Aalto University

Process system analysis and control

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Process system analysis and control

Process system analysis and control

There is a wide spectrum of topics around **process system analysis and control**

A (process) system

A **(process) system** can be defined as a set of elements (or components) that cooperate in order to perform a specific functionality which would be otherwise impossible to attain for the individual components alone

This definition is very fine, but it does not highlight one important element

- There is no notion of the **dynamical behaviour** of the system

A central paradigm will be that systems are subjected to external stimuli

⇝ Stimuli influence the temporal evolution of the system itself

A (process) system, **reloaded**

A (process) system is a physical entity, typically consisting of different interacting elements (or components), that responds to external stimuli according to some determined, or specific, dynamical behaviour

Process system analysis and control (cont.)

- System **modelling** and identification
- System **analysis** and **control**
- System **optimisation**
- System verification
- System diagnosis

Process systems theory

We study how to analyse and control, mathematically, a variety of process systems

Our scope is to understand their dynamical behaviour

- We want to operate them appropriately
- We want to design control devices

A methodological approach, both formal and system (process) independent

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What sort of systems and what sort of elements/components?

- Examples from chemical process engineering
- Modern examples, as natural extensions

System modelling and identification

Process system analysis and control

Modelling

To study a(ny) system, the availability of a **mathematical model** is a crucial point

⇝ Models provide a quantitative description of the behaviour of the system

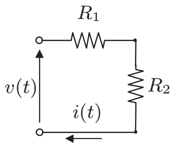
The model is often constructed on the knowledge of the component devices

- Some knowledge of the laws the system obeys to must be available

Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow $i(t)$[A] through system depends on tension $v(t)$[V]
- This dependence is assumed to be valid for any point t in time



- $R_1 = 1[\Omega]$
- $R_2 = 3[\Omega]$

Both resistors will follow Ohm's law

⇝ $v(t) = (R_1 + R_2)i(t) = 4i(t)$

⇝ The dependence is linear

⇝ (Assumptions!)

The potential difference ('voltage') across an ideal conductor is proportional to the current that flows through it, the proportionality constant is known as 'resistance'

Identification

Oftentimes, we only have an incomplete knowledge about the system's components

- The model must be constructed from observations
- By using observations of the system behaviour

Case A) We have a knowledge on the type/number of component devices

- Not all of their parameters are known
- System observations are available

⇝ **White-box identification**

Case B) We have no knowledge on the components and their parameters

- Observations of the system are available

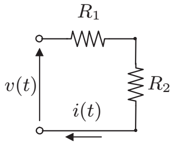
⇝ **Black-box identification**

Identification (cont.)

Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow $i(t)$[A] through system depends on tension $v(t)$[V]



- $R_1 = \textbf{?}$
- $R_2 = \textbf{?}$

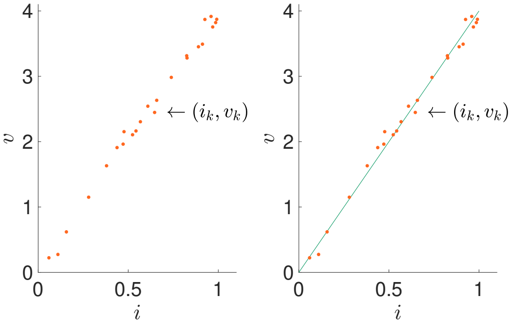
$\rightsquigarrow\ v(t) = (R_1 + R_2)i(t) = Ri(t)$

Both resistors can still be assumed to follow Ohm's laws

- $R = R_1 + R_2$ is now an unknown model parameter
- R can/should/must be identified from data

Identification (cont.)

We can observe the system by collecting K pairs of measurements $\{(v_k, i_k)\}_{k=1}^{K}$



$$v_k = R i_k + \varepsilon_k$$

Often (always), such points will not be perfectly aligned along a line of slope R

- **Disturbances** alter the behaviour to the system
- **Measurement errors** are always present

We choose R corresponding to the line that *best* approximates the measurements

System analysis, control and optimisation

Process system analysis and control

Analysis

Example

The marine ecosystem is described through the time evolution of its fauna and flora

- Birth-growth-dead processes

They *recently* spoke about reducing CO_2 emissions by injecting it into the sea

- CO_2 dissolves in sea water

The behaviour of the ecosystem is influenced by a large number of factors

- Climate, food availability, human predators, pollutants, ...

The lack of a valid model limits our understanding of the system

- We do not know the response of the ecosystem

Systems analysis is understanding the system and forecasting its future behaviour

⇝ Autonomously and in response to the external stimuli it is subjected to

The availability of a mathematical model of the system is fundamental

- It is needed to approach the problem in a quantitative manner

Control

The objective of **control** is about imposing a desired behaviour to a system

We need to explicitly formulate what we mean by 'desired behaviour'

- ⇝ The **specifications** that such behaviour must satisfy

We need to design a device for implementing this task, a **controller**

- ⇝ The scope of a controller is to stimulate the system
- ⇝ Drive its evolution toward the desired behaviour

Control (cont.)

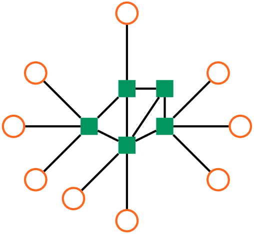
Example

Consider a conventional network for the distribution of drinking water in urban setups

- Water pressure must be kept constant throughout the network

We can measure the pressure at various network locations

- Locations have nominal (target) pressure values



Specs suggest that instantaneous pressure variations should be kept at $\pm 10\%$ of nominal value

Two stimuli act on the system (and affect it)

- ⇝ The flow-rate of water that is withdrawn
- ⇝ The pressure imposed by the pumps

We cannot control water withdrawals, they are understood as disturbances

Pump pressures can/must be **manipulated** to meet the specifications

- The adjustment of the pumps is performed by a controller

Optimisation

We want to achieve a certain system's behaviour, while optimising a performance index

- **Optimisation** can be understood as a special case of control

We impose a desired behaviour to the system, while optimising a **performance index**

- The index measures the quality of the behaviour of the system
- (In economic, environmental and/or operational terms)

Optimisation (cont.)

Example

Consider the suspension system of a vehicle for human locomotion, a conventional car

These systems are designed to satisfy two different needs

- An appropriate level of passengers' comfort
- Good handling in all types of conditions

Modern cars have suspensions based on 'semi-active' technology (fancy springs)

- A controller (dynamically, in real-time) changes the damping factor
- These actions guarantees (a compromise between) the two needs

The optimiser/controller accounts for cabin and wheel oscillations

System validation and diagnosis

Process system analysis and control

Validation

Example

Consider an elevator, the system is controlled to guarantee correct responses to requests

Formal verification can be used to guarantee the correct functioning

- The controller is a so called abstract machine
- Programmable logic controller (PLC)

Suppose that a model of the system under study is available (someone derived it)

- Suppose that a set of desired properties can be formally expressed
- **Validation** checks whether a model satisfies such properties

Fault diagnosis

Example

We understand the human body as a complex system, it is subjected to many faults

- We conventionally call these faults diseases

Consider the presence of fever, or another anomalous condition

- Symptoms reveal the presence of a disease

A doctor, once identified the pathology, prescribes a therapy

Systems deviate from nominal behaviour because of occurrence of faults

- ⇝ We need to detect the presence of an anomaly
- ⇝ We need to identify the typology of fault
- ⇝ We need to devise a corrective action

Fault diagnosis

Classification of systems/models

Classification

The diversity of systems leads to a number of methodological (modelling) approaches

- Each approach pertains a particular class of models

Conventional methodological approaches and dynamical model/system classification

Models, by general typology

⇝ **Time-evolving systems**

- Discrete-event systems
- Hybrid systems

Models, by representation

⇝ **State-space models**

- Input-output models

Time-evolving systems

Time-evolving systems

The system/model behaviour is described with functions

- The independent variable is time (t or k)
- The dependent variable varies (duh!)

