq 0$

The values taken by $f(t)$ for $t \neq 0$ are not significant (as the impulse is zero)

Derivative of the impulse

Canonical signals

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivative of the impulse

We use the limit reasoning to define higher-order derivatives of the impulse

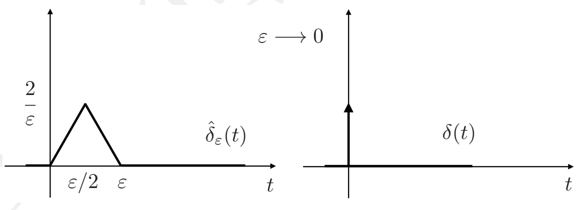
Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivative of the impulse (cont.)

Consider the function $\hat{\delta}_\varepsilon(t)$

$$\widehat{\delta}_\varepsilon(t) = \begin{cases} 4t/\varepsilon^2, & t \in [0, \varepsilon/2) \\ 4/\varepsilon - 4t/\varepsilon^2, & t \in [\varepsilon/2, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

The impulse can be re-defined



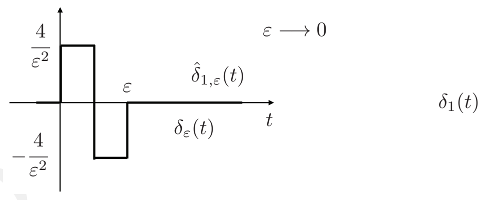
$$\rightsquigarrow \quad \delta(t) = \lim_{\varepsilon \to 0} \widehat{\delta}_\varepsilon(t)$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivative of the impulse (cont.)

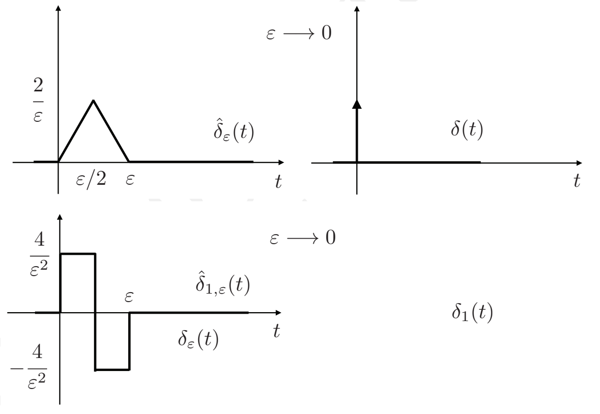
$$\widehat{\delta}_\varepsilon(t) = \begin{cases} 4t/\varepsilon^2, & t \in [0, \varepsilon/2) \\ 4/\varepsilon - 4t/\varepsilon^2, & t \in [\varepsilon/2, \varepsilon) \\ 0, & \text{elsewhere} \end{cases} ; \quad \rightsquigarrow \quad \delta(t) = \lim_{\varepsilon \to 0} \widehat{\delta}_\varepsilon(t)$$

We define the first-order derivative of the impulse $\delta(t)$



$$\rightsquigarrow \quad \delta_1(t) = \frac{d}{dt}\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\lim_{\varepsilon \to 0} \widehat{\delta}_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}t}\delta_{1,\varepsilon}(t)$$

Derivative of the impulse (cont.)



Derivative of the impulse (cont.)

The higher-order ($k > 1$) derivatives of the impulse

$$\rightsquigarrow \quad \delta_k(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta_{k-1}(t)$$

The family of canonical signals

Canonical signals

The family of canonical signals

For $k \in \mathcal{Z}$, we can define a family of canonical signals, $\delta_k(t)$

- $\rightsquigarrow$ $\delta_0(t) = \delta(t)$, the impulse ($k = 0$)
- $\rightsquigarrow$ $k < 0$, the integrals of the impulse
- $\rightsquigarrow$ $k > 0$, the derivatives of the impulse

Such signals are linearly independent

The family of canonical signals (cont.)

Definition

Linear dependence of scalar functions

Consider a set of scalar real functions $f_1(t), f_2(t), \dots, f_n(t)$

$$f_i(t) : \mathcal{R} \rightarrow \mathcal{R}$$

Functions $\{f_i\}_{i=1}^n$ *are said to be* ***linearly dependent*** *over the interval* $[t_1, t_2]$, *if and only if there exist a set of real numbers* $\alpha_1, \alpha_2, \dots, \alpha_n$ *that are not all equal to zero and such that*

$$\rightsquigarrow \quad \alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_n f_n(t) = 0, \quad \forall t \in [t_1, t_2]$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Example

