

CHEM-E7190
2021

Process systems

System modelling
and identification

Analysis, control
and optimisation

System validation
and diagnosis

Classification of
systems/models

Controlled
process systems

System/model
representation

Input-output
representation

State-space
representation



Aalto University

Process system analysis and control

CHEM-E7190 (was E7140) | 2021

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Process system analysis and control

There is a wide spectrum of topics around **process system analysis and control**

A (process) system

A **(process) system** can be defined as a set of elements (or components) that cooperate in order to perform a specific functionality which would be otherwise impossible to attain for the individual components alone

This definition is very fine, but it does not highlight one important element

- There is no notion of the **dynamical behaviour** of the system

A central paradigm will be that systems are subjected to external stimuli

↪ Stimuli influence the temporal evolution of the system itself

A (process) system, reloaded

A (process) system is a physical entity, typically consisting of different interacting elements (or components), that responds to external stimuli according to some determined, or specific, dynamical behaviour

Process system analysis and control (cont.)

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- ↪ System **modelling** and identification
- ↪ System **analysis** and **control**
- ↪ System **optimisation**
- ↪ System verification
- ↪ System diagnosis

Process systems theory

We study how to analyse and control, mathematically, a variety of process systems

Our scope is to understand their dynamical behaviour

- ↪ We want to operate them appropriately
- ↪ We want to design control devices

A methodological approach, both formal and system (process) independent

What sort of systems and what sort of elements/components?

- Examples from chemical process engineering
- Modern examples, as natural extensions

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System modelling and identification

Process system analysis and control

Modelling

To study a(ny) system, the availability of a **mathematical model** is a crucial point

↪ Models provide a quantitative description of the behaviour of the system

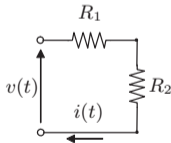
The model is often constructed on the knowledge of the component devices

- Some knowledge of the laws the system obeys to must be available

Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow $i(t)$ [A] through system depends on tension $v(t)$ [V]
- This dependence is assumed to be valid for any point t in time



- $R_1 = 1[\Omega]$
- $R_2 = 3[\Omega]$

Both resistors will follow Ohm's law

$$\rightsquigarrow v(t) = (R_1 + R_2)i(t) = 4i(t)$$

↪ The dependence is linear

↪ (Assumptions!)

The potential difference ('voltage') across an ideal conductor is proportional to the current that flows through it, the proportionality constant is known as 'resistance'

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Identification

Oftentimes, we only have an incomplete knowledge about the system's components

- The model must be constructed from observations
 - By using observations of the system behaviour
-

Case A) We have a knowledge on the type/number of component devices

- Not all of their parameters are known
- System observations are available

↪ **White-box identification**

Case B) We have no knowledge on the components and their parameters

- Observations of the system are available

↪ **Black-box identification**

Identification (cont.)

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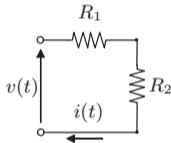
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Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow $i(t)$ [A] through system depends on tension $v(t)$ [V]



- $R_1 = ?$
- $R_2 = ?$

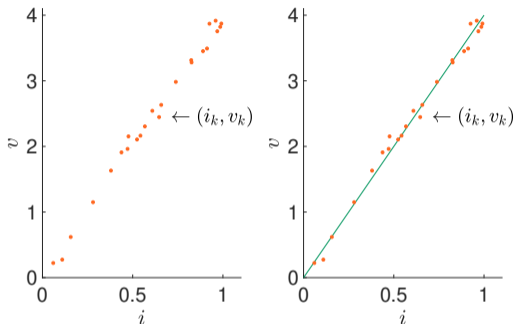
$$\rightsquigarrow v(t) = (R_1 + R_2)i(t) = Ri(t)$$

Both resistors can still be assumed to follow Ohm's laws

- $R = R_1 + R_2$ is now an unknown model parameter
- R can/should/must be identified from data

Identification (cont.)