Functions of time are also called signals

Continuous time-evolving systems

⇝ The time variable varies continuously

Discrete time-evolving systems

⇝ The time variable takes discrete values

A particular case of (continuous or discrete) time-evolving systems

⇝ The signal that can only take values in a discrete set

⇝ **Digital time-evolving systems**

Time-evolving systems (cont.)

The evolution of any dynamical models is completely based on the passage of time

Signals associated to model behaviour satisfy **differential**/**difference equations**

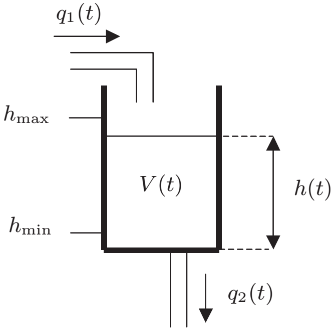
- These equations specify a relation betweens functions and their derivatives

Example

Continuous time-evolving systems

Consider a surge tank in which the volume of liquid $V(t)$ [m^3] varies over time

- This variation is only due to input and output flows, $q_1(t)$ and $q_2(t)$
- (Inflow and outflow with externally operated pumps)



(The tank cannot be emptied/overflooded)

⇝ Output flow-rate $q_2(t) \geq 0$ [$m^3 s^{-1}$]
⇝ Input flow-rate $q_1(t) \geq 0$ [$m^3 s^{-1}$]

$$\rightsquigarrow \quad \frac{\mathrm{d}V(t)}{\mathrm{d}t} = q_1(t) - q_2(t)$$

We are interested in the evolution of V

- Function $V(t)$

The **differential equation** relates continuous-time functions $V(t)$, $q_1(t)$, and $q_2(t)$

Time-evolving systems(cont.)

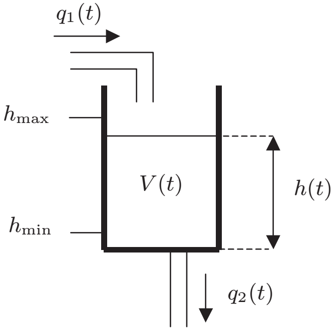
Example

Discrete time-evolving systems

Consider a surge tank in which the volume of liquid $V(t)$ [m^3] varies over time

- Suppose that measurements are not continuously available
- Sensor acquisitions only at Δt-apart units of time

We are still interested in the evolution of V, at times $\{0, \Delta t, 2\Delta t, \cdots, k\Delta t, \cdots\}$



We can consider discrete-time functions

For $k = 0, 1, 2, \ldots,$ we define

- $\rightsquigarrow$ $V(k) = V(k\Delta t)$
- $\rightsquigarrow$ $q_1(k) = q_1(k\Delta t)$
- $\rightsquigarrow$ $q_2(k) = q_2(k\Delta t)$

Time-evolving systems(cont.)

We can approximate the derivative in the balance equaton with the difference quotient

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \approx \frac{\Delta V}{\Delta t} = \frac{V(k+1) - V(k)}{\Delta t} = q_1(k) - q_2(k)$$

Multiply both sides by Δt

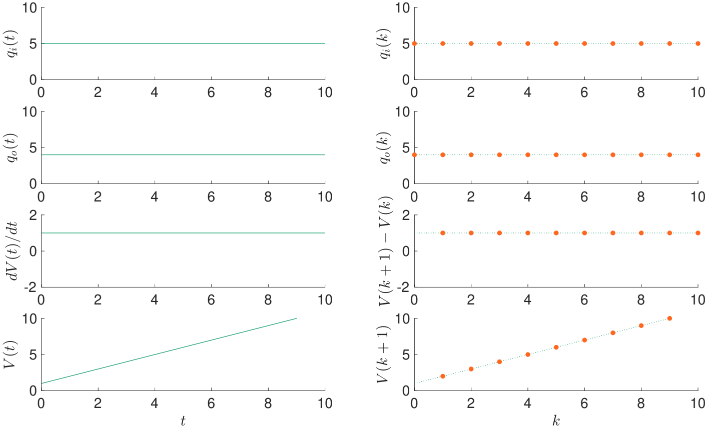
$$V(k+1) - V(k) = \big[q_1(k) - q_2(k)\big]\Delta t$$

Or, equivalently

$$\rightsquigarrow \quad V(k+1) = V(k) + [q_1(k) - q_2(k)]\Delta t$$

The **difference equation** relates discrete-time functions $V(k)$, $q_1(k)$, and $q_2(k)$

Time-evolving systems(cont.)



Discrete-event systems

Discrete-event systems

These are systems whose *state* variables take logical or symbolic values (not numeric)

Their dynamic behaviour is characterised by the occurrence of instantaneous events

- Events occur at irregular (perhaps unknown beforehand) times
- The occurrence of events triggers the evolution in time

The behaviour of such systems is represented (modelled) in terms of **states** and **events**

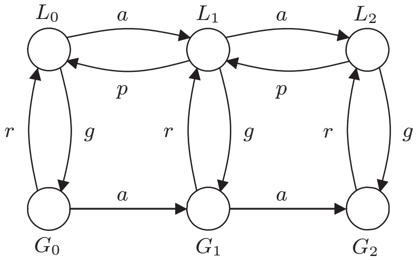
Discrete-event systems (cont.)

Example

Discrete-event systems

Consider a depot where mechanical parts are awaiting to be processed by a machine

- The number of parts awaiting to be processed cannot be larger than 2
- The machine can be either healthy (working) or faulty (stopped)



The complete state of the system

$$(\{0, 1, 2\} \times \{\mathrm{H}, \mathrm{F}\})$$

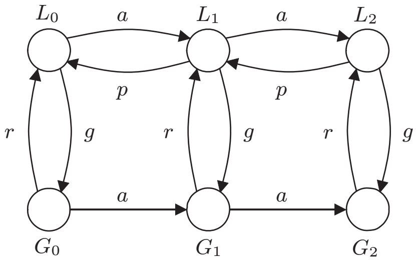
- Number of awaiting parts

$$\{0, 1, 2\}$$

- Status of the machine

$$\{\mathrm{H}, \mathrm{F}\}$$

Discrete-event systems (cont.)



Six possible states (nodes)

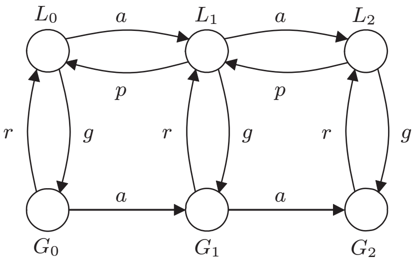
- L_0, L_1 and L_2
- G_0, G_1 and G_2

- L_0, the machine is working and the depot is empty
- L_1, the machine is working and there is one part in the depot
- L_2, the machine is working and there are two parts in the depot

- G_0, the machine is not working and the depot is empty
- G_1, the machine is not working and there is one part in the depot
- G_2, the machine is not working and there are two parts in the depot

Discrete-event systems (cont.)

The events the system can be subjected to are all possible causes of changes in state

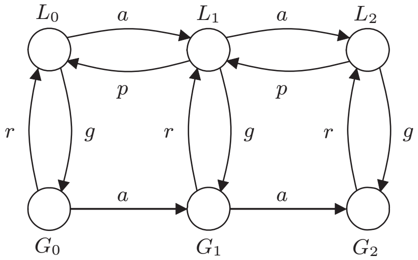


Four possible events (transitions)

- a and p
- g and r

- a, a new part arrives to the depot
- p, the machines takes one part from the depot
- g, the machine gets faulty
- r, the machine gets fixed

Discrete-event systems (cont.)