Consider two functions $f_1(t)$ and $f_2(t)$

$$f_1(t) = t, \quad t \in (-\infty, \infty)$$

$$f_2(t) = |t| = \begin{cases} -t, & t \in (-\infty, 0] \\ t, & t \in (0, \infty) \end{cases}$$

The two functions are linearly dependent on each interval $[t_1, t_2]$ with $t_2 \leq 0$

- Let $\alpha_1 = \alpha_2 \neq 0$, we have $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$, for every $t \in [t_1, t_2]$

The two functions are linearly dependent on each interval $[t_1, t_2]$ with $t_1 \geq 0$

- Let $\alpha_1 = \alpha_2 \neq 0$, we have $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$, for every $t \in [t_1, t_2]$

The two functions are linearly independent on $[t_1, t_2]$, $t_1 < 0$ and $t_2 > 0$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Remark

Two or more functions can be linearly dependent in an interval

- Yet, they can be linearly independent in a larger interval

Conversely, linear independence in a given interval implies linear independence in any larger interval of which the initial interval is a subset

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Consider the function

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \delta_k(t)$$

Suppose that such a function is identically null over $[a, b]$, with $a \neq b$

$\rightsquigarrow$ Then, $a_k = 0$ for all $k \in \mathcal{Z}$

Derivatives of a discontinuous function

Signals and distributions

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivatives of a discontinuous function

We can formally calculate the derivative of discontinuous functions

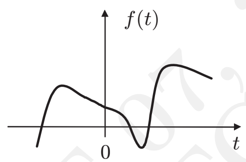
Discontinuous functions are common in systems analysis

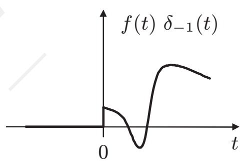
- Zero for $t < 0$ and continuous for $t \geq 0$
- Discontinuity in the origin

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivatives of a discontinuous function (cont.)

Let $f(t)$ be a continuous function





We are interested in calculating the derivative of function $f(t)\delta_{-1}(t)$

- If $f(0) \neq 0$, then $f(t)\delta_{-1}(t)$ has a discontinuity in $t = 0$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivatives of a discontinuous function (cont.)

The first-order derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(t)\delta_{-1}(t)\Big] = \Big[\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big]\delta_{-1}(t) + f(t)\underbrace{\Big[\frac{\mathrm{d}}{\mathrm{d}t}\delta_{-1}(t)\Big]}_{\delta(t)}$$

$$\rightsquigarrow \quad = \dot{f}(t)\delta_{-1}(t) + f(0)\delta(t)$$

It is the first-order derivative of the original function multiplied by $\delta_{-1}(t)$

- Plus the impulse at the origin multiplied by $f(0)$

Derivatives of a discontinuous function (cont.)

The second-order derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[f(t)\delta_{-1}(t)\Big] = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\dot{f}(t)\Big]\delta_{-1}(t) + \dot{f}(t)\Big[\frac{\mathrm{d}}{\mathrm{d}t}\delta_{-1}(t)\Big] + f(0)\Big[\frac{\mathrm{d}}{\mathrm{d}t}\delta(t)\Big]$$
$$\rightsquigarrow \quad = \ddot{f}(t)\delta_{-1}(t) + \dot{f}(0)\delta(t) + f(0)\delta_1(t)$$

It is the second-order derivative of the original function multiplied by δ_{-1}

- Plus the impulse at the origin multiplied by $\dot{f}(0)$
- Plus $\delta_1(t)$ multiplied by $f(0)$

Derivatives of a discontinuous function (cont.)

Higher-order derivatives are calculated analogously

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t)\delta_{-1}(t) = f^{(k)}\delta_1(t) + f^{(k-1)}(0)\delta(t) + \cdots + f(0)\delta_{k-1}(t)$$
$$\rightsquigarrow \quad = f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)$$

We used $\delta_0(t) = \delta(t)$

Derivatives of a discontinuous function (cont.)

Example

Consider the function

$$f(t) = \cos(t)\delta_{-1}(1)$$

We are interested in its derivatives

The first-order derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\cos t\delta_{-1}(t)\Big] = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\cos(t)\Big] + \sin(0)\delta_t + \cos(0)\delta_1(t)$$
$$= -\sin(t)\delta_{-1}(t) + \delta_1(t)$$

The second-order derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[\cos(t)\delta_{-1}(t)\Big] = \Big[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(t)\Big]\delta_{-1}(t) - \sin(0)\delta(t) + \cos(0)\delta_1(t)$$
$$= -\cos(t)\delta_{-1}(t) + \delta_{-1}(t)$$

■

Derivatives of a discontinuous function (cont.)