We can observe the system by collecting K pairs of measurements $\{(v_k, i_k)\}_{k=1}^K$



$$v_k = Ri_k + \varepsilon_k$$

Often (always), such points will not be perfectly aligned along a line of slope R

↪ **Disturbances** alter the behaviour of the system

↪ **Measurement errors** are always present

We choose R corresponding to the line that *best* approximates the measurements

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Example

The marine ecosystem is described through the time evolution of its fauna and flora

- Birth-growth-dead processes

They *recently* spoke about reducing CO₂ emissions by injecting it into the sea

- CO₂ dissolves in sea water

The behaviour of the ecosystem is influenced by a large number of factors

- Climate, food availability, human predators, pollutants, ...

The lack of a valid model limits our understanding of the system

- We do not know the response of the ecosystem

Systems analysis is understanding the system and forecasting its future behaviour

↪ Autonomously and in response to the external stimuli it is subjected to

The availability of a mathematical model of the system is fundamental

- It is needed to approach the problem in a quantitative manner

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The objective of **control** is about imposing a desired behaviour to a system

We need to explicitly formulate what we mean by ‘desired behaviour’

↪ The **specifications** that such behaviour must satisfy

We need to design a device for implementing this task, a **controller**

↪ The scope of a controller is to stimulate the system

↪ Drive its evolution toward the desired behaviour

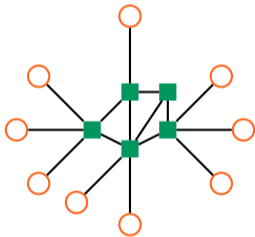
Example

Consider a conventional network for the distribution of drinking water in urban setups

- Water pressure must be kept constant throughout the network

We can measure the pressure at various network locations

- Locations have nominal (target) pressure values



Specs suggest that instantaneous pressure variations should be kept at $\pm 10\%$ of nominal value

Two stimuli act on the system (and affect it)

- ↪ The flow-rate of water that is withdrawn
- ↪ The pressure imposed by the pumps

We cannot control water withdrawals, they are understood as disturbances

Pump pressures can/must be **manipulated** to meet the specifications

- The adjustment of the pumps is performed by a controller

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We want to achieve a certain system's behaviour, while optimising a performance index

- **Optimisation** can be understood as a special case of control

We impose a desired behaviour to the system, while optimising a **performance index**

- The index measures the quality of the behaviour of the system
- (In economic, environmental and/or operational terms)

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Example

Consider the suspension system of a vehicle for human locomotion, a conventional car

These systems are designed to satisfy two different needs

- ↪ An appropriate level of passengers' comfort
- ↪ Good handling in all types of conditions

Modern cars have suspensions based on 'semi-active' technology (fancy springs)

- A controller (dynamically, in real-time) changes the damping factor
- These actions guarantees (a compromise between) the two needs

The optimiser/controller accounts for cabin and wheel oscillations



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Example

Consider an elevator, the system is controlled to guarantee correct responses to requests

Formal verification can be used to guarantee the correct functioning

- The controller is a so called abstract machine
- Programmable logic controller (PLC)

Suppose that a model of the system under study is available (someone derived it)

- Suppose that a set of desired properties can be formally expressed
- **Validation** checks whether a model satisfies such properties

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Fault diagnosis

Example

We understand the human body as a complex system, it is subjected to many faults

- We conventionally call these faults diseases

Consider the presence of fever, or another anomalous condition

- Symptoms reveal the presence of a disease

A doctor, once identified the pathology, prescribes a therapy

Systems deviate from nominal behaviour because of occurrence of faults

- ↪ We need to detect the presence of an anomaly
- ↪ We need to identify the typology of fault
- ↪ We need to devise a corrective action

Fault diagnosis

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Classification

The diversity of systems leads to a number of methodological (modelling) approaches

- Each approach pertains a particular class of models

Conventional methodological approaches and dynamical model/system classification

Models, by general typology

↪ **Time-evolving systems**

- Discrete-event systems
- Hybrid systems

Models, by representation

↪ **State-space models**

- Input-output models

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Time-evolving systems

Time-evolving systems

The system/model behaviour is described with functions

- The independent variable is time (t or k)
- The dependent variable varies (duh!)