Event a (new part arrives) can only occur when the depot does not have two parts

$$a \rightsquigarrow \begin{cases} L_i \to L_{i+1} \\ G_i \to G_{i+1} \end{cases}$$

Event p (machine takes one part) can only occur when the deport is not empty

$$p \rightsquigarrow \begin{cases} L_i \to L_{i-1} \end{cases}$$

Event g and r determine the switches $L_i \to G_i$ and $G_i \to L_i$, respectively

Hybrid systems

Hybrid models can combine time-evolving dynamics and discrete-event dynamics

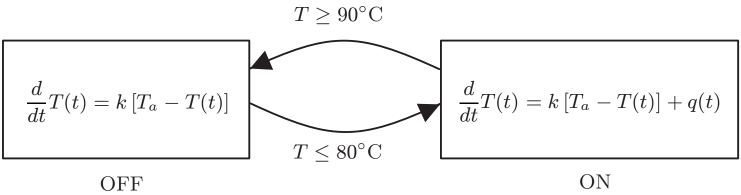
- ⇝ They are the most general class of dynamical systems

Example

Hybrid systems

Consider a modern but mild sauna, a cabin where the temperature is regulated

- A thermostat controls a stove used as heat generator
- Keep the temperature between 80°C and 90°C



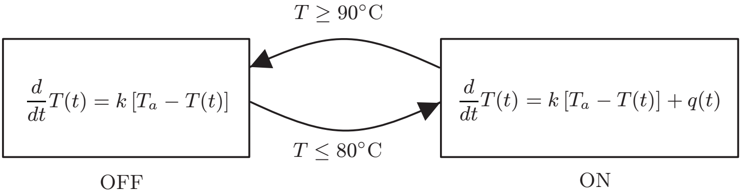
The thermostat can be represented using a discrete-event model

- Switch $\{\text{ON}, \text{OFF}\}$

The cabin can be represented using a time-evolving model

- Temperature $T(t)$

Hybrid systems(cont.)



Suppose that the state of the thermostat is OFF, $T(t)$ in the cabin decreases

- Heat is exchanged with the outside $[T_a < T(t)]$

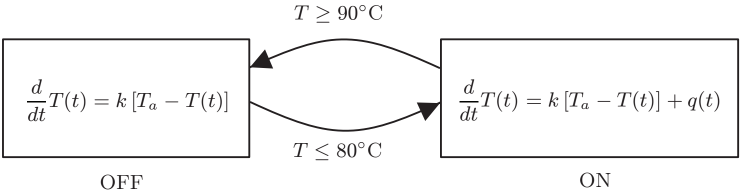
$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}T(t) = k\big[T_a - T(t)\big], \quad \text{with } k > 0$$

Suppose that the state of the thermostat is ON, $T(t)$ in the cabin increases

- Heat is exchanged with the outside $[T_a < T(t)]$
- Heat is generated by the stove $q(t)$

$$\rightsquigarrow \quad \frac{d}{dt}T(t) = k\big[T_a - T(t)\big] + q(t)$$

Hybrid systems(cont.)



The state of the system is $x = (l, T)$

- A logical variable $l \in \{\text{ON}, \text{OFF}\}$, representing the discrete state
- A real function $T(t) \in \mathcal{R}^+$, representing the continuous state

Controlled process systems

Controlled process systems

A process is a set of units (reactors, distillation columns, pumps, compressors, ...)

- These units constitute the main plant elements
- (Auxiliary and complementary elements)

Objectives of the plant/process

- ⇝ Receive raw materials, and use sources of energy to produce products
- ⇝ In the most economic and, sustainable, environmentally aware way

Plant/process requirements

- Safety (people and the environment)
- Operation constraints (mass, energy capacities)
- Production specification (desired product quality and quantity)

Controlled process systems (cont.)

The satisfaction of the objectives and requirements requires external intervention

- ⇝ Generally, the **process automation system**
- ⇝ Specifically, the **process control system**

The process control system is designed to fulfil some basic and yet critical tasks

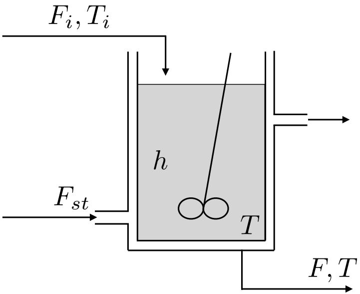
- Reduce the influence of **external disturbances** on the process
- Ensure the **stability** and **performances** of the process

Controlled process systems (cont.)

Example

Heating tank

Consider a perfectly mixed tank in which some liquid is heated using steam circulation



- Input liquid flowrate, $F_i(t)$
- Input liquid temperature, $T_i(t)$
- Output liquid flowrate, $F(t)$
- Output liquid temperature, $T(t)$
- Liquid level in the tank, $h(t)$
- Steam flowrate, $F_{st}(t)$

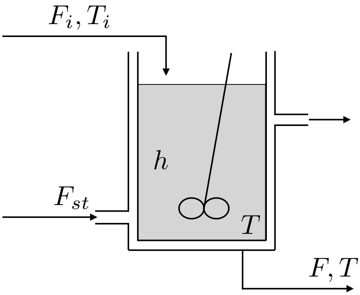
The objective of the process is to maintain the liquid temperature at desired value, T_d

- Another objective is to maintain the liquid level at some desired value h_d

Controlled process systems (cont.)

To operate such a system, first we need to go through a predefined startup procedure

- The startup procedure brings the system to some steady-state (SS) conditions
- In steady-state, all the variables remain constant, stationary, over time (t)



In steady-state conditions, we have

⇝ $T(t) = \text{constant}$

⇝ $h(t) = \text{constant}$

⇝ $\cdots$

Suppose that there are no changes in inflow and steam (F_i, T_i and F_{st} are constant)

⇝ Then, the system will remain in steady-state conditions

⇝ The temperature T will stay stationary

⇝ (The level h, and thus also F will)

Controlled process systems (cont.)

In this ideal (unrealistic) situation, this means that there is no need of a control system

- Given that the steady-state corresponds to the desired value(s) of T (and h)

This scenario is implausible as the inflow and steam variables will necessarily change

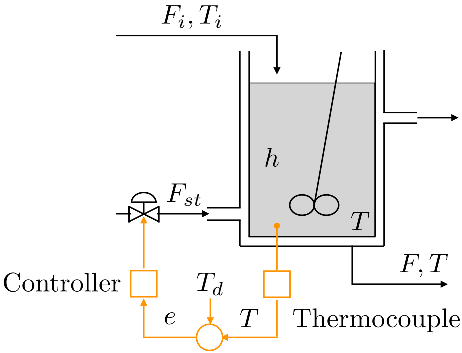
- We do not have any control on the inflow flow-rate and temperature
- The value of these variables depends on upstream processes
- Even in the most optimistic cases, they sightly change

As a consequence, the system variables may drift away from these desired values

- We need to intervene on the system to bring it back
- A controller is the device designed for this task

Controlled process systems (cont.)

Consider the problem of controlling the temperature T of the liquid in the tank, at T_d



1. Read the temperature of the liquid in the tank

$$T(t)$$

2. Compare this value with some desired value T_d

⇝ (Compute a difference)

$$e(t) = T_d - T(t)$$

3. The error is used to compute the control action
4. Control action is implemented in the steam valve

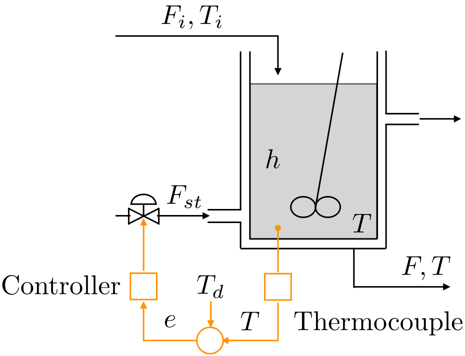
Suppose that the error is positive, $e(t) = T_d - T(t) > 0$, the controller opens the valve

- We need to steer the system's temperature $T(t)$ towards T_s
- The controller will increase the steam flow-rate $F_{st}(t)$

Controlled process systems (cont.)

Consider a system at steady-state and suppose that an increase of inflow occurs ($F_i \uparrow$)

- Other variables being constant, the temperature of the liquid decreases ($T \downarrow$)
- Comparison with the desired value gives a positive error ($e = T_d - T > 0$)



- The control action is to request for more steam by increasing its flow-rate
- This is again practically implemented by opening the steam valve ($F_{st} \uparrow$)

Controlled process systems (cont.)