Example

Consider the cisoid function

$$f(t) = te^{(at)}\delta_{-1}(t)$$

We are interested in its derivatives

The first-order derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[te^{(at)}\delta_{-1}(t)\Big] = e^{(at)}\delta_{-1}(t) + ate^{(at)} + \Big[te^{(at)}\Big]_{t=0}\delta(t)$$
$$= (1-at)e^{(at)}\delta_{-1}(t)$$

The second-order derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big[te^{(at)}\delta_{-1}(t)\Big] = ae^{(at)}\delta_{-1}(t) + a(1+at)e^{(at)}\delta_{-1}(t)$$
$$+ \Big[(1+at)e^{(at)}\Big]_{t=0}\delta(t)$$
$$= (2a + a^2t)e^{(at)}\delta_{-1}(t) + \delta(t)$$

■

Convolution integrals

Signals and distributions

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals

The convolution integral is an important operator

- Largely utilised in various field

$\rightsquigarrow$ System and signal analysis

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals(cont.)

Definition

Convolution

Consider the two functions

$$f, g : \mathcal{R} \rightarrow \mathcal{C}$$

*The **convolution** of f with g is a function $h : \mathcal{R} \rightarrow \mathcal{C}$ in the real variable t*

$$h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau$$

Function $h(t)$ is built by using the operator **convolution integral**

$\rightsquigarrow \quad \star$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

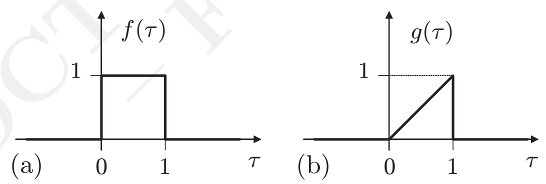
Convolution integrals (cont.)

Example

Consider the two functions

$$f(\tau) = \begin{cases} 1, & \tau \in [0, 1] \\ 0, & \text{elsewhere} \end{cases}$$

$$g(\tau) = \begin{cases} \tau, & \tau \in [0, 1] \\ 0, & \text{elsewhere} \end{cases}$$



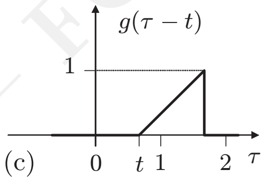
Convolution integrals (cont.)

Suppose that we want to calculate the function

$$g(\tau - t) = \begin{cases} \tau - t, & \tau \in [t, t+1] \\ 0, & \text{elsewhere} \end{cases}$$

We shift $g(\tau)$ by the quantity t

- If $t > 0$, to the right
- If $t < 0$, to the left



Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2

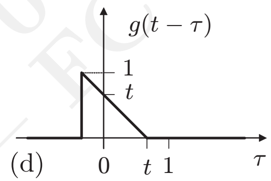
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

Suppose that we want to calculate the function

$$g(t - \tau) = \begin{cases} t - \tau, & \tau \in [t-1, t] \\ 0, & \text{elsewhere} \end{cases}$$

We flip $g(\tau - t)$ around $\tau = t$ (vertically)



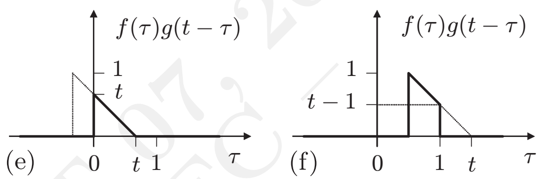
Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

We can now calculate the product function

$$\rightsquigarrow \quad f(\tau)g(t - \tau)$$



$$h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)\mathrm{d}\tau$$

- For $t \in [0, 1]$, area $0.5t^2$
- For $t \in [1, 2]$, area $0.5 - 0.5(t-1)^2$
- Zero elsewhere

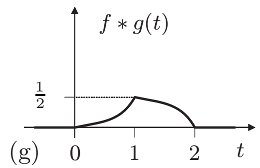
Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

We thus have,

$$f \star g(t) = \begin{cases} 0.5t^2, & t \in [0, 1] \\ 0.5 - 0.5(t-1)^2, & t \in [1, 2] \\ 0, & \text{elsewhere} \end{cases}$$



Convolution integrals (cont.)