Functions of time are also called signals

Continuous time-evolving systems

↪ The time variable varies continuously

Discrete time-evolving systems

↪ The time variable takes discrete values

A particular case of (continuous or discrete) time-evolving systems

↪ The signal that can only take values in a discrete set

↪ **Digital time-evolving systems**

Time-evolving systems (cont.)

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The evolution of any dynamical models is completely based on the passage of time

Signals associated to model behaviour satisfy **differential/difference equations**

- These equations specify a relation between functions and their derivatives

Process systems

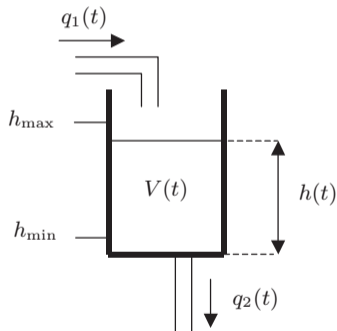
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Example

Continuous time-evolving systems

Consider a surge tank in which the volume of liquid $V(t)$ [m^3] varies over time

- This variation is only due to input and output flows, $q_1(t)$ and $q_2(t)$
- (Inflow and outflow with externally operated pumps)



(The tank cannot be emptied/overflowed)

\rightsquigarrow Output flow-rate $q_2(t) \geq 0$ [m^3s^{-1}]

\rightsquigarrow Input flow-rate $q_1(t) \geq 0$ [m^3s^{-1}]

$$\rightsquigarrow \frac{dV(t)}{dt} = q_1(t) - q_2(t)$$

We are interested in the evolution of V

- Function $V(t)$

The **differential equation** relates continuous-time functions $V(t)$, $q_1(t)$, and $q_2(t)$

Time-evolving systems(cont.)

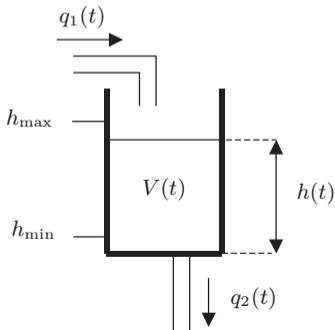
Example

Discrete time-evolving systems

Consider a surge tank in which the volume of liquid $V(t)$ [m³] varies over time

- Suppose that measurements are not continuously available
- Sensor acquisitions only at Δt -apart units of time

We are still interested in the evolution of V , at times $\{0, \Delta t, 2\Delta t, \dots, k\Delta t, \dots\}$



We can consider discrete-time functions

For $k = 0, 1, 2, \dots$, we define

$$\rightsquigarrow V(k) = V(k\Delta t)$$

$$\rightsquigarrow q_1(k) = q_1(k\Delta t)$$

$$\rightsquigarrow q_2(k) = q_2(k\Delta t)$$

Time-evolving systems(cont.)

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We can approximate the derivative in the balance equation with the difference quotient

$$\frac{dV(t)}{dt} \approx \frac{\Delta V}{\Delta t} = \frac{V(k+1) - V(k)}{\Delta t} = q_1(k) - q_2(k)$$


Multiply both sides by Δt

$$V(k+1) - V(k) = [q_1(k) - q_2(k)]\Delta t$$

Or, equivalently

$$\rightsquigarrow V(k+1) = V(k) + [q_1(k) - q_2(k)]\Delta t$$

The **difference equation** relates discrete-time functions $V(k)$, $q_1(k)$, and $q_2(k)$



Time-evolving systems(cont.)