Stability

Consider the time evolution of a (set of) variable(s) of system originally at steady-state

- At some point in time, the system is perturbed (some change occurs)

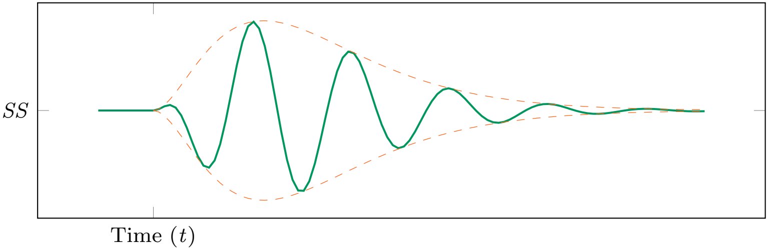
⇝ The system will respond to the perturbation (move away from SS)

⇝ (Its variables will start varying, changing their value)

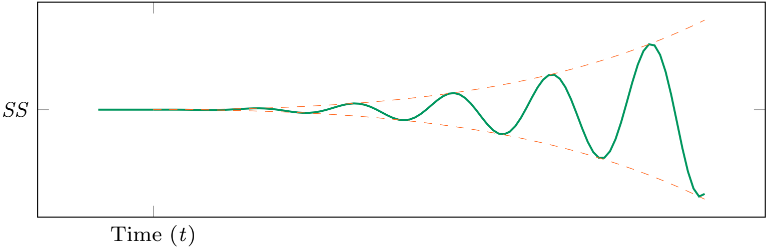
A system is stable if its variable(s) return autonomously to their steady-state value(s)

- A stable process is also said to be a self-regulating process
- A stable process would not need a controller, in general
- (If the steady-state condition is the desired state)
- (And, if we have an infinite amount of time)

Stable



Unstable



Controlled process systems (cont.)

Performance

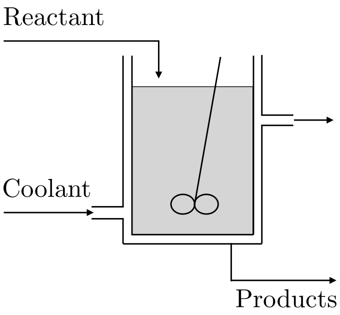
Consider a process for which operational safety and production specifications are met

- The next important objective to be satisfied is (profit) optimisation

Example

Continuously stirred-tank reactor

Consider a jacketed continuous stirred tank reactor, reaction sequence $A \to B \to C$



The reaction develops heat, exothermic

⇝ To be removed with some coolant

- Reactant A enters the process
- Products leave the process
- B is the desired product
- C is undesired

Interest to maximise profit over time

$$\varphi = \int_0^t f[\text{profit }(B), \text{cost}(A + \text{coolant})]\mathrm{d}t$$

Controlled process systems (cont.)

Classification of (process) variables

We considered two types of process variables, **input variables** and **output variables**

- Inputs are understood as entering (as in 'stimulating') the system
- Outputs are understood as exiting the system (as in 'responses')

Process variables
- **Inputs (I)**
  - **Disturbances (LV)**
    - **Measurable**
    - **Unmeasurable**
  - **Controls (MV)**
- **Outputs (O)**
  - **Measurable**
  - **Unmeasurable**

The **controlled variables (CV)** are the third type of variables involved in control

- They are those variables that we would want to maintain at a desired value
- They often, but not necessarily, correspond to the measured outputs

Controlled process systems (cont.)

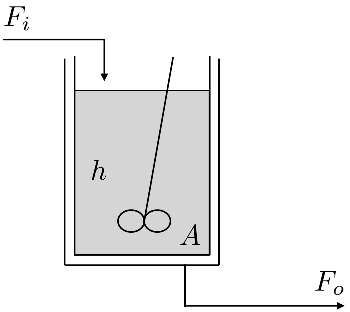
Example

Liquid tank

Consider a cylindrical tank used to store some desired volume of liquid (that is, $h = h_d$)

- Liquid enters with flow-rate F_i and the outflow has flow-rate, F_o
- The cross-sectional area A of the tank is constant

The liquid level h is the controlled variable (CV), what are the I and O variables?



A single input variable (I)

- F_i, often measurable

A single output variable (O)

- h, measurable

F_o is also often measurable

- It can also be an input
- It can be an output

Controlled process systems (cont.)

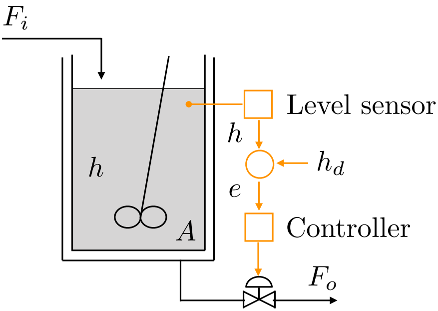
We measure the CV using a level sensor, then we compare its value with a target value

- ⇝ This generates an error $e = h_d - h$ which is passed to the controller

Case 1

One possible control variable (MV) is the outflow flow-rate

- ⇝ The control action is implement in its control valve



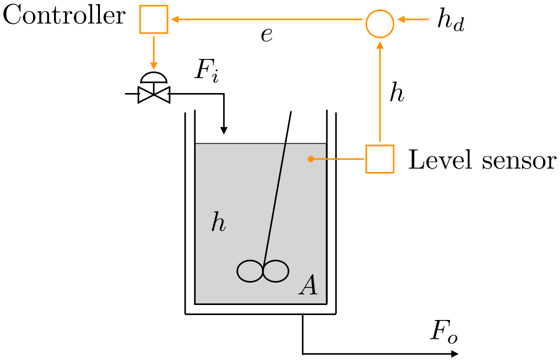
- ⇝ $h(t)$, controlled variable (CV)
- ⇝ $F_o(t)$, control variable (MV)
- ⇝ $F_i(t)$, disturbance (LV)

Controlled process systems (cont.)

Case 2

One alternative control variable (MV) is the inflow flow-rate

- The control action is implement in its control valve



⇝ h, controlled variable

⇝ F_i, control variable

⇝ F_o, disturbance?

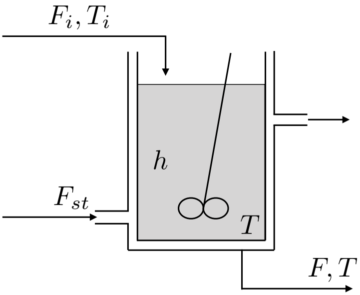
- (F_o, now input)

Controlled process systems (cont.)

Example

Heating tank

Consider a perfectly mixed tank in which some liquid is heated using steam circulation

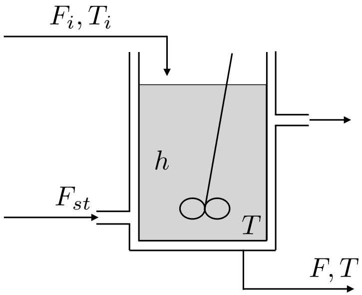


- Input liquid flowrate, $F_i(t)$
- Input liquid temperature, $T_i(t)$
- Output liquid flowrate, $F(t)$
- Output liquid temperature, $T(t)$
- Liquid level in the tank, $h(t)$
- Steam flowrate, $F_{st}(t)$

The objective of the process is to keep the liquid temperature T at desired value, T_d

- Another objective is to maintain the liquid level h at some desired value h_d

Controlled process systems (cont.)



The objective of the process is to keep the liquid temperature T at target value, T_d

- Adjust the steam flow-rate (F_{st}, MV), F_i and T_i are disturbances (LV)
- Are there alternative control structures usable for the task?