Theorem

The convolution operator is commutative

$$f \star g(t) = g \star f(t)$$

Proof

Let $\rho = t - \tau$, then write

$$f \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) \mathrm{d}\tau = \int_{-\infty}^{+\infty} f(t-\rho) g(\rho) \mathrm{d}\rho$$
$$= g \star f(t)$$

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Signals and distributions
UFC/DC CK0255|TIP8244 2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

Definition

Convolution and differentiation/integration

Consider the two functions

$$f, g : \mathcal{R} \rightarrow \mathcal{C}$$

Their derivatives

$$\dot{f}(t) = \frac{d}{dt} f(t)$$
$$\dot{g}(t) = \frac{d}{dt} g(t)$$

Their integrals

$$\mathcal{F}(t) = \int_{-\infty}^{t} f(\tau) d\tau$$
$$\mathcal{G}(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

Signals and distributions
UFC/DC CK0255|TIP8244 2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

The following statements are true

(1) The derivative of the convolution between two functions is given by the convolution of one function with the derivative of the other function

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$$

(2) The integral of the convolution between two functions is given by the convolution of one function with the integral of the other function

$$\rightsquigarrow \quad \int_{-\infty}^{t} f \star g(\tau) \mathrm{d}\tau = f \star \mathcal{G}(t) = \mathcal{F} \star g(t)$$

(3) The integral of a convolution between two function does not change if one of the two operands is derived and the other one is integrated

$$\rightsquigarrow \quad f \star g(t) = \mathcal{F} \star \dot{g}(t) = \dot{f} \star \mathcal{G}(t)$$

Signals and distributions
UFC/DC CK0255|TIP8244 2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

Proof

To demonstrate (1), observe that we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) \mathrm{d}\tau = \int_{-\infty}^{+\infty} f(\tau) \frac{\mathrm{d}}{\mathrm{d}t} g(t-\tau) \mathrm{d}\tau$$
$$= \int_{-\infty}^{+\infty} f(\tau) \dot{g}(t-\tau) \mathrm{d}\tau = f \star \dot{g}(t)$$

Because of the commutative property $f \star g(t) = g \star f(t)$, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t} g \star f(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t} f(t-\tau) g(\tau) \mathrm{d}\tau$$
$$= \int_{-\infty}^{+\infty} \dot{f}(t-\tau) g(\tau) \mathrm{d}\tau = g \star \dot{f}(t) = \dot{f} \star g(t)$$

Convolution integrals (cont.)

To demonstrate (2), where the three functions are identical, we use (1)

Observe that all three functions when evaluated for $t = -\infty$ are null

- Whereas their derivatives are equal, for all values of t

This is because of the definition of integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{0} f * g(\tau)d\tau = f * g(t)$$

And, because

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star \mathcal{G}(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\Big](t) = f \star g(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \star g(t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\Big] \star g(t) = f \star g(t)$$

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution integrals (cont.)

To demonstrate (3), we use (1) again

$\mathcal{F} \star \dot{g}(t)$ is obtained from (1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \star g(t) = \mathcal{F} \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}g\Big](t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\Big] \star g(t)$$

$$\rightsquigarrow \quad \mathcal{F} \star \dot{g}(t) = f \star g(t)$$

$\dot{f} * \mathcal{G}(t)$ is obtained by differentiating $f \star \mathcal{G}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star \mathcal{G}(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\Big](t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t}f\Big] \star \mathcal{G}(t)$$

$$\rightsquigarrow \quad f \star g(t) = \dot{f} \star \mathcal{G}(t)$$

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Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution with canonical signals

Signals and distributions

Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution with canonical signals

Theorem

Convolution with the impulse

Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$ continuous in t

We have,

$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$

For any interval (t_a, t_b) containing t, we have

$$f(t) = \int_{t_a}^{t_b} f(\tau)\delta(t-\tau)d\tau$$

Convolution with canonical signals (cont.)

Proof

Observe that $\delta(t-\tau) = \delta(\tau - t)$ is an impulse centred in $\tau = t$

Thus,

$$\int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} \underbrace{f(t)\delta(t-\tau)}_{f(t)\delta(t-T)=f(T)\delta(t-T)} \mathrm{d}\tau$$

$$= f(t) \underbrace{\int_{-\infty}^{+\infty} \delta(t-\tau)\mathrm{d}\tau}_{\int_{-\infty}^{+\infty}\delta(t)\mathrm{d}t=\int_{0^-}^{0^+}\delta(t)dt=1} = f(t)$$

The second part is derived from the first one, as $\delta(t-\tau) = 0$ for $\tau \neq t$

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Signals and distributions
UFC/DC
CK0255|TIP8244
2018.2
Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Convolution with canonical signals (cont.)

Theorem

Consider a continuous function $f : \mathcal{R} \to \mathcal{R}$ with k continuous derivatives

We have,

$$\frac{d^k}{dt^k} f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta_k(t-\tau)d\tau$$

■

Proof

Observe that $f(t) = f \star \delta(t)$

By repeatedly differentiating and using that $\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\delta\Big](t) = f \star \delta_1(t) \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_1(t) = f \star \delta_2(t) \\
\cdots &= \cdots \\
\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_{k-1}(t) = f \star \delta_k(t)
\end{aligned}$$

■