Another important objective is to maintain the liquid level h at a desired value h_d

- Adjust the outflow flow-rare (F, MV), F_i is a disturbance (LV)
- Adjust the inflow flow-rate (F_i, MV), F may be a disturbance (LV)

Representation of systems/models

Model representations

We provided fundamental concepts for the analysis of time-evolving systems/models

- Evolution from the passing of time, focus on continuous-time models

A fundamental step to use formal techniques to study time-evolving systems/models

$\rightsquigarrow$ We describe the system/model behaviour in terms of functions

---

For given input functions, we are interested in studying how the system evolves in time

- This can be done by analysing the system's representation

Model representations (cont.)

We introduce the two main forms that are used for describing such systems/models

- **Input-output** (**IO**) representation

⇝ **State-space** (**SS**) representation

The mathematic formulations and examples specific for continuous-space systems

- Yet another classification based on properties of the representation

Input-output representation

Model representations

Input-output representation

Consider the quantities involved in the input-output (IO) representation of a system

Causes

⇝ Quantities that are generated outside the system

- Their evolution influences the system behaviour
- Not influenced by the system behaviour

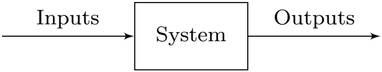
Effects

⇝ Quantities whose behaviour is influenced by the causes

- Their evolution depends on the nature of the system

By convention,

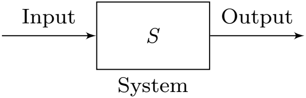
$$\begin{cases} \textbf{Causes} & \rightsquigarrow & \textbf{Inputs} \\ \textbf{Effects} & \rightsquigarrow & \textbf{Outputs} \end{cases}$$



Input-output representation (cont.)

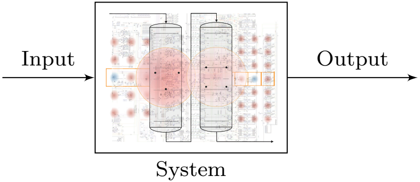
A (process) system

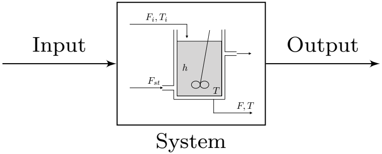
The system/model S can be seen as an operator or a processing/computing unit

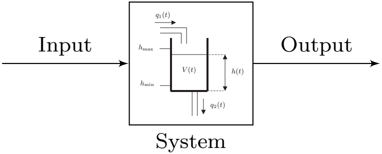


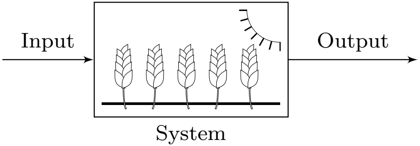
- The system assigns a specific evolution to the output variables (effects)
- One for each possible evolution of the input variables (causes)

Input-output representation (cont.)





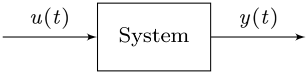




Input-output representation (cont.)

A system/model can have more than one (N_u) input and more than one (N_y) output

- Both inputs and outputs will assumed to be measurable



⇝ N_u **inputs** $u(t)$, in $\mathcal{R}^{N_u}$

$$u(t) = \big[u_1(t) \cdots u_{N_u}(t)\big]'$$

⇝ N_y **outputs** $y(t)$, in $\mathcal{R}^{N_y}$

$$y(t) = \big[y_1(t) \cdots y_{N_y}(t)\big]'$$

Manipulable inputs

- They can be used for control

Non-manipulable inputs

- The disturbances

Input-output representation (cont.)

Example

A car (IO representation)

Let the position and speed of a car be the output variables, $y(t) \in \mathcal{R}^{N_y=2}$

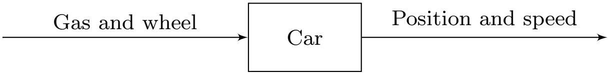
- They are both measurable

As input variables, we can consider wheel and gas position, $u(t) \in \mathcal{R}^{N_u=2}$

- They are both measurable
- They are both manipulable

By acting on the input variables, we influence the behaviour of the output

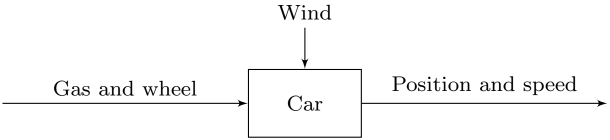
- How the outputs change depend on the specific system (car)
- (More precisely, on the system's dynamics)



Input-output representation (cont.)

Wind speed could be considered as an additional input variable to the system car

- It may be measurable, but it is hardly manipulable
- We treat it as non-manipulable input, disturbance



In summary, we have $N_u = 2 + 1 = 3$ inputs and $N_y = 2$ outputs

$\rightsquigarrow$ A Multiple-Input-Multiple-Output (MIMO) system

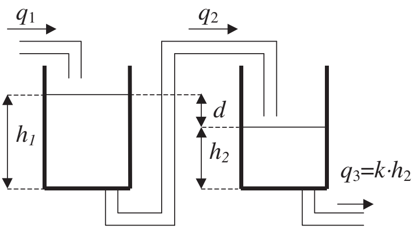
Input-output representation (cont.)

Example

Two tanks (IO representation)

Consider a system consisting of two cylindric liquid tanks, same cross section B [m^2]

- A main inflow to tank 1, a main outflow from tank 2
- The outflow from tank 1 is the inflow to tank 2



First liquid tank

- Inflow, rate q_1 [$\mathrm{m}^3\mathrm{s}^{-1}$]
- Outflow, rate q_2 [$\mathrm{m}^3\mathrm{s}^{-1}$]
- h_1 is the liquid level [m]

Second liquid tank

- Inflow, rate q_2 [$\mathrm{m}^3\mathrm{s}^{-1}$]
- Outflow, rate q_3 [$\mathrm{m}^3\mathrm{s}^{-1}$]
- h_2 is the liquid level [m]

We can characterise the involved system variables using standard process notation

Input-output representation (cont.)

Suppose that flow-rates $q_1(t)$ and $q_2(t)$ can be set to desired values (pumps), at each t

Also, suppose that $q_3(t)$ depends linearly on the liquid level in the tank, $h_2(t)$

- $q_3(t) = k \cdot h_2(t)$ $[\mathrm{m}^3\mathrm{s}^{-1}]$, with k $[\mathrm{m}^2\mathrm{s}^{-1}]$ some appropriate constant

Inputs, q_1 and q_2

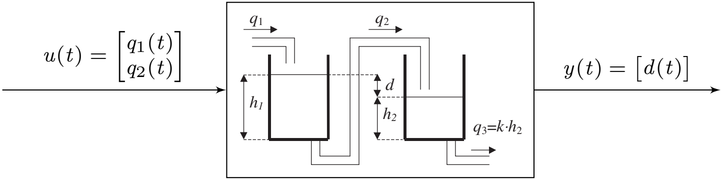
⇝ Measurable and manipulable

⇝ They influence the liquid levels in the tanks

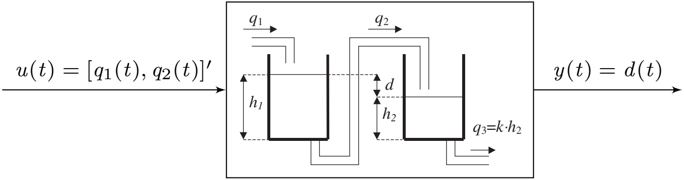
Output, $d = h_1 - h_2$

⇝ Measurable but it cannot be manipulated

⇝ But, it is influenced by the inputs



Input-output representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \dfrac{\mathrm{d}\, V_1(t)}{\mathrm{d}t} = q_1(t) - q_2(t) \\ \dfrac{\mathrm{d}\, V_2(t)}{\mathrm{d}t} = q_2(t) - q_3(t) = q_2(t) - kh_2(t) \end{cases}$$

We can set $h_1 = V_1/B$, $h_2 = V_2/B$, and $q_3 = kh_2$

$$\rightsquigarrow \quad \begin{cases} \dot{h}_1(t) = \dfrac{1}{B}q_1(t) - \dfrac{1}{B}q_2(t) \\ \dot{h}_2(t) = \dfrac{1}{B}q_2(t) - \dfrac{1}{B}q_3(t) = \dfrac{1}{B}q_2(t) - \dfrac{k}{B}h_2(t) \end{cases}$$

Input-output representation (cont.)

$$\rightsquigarrow \quad \begin{cases} \dot{h}_1(t) = \dfrac{1}{B}q_1(t) - \dfrac{1}{B}q_2(t) \\ \dot{h}_2(t) = \dfrac{1}{B}q_2(t) - \dfrac{1}{B}q_3(t) = \dfrac{1}{B}q_2(t) - \dfrac{k}{B}h_2(t) \end{cases}$$

By taking the first derivative of $y(t) = h_1(t) - h_2(t)$, we have

$$\begin{aligned} \dot{y}(t) &= \dot{h}_1(t) - \dot{h}_2(t) \\ &= \underbrace{\left[\frac{1}{B}q_1(t) - \frac{1}{B}q_2(t)\right]}_{\dot{h}_1(t)} - \underbrace{\left[\frac{1}{B}q_2(t) - \frac{k}{B}h_2(t)\right]}_{\dot{h}_2(t)} \\ &= \frac{1}{B}q_1(t) - 2\frac{1}{B}q_2(t) + \frac{k}{B}h_2(t) \\ &= \frac{1}{B}u_1(t) - 2\frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)] \end{aligned}$$

Input-output representation (cont.)

$$\dot{y}(t) = \frac{1}{B}u_1(t) - 2\frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)]$$

By taking the second derivative of $y(t)$, we have

$$\begin{aligned}\ddot{y}(t) &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B}\dot{h}_1(t) - \frac{k}{B}\dot{y}(t) \\ &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \underbrace{\frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t)}_{\frac{k}{B}\dot{h}_1(t)} - \frac{k}{B}\dot{y}(t)\end{aligned}$$

We used $\dot{h_1}(t) = (u_1(t) - u_2(t))\,/B$

Input-output representation (cont.)

$$\ddot{y}(t) = \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t) - \frac{k}{B}\dot{y}(t)$$

Rearranging terms, the IO system's representation is an ordinary differential equation

$$\rightsquigarrow \quad \underbrace{\ddot{y}(t) + \frac{k}{B}\dot{y}(t)}_{y \text{ and its derivatives}} \underbrace{-\frac{1}{B}\dot{u}_1(t) + \frac{2}{B}\dot{u}_2(t) - \frac{k}{B^2}u_1(t) + \frac{k}{B}u_2(t)}_{u \text{ and its derivatives}} = 0$$

The system model we have derived for the two-tank process is in the general IO form

Input-output representation (cont.)

The IO model of a system is a relationship between the system output $y(t) \in \mathcal{R}^{N_y}$ and its derivatives, the system input $u(t) \in \mathcal{R}^{N_u}$ and its derivatives, a differential equation

The IO model of a Single-Input Single-Output (SISO, $N_y = 1, N_u = 1$) system

$$h\Big[\underbrace{y(t), \dot{y}(t), \ldots, y^{(n)}(t)}_{\text{output}}, \underbrace{u(t), \dot{u}(t), \ldots, u^{(m)}(t)}_{\text{input}}, \underbrace{t}_{\text{time}}\Big] = 0$$

h is a multi-parametric function that depends on the system

- n is the maximum order of derivation of the output
- m is the maximum order of derivation of the input

The **order of the system** (**model**) is n

- $\dot{y}(t) = \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}$, $\ddot{y}(t) = \dfrac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}$ and $y^{(n)}(t) = \dfrac{\mathrm{d}^n y(t)}{\mathrm{d}t^n}$
- $\dot{u}(t) = \dfrac{\mathrm{d}u(t)}{\mathrm{d}t}$, $\ddot{u}(t) = \dfrac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2}$ and $u^{(m)}(t) = \dfrac{\mathrm{d}^m u(t)}{\mathrm{d}t^m}$

Input-output representation (cont.)

Linear and linear time-invariant differential equation

Consider the differential equation

$$h\Big[y(t), \dot{y}(t), \ldots, y^{(n)}(t), u(t), \dot{u}(t), \ldots, u^{(m)}(t), t\Big] = 0$$

The equation is linear if and only if function h is a linear combination of the output and its derivatives $y(t), \dot{y}(t), \ldots, y^{(n)}(t)$, and of the input and its derivatives $u(t), \dot{u}(t), \ldots, u^{(m)}(t)$

$$\alpha_0(t)y(t) + \alpha_1(t)\dot{y}(t) + \cdots + \alpha_n(t)y^{(n)}(t) + \beta_0(t)u(t) + \beta_1(t)\dot{u}(t) + \cdots + \beta_m(t)u^{(m)}(t) = 0$$

$\rightsquigarrow$ A zero-sum weighted sum of inputs, outputs, and respective derivatives

The equation is linear and time-invariant if and only if the function h is a time-independent linear combination of the output, the input and their derivatives

$$\alpha_0 y(t) + \alpha_1 \dot{y}(t) + \cdots + \alpha_n y^{(n)}(t) + \beta_0 u(t) + \beta_1 \dot{u}(t) + \ldots + \beta_m u^{(m)}(t) = 0$$

$\rightsquigarrow$ A zero-sum weighted sum of inputs, outputs, and derivatives

Input-output representation (cont.)

The IO model of a Multiple-Input Multiple-Output (MIMO, $N_y > 1, N_u > 1$) system

$$
\begin{cases}
h_1\Big[\underbrace{y_1(t), \dot{y}_1(t), \ldots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_1,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_{N_u}(t), \ldots, u_{N_u}^{(m_1,N_u)}(t)}_{\text{input } N_u}, t\Big] \\
= 0 \\
h_2\Big[\underbrace{y_2(t), \dot{y}_2(t), \ldots, y_2^{(n_1)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_1,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_{N_u}(t), \ldots, u_{N_u}^{(m_1,N_u)}(t)}_{\text{input } N_u}, t\Big] \\
= 0 \\
\vdots \\
h_{N_y}\Big[\underbrace{y_{N_y}(t), \dot{y}_{N_y}(t), \ldots, y_{N_y}^{(n_{N_y})}(t)}_{\text{output } N_y}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_{N_y},1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_{N_u}(t), \ldots, u_{N_u}^{(m_{N_y},N_u)}(t)}_{\text{input } N_u}, t\Big] \\
= 0
\end{cases}
$$

Each h_i $(i = 1, \ldots, N_y)$ is a multi-parametric function depending on the system

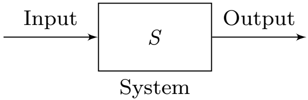
- n_i, max order of derivation of the i-th component of output $y_i(t)$
- m_i, max order of derivation of the i-th component of input $u_i(t)$

A total of N_y differential equations

State-space representation

Model representations

State-space representation



For a given behaviour of the inputs, system S defines the behaviour of the outputs

- ⇝ The system's output at time t is not only dependent on the input at time t
- ⇝ It also depends on the past of the system, through its current state

State-space representation (cont.)

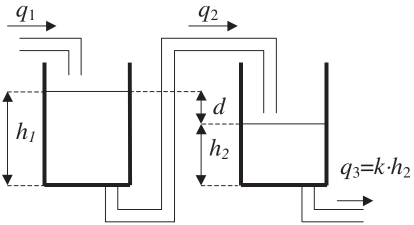
Example

Two tanks (SS representation)

Consider a system consisting of two cylindric liquid tanks, same cross-section B [m^2]

Let $d_0 = h_{1,0} - h_{2,0} > 0$ be some positive value of the output at time t_0

- (Equivalent to let $h_{1,0}$ and $h_{2,0}$ be different liquid levels at t_0)



Suppose that all input variables (q_1 and q_2) are zero at time t_0

- $q_{1,0} = 0$
- $q_{2,0} = 0$

Output $d(t)$ at any time $t > t_0$ does not depend only on input values $q_1(t)$ and $q_2(t)$

- Yet $y(t)$ will vary over the entire interval $[t_o, t]$

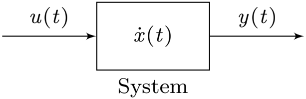
⇝ ... regardless of $u(t)$

State-space representation (cont.)

We can take this observation into account by introducing an *intermediate* variable

This system variable can be understood to *exist* between inputs and outputs

- ⇝ The **state** variable of the system
- ⇝ We use $x(t)$ to denote it



- N_u **inputs** $u(t)$, in $\mathcal{R}^{N_u}$

$$u(t) = [u_1(t) \cdots u_{N_u}(t)]'$$

- N_y **outputs** $y(t)$, in $\mathcal{R}^{N_y}$

$$y(t) = [y_1(t) \cdots y_{N_y}(t)]'$$

- N_x **states** $x(t)$, in $\mathcal{R}^{N_x}$

$$x(t) = [x_1(t) \cdots x_{N_x}(t)]'$$

The state variable condenses information about the past and present of the system

State-space representation (cont.)

Definition

State variable

The **state variable** of a system/model at time t_0 is a variable that contains the necessary information to univocally determine the behaviour of output $y(t)$ for $t \geq t_0$

1. Given the behaviour of input $u(t)$, for $t \geq t_0$
2. Given the state itself at t_0, $x(t_0)$

The **state** $x(t) = [x_1(t) \cdots x_{N_x}(t)]^T$ is a vector (a point in space) with N_x components

- ⇝ We say that N_x is the order of the system/model
- (In the state-space representation)

State-space representation (cont.)

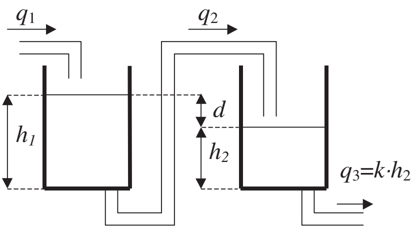
In general, it is possible to use different physical/non-physical entities as state variables

- The state variable is neither univocally defined, nor it is determined
- It is anything that can be seen as an *internal cause* of evolution
- (Again, in general)

State-space representation (cont.)

Example

Two tanks (SS representation)



First tank

- Inflow, rate q_1 [m^3s^{-1}]
- Outflow, rate q_2 [m^3s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Inflow, rate q_2 [m^3s^{-1}]
- Outflow, rate q_3 [m^3s^{-1}]
- h_2 is the liquid level [m]

Suppose that flow-rates $q_1(t)$ and $q_2(t)$ can be set to some desired value (pumps)

Also, suppose that $q_3(t0$ depends linearly on the liquid level in the tank, $h_2(t)$

- $q_3(t) = k \cdot h_2(t0$ [m^3s^{-1}], with k [m^2s^{-1}] some appropriate constant

State-space representation (cont.)

Inputs, q_1 and q_2

- Measurable and manipulable
- They influence the liquid levels in the tanks

Output, $d = h_1 - h_2$

- Measurable but it cannot be manipulated
- But, it is influenced by the inputs

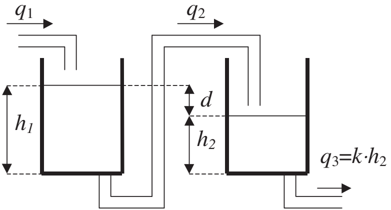
As for the state variables, we can select the liquid volume in the tanks, $V_1(t)$ and $V_2(t)$

$$u(t) = [q_1(t), q_2(t)]' \longrightarrow \boxed{\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}} \longrightarrow y(t) = d(t)$$

States, $x_1 = V_1$ and $x_2 = V_2$

- Measurable but cannot be manipulated
- They are influenced by the inputs
- $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

State-space representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} = q_1(t) - q_2(t) \\ \dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} = q_2(t) - q_3(t) = q_2(t) - k\,\underbrace{\dfrac{V_2(t)}{B}}_{h_2(t)} \end{cases}$$

By the definition of the output,

$$\begin{aligned} d(t) &= h_1(t) - h_2(t) \\ &= \frac{V_1(t)}{B} - \frac{V_2(t)}{B} \end{aligned}$$

State-space representation (cont.)

Summarising, we have

$$\begin{cases} \begin{cases} \dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} = \underbrace{q_1(t) - q_2(t)}_{f_1(V_1(t),V_2(t),q_1(t),q_2(t))} \\ \dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} = \underbrace{q_2(t) - k\dfrac{V_2(t)}{B}}_{f_2(V_1(t),V_2(t),q_1(t),q_2(t))} \end{cases} \\ d(t) = \underbrace{\dfrac{V_1(t)}{B} - \dfrac{V_2(t)}{B}}_{g(V_1(t),V_2(t),q_1(t),q_2(t))} \end{cases}$$

Diagrammatically,

$$u(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \longrightarrow \boxed{\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} f_1(\cancel{V_1(t)}, \cancel{V_2(t)}, q_1(t), q_2(t) \\ f_2(\cancel{V_1(t)}, V_2(t), \cancel{q_1(t)}, q_2(t) \end{bmatrix} \\ d(t) = g\left(V_1(t), V_2(t), \cancel{q_1(t)}, \cancel{q_2(t)}\right) \end{cases}} \longrightarrow y(t) = d(t)$$

State-space representation (cont.)

Rearranging terms, the state-space representation of the two-tank system

$$\rightsquigarrow \quad \begin{cases} \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t) \\ \dot{x}_2(t) = -k/Bx_2(t) + u_2(t) \end{cases} \\ \\ y(t) = x_1(t)/B - x_2(t)/B \end{cases}$$

The model is set of ordinary differential equations and an algebraic equation

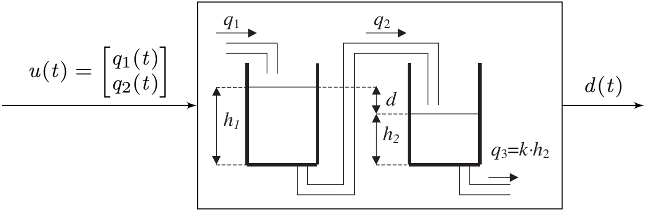
- State variables, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$
- Input (control) variables $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$
- Output (measurement) variables $= u(t) = \begin{bmatrix} y(t) \end{bmatrix} = \begin{bmatrix} d(t) \end{bmatrix}$

Quantity k and B are constant (do not change w/ time), system parameters

---

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}V_1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}V_2(t) \end{bmatrix}$$

State-space representation (cont.)



Equivalently,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \longrightarrow \left\{ \begin{array}{l} \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t) \\ \dot{x}_2(t) = -(k/B)x_2(t) + u_2(t) \end{cases} \\ \\ y(t) = x_1(t)/B - x_2(t)/B \end{array} \right. \longrightarrow y(t)$$

State-space representation (cont.)

The SS model of a system describes how the evolution (the change in time) $\dot{x}(t) \in \mathcal{R}^{N_x}$ of the system state depends on the state $x(t) \in \mathcal{R}^{N_x}$ itself and on the input $u(t) \in \mathcal{R}^{N_u}$

- The **state equation**
- A set of differential equations

$$\begin{cases} \dot{x}_1(t) = f_1\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \\ \dot{x}_2(t) = f_2\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \\ \quad \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \end{cases}$$

The SS model of a system also describes how the system output $y(t) \in \mathcal{R}^{N_y}$ depends on system state $x(t) \in \mathcal{R}^{N_x}$ and on system input $u(t) \in \mathcal{R}^{N_u}$

- The **output transformation**
- A set of algebraic equations

$$\begin{cases} y_1(t) = g_1\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \\ y_2(t) = g_2\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \\ \quad \vdots \\ y_{N_y}(t) = g_{N_y}\left[x_1(t), \dots, x_{N_x}(t), u(t), t\right] \end{cases}$$

For compactness, we used $u(t) = [u_1(t), u_2(t), \dots, u_{N_u}(t)]$

State-space representation (cont.)

The state equation is a set of N_x first-order ordinary differential equations

- Regardless of the fact that the system is SISO or MIMO

The output transformation is a scalar or vectorial algebraic equation

- Depending on the number p of output variables

State-space representation (cont.)

The SS model of a SISO ($y(t) \in \mathcal{R}^{Ny=1}$ and $u(t) \in \mathcal{R}^{Nu=1}$) system with N_x states

$$\begin{cases} \begin{cases} \dot{x}_1(t) = f_1\left[x_1(t), \ldots, x_{N_x}(t), u(t), t\right] \\ \dot{x}_2(t) = f_2\left[x_1(t), \ldots, x_{N_x}(t), u(t), t\right] \\ \quad \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}\left[x_1(t), \ldots, x_{N_x}(t), u(t), t\right] \end{cases} \\ \\ y(t) = g\left[x_1(t), \ldots, x_{N_x}(t), u(t), t\right] \end{cases}$$

Let $\dot{x}(t) \in \mathcal{R}^{N_x}$ be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} \dot{x}(t) = f\left[x(t), u(t), t\right] \\ y(t) = g\left[x(t), u(t), t\right] \end{cases}$$

f is a multi-parametric vectorial function with i-th component f_i, $i = 1, \ldots, N_x$

State-space representation (cont.)

The SS model of a MIMO ($y(t) \in \mathcal{R}^{Ny \neq 1}$ and $u(t) \in \mathcal{R}^{Nu \neq 1}$) system with N_x states

$$\begin{cases} \begin{cases} \dot{x}_1(t) = f_1\left[x_1(t), \dots, x_n(t), u_1(t), \dots, u_{N_u}(t), t\right] \\ \dot{x}_2(t) = f_2\left[x_1(t), \dots, x_n(t), u_1(t), \dots, u_{N_u}(t), t\right] \\ \quad \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}\left[x_{N_x}(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t\right] \end{cases} \\ \begin{cases} y_1(t) = g_1\left[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t\right] \\ y_2(t) = g_2\left[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t\right] \\ \quad \vdots \\ y_{N_y}(t) = g_{N_y}\left[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t\right] \end{cases} \end{cases}$$

Let $\dot{x}(t) \in \mathcal{R}^{N_x}$ be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} \dot{x}(t) = f\left[x(t), u(t), t\right] \\ y(t) = g\left[x(t), u(t), t\right] \end{cases}$$

f and g are multi-parametric vectorial functions depending on the system

- f_i with $i = 1, \dots, N_x$ and g_i with $i = 1, \dots, N_y$

State-space representation (cont.)

Linear and linear time-invariant SS representation

A necessary and sufficient condition for a system to be linear is that state equation and output transformation in the SS model are linear equations

$$
\begin{cases}
\begin{cases}
\dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,N_x}(t)x_{N_x}(t) + b_{1,1}(t)u_1(t) + \cdots + b_{1,N_u}(t)u_{N_u}(t) \\
\dot{x}_2(t) = a_{2,1}(t)x_1(t) + \cdots + a_{2,N_x}(t)x_{N_x}(t) + b_{2,1}(t)u_1(t) + \cdots + b_{2,N_u}(t)u_{N_u}(t) \\
\quad \vdots \\
\dot{x}_{N_x}(t) = \\
\qquad a_{N_x,1}(t)x_1(t) + \cdots + a_{N_x,N_x}(t)x_{N_x}(t) + b_{N_x,1}(t)u_1(t) + \cdots + b_{N_x,N_u}(t)u_{N_u}(t)
\end{cases} \\
\begin{cases}
y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,N_x}(t)x_{N_x}(t) + d_{1,1}(t)u_1(t) + \cdots + d_{1,N_u}(t)u_{N_u}(t) \\
y_2(t) = c_{2,1}(t)x_1(t) + \cdots + c_{2,N_x}(t)x_{N_x}(t) + d_{2,1}(t)u_1(t) + \cdots + d_{2,N_u}(t)u_{N_u}(t) \\
\quad \vdots \\
y_{N_y}(t) = \\
\qquad c_{N_y,1}(t)x_1(t) + \cdots + c_{N_y,N_x}(t)x_{N_x}(t) + d_{N_y,1}(t)u_1(t) + \cdots + d_{N_y,N_u}(t)u_{N_u}(t)
\end{cases}
\end{cases}
$$

$$\rightsquigarrow \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\rightsquigarrow A(t) = \{a_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_x}$$

$$\rightsquigarrow B(t) = \{b_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_u}$$

$$\rightsquigarrow C(t) = \{c_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_x}$$

$$\rightsquigarrow D(t) = \{d_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_u}$$

Coefficient matrices $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are time dependent (varying)

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ a_{N_x,1}(t) & \cdots & a_{N_x,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} b_{1,1}(t) & \cdots & b_{1,N_y}(t) \\ \vdots & \ddots & \vdots \\ b_{N_x,1}(t) & \cdots & b_{N_x,N_y}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} = \begin{bmatrix} c_{1,1}(t) & \cdots & c_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ c_{N_y,1}(t) & \cdots & c_{N_y,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} d_{1,1}(t) & \cdots & d_{1,N_u}(t) \\ \vdots & \ddots & \vdots \\ d_{N_y,1}(t) & \cdots & d_{N_y,N_u}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\rightsquigarrow \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$\rightsquigarrow \; A = \{a_{i,j}\} \in \mathcal{R}^{N_x \times N_x}$

$\rightsquigarrow \; B = \{b_{i,j}\} \in \mathcal{R}^{N_x \times N_u}$

$\rightsquigarrow \; C = \{c_{i,j}\} \in \mathcal{R}^{N_y \times N_x}$

$\rightsquigarrow \; D = \{d_{i,j}\} \in \mathcal{R}^{N_y \times N_u}$

Coefficient matrices A, B, C and D are time independent (fixed)

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,N_y} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_y} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,N_x} \\ \vdots & \ddots & \vdots \\ c_{N_y,1} & \cdots & c_{N_y,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} d_{1,1} & \cdots & d_{1,N_u} \\ \vdots & \ddots & \vdots \\ d_{N_y,1} & \cdots & d_{N_y,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

State-space representation (cont.)

Common to choose as state those variables that characterise energy within the system

Consider a system in which there is energy stored, its state is not zero

- The system will evolve even in the absence of external inputs

The state can be understood as a possible (internal) cause of evolution

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- For a cylindric tank of base B and liquid level $h(t)$, the potential energy at time t is $E_p(t) = 1/2\rho g V^2(t)/B$, with ρ the density of the liquid and $V(t) = Bh(t)$. $V(t)$ or equivalently $h(t)$ can be used as state variable

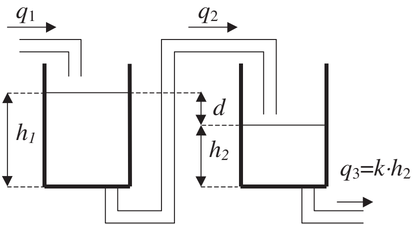
- For a spring with elastic constant k, the potential energy at time t is $E_k(t) = 1/2kz^2(t)$ with $z(t)$ the spring deformation with respect to an equilibrium position. $z(t)$ can be used as state variable

- For a mass m moving with speed $v(t)$ on a plane, the kinetic energy at time t is $E_m(t) = 1/2mv^2(t)$. $v(t)$ can be used as state of the system

State-space representation (cont.)

Example

Two tanks (SS representation, reloaded)



First tank

- Inflow, rate q_1 $[m^3s^{-1}]$
- Outflow, rate q_2 $[m^3s^{-1}]$
- h_1 is the liquid level [m]

Second tank

- Inflow, rate q_2 $[m^3s^{-1}]$
- Outflow, rate q_3 $[m^3s^{-1}]$
- h_2 is the liquid level [m]

Each of the tanks can store a certain amount of potential energy

- The amount of energy depends on the liquid volumes

The complete (two-tank) system has order $N_x = 2$