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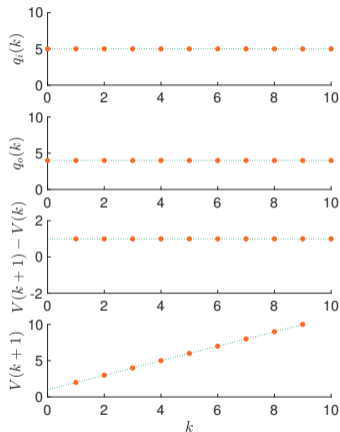
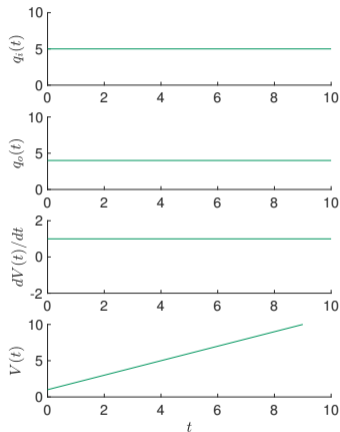
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Discrete-event systems

Discrete-event systems

These are systems whose *state* variables take logical or symbolic values (not numeric)

Their dynamic behaviour is characterised by the occurrence of instantaneous events

~> Events occur at irregular (perhaps unknown beforehand) times

~> The occurrence of events triggers the evolution in time

The behaviour of such systems is represented (modelled) in terms of **states** and **events**

Discrete-event systems (cont.)

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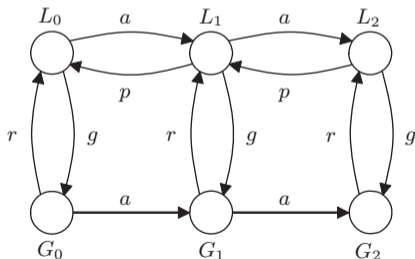
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Example

Discrete-event systems

Consider a depot where mechanical parts are awaiting to be processed by a machine

- The number of parts awaiting to be processed cannot be larger than 2
- The machine can be either healthy (working) or faulty (stopped)

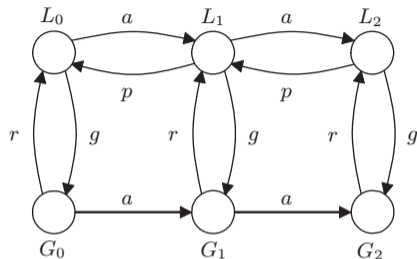


The complete state of the system

$$(\{0, 1, 2\} \times \{H, F\})$$

- Number of awaiting parts
 $\{0, 1, 2\}$
- Status of the machine
 $\{H, F\}$

Discrete-event systems (cont.)



Six possible states (nodes)

- L_0 , L_1 and L_2
 - G_0 , G_1 and G_2
-
- L_0 , the machine is working and the depot is empty
 - L_1 , the machine is working and there is one part in the depot
 - L_2 , the machine is working and there are two parts in the depot
-
- G_0 , the machine is not working and the depot is empty
 - G_1 , the machine is not working and there is one part in the depot
 - G_2 , the machine is not working and there are two parts in the depot

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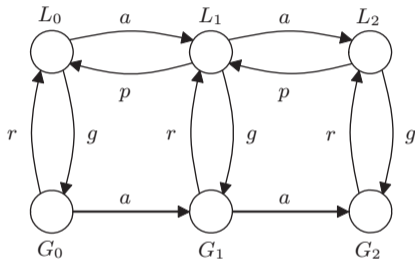
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The events the system can be subjected to are all possible causes of changes in state

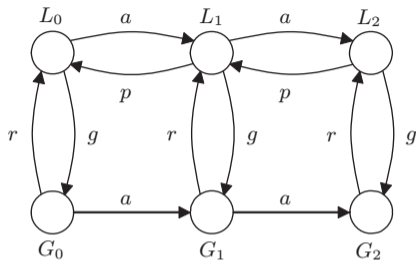


Four possible events (transitions)

- a and p
- g and r

- a , a new part arrives to the depot
- p , the machines takes one part from the depot
- g , the machine gets faulty
- r , the machine gets fixed

Discrete-event systems (cont.)



Event a (new part arrives) can only occur when the depot does not have two parts

$$a \rightsquigarrow \begin{cases} L_i \rightarrow L_{i+1} \\ G_i \rightarrow G_{i+1} \end{cases}$$

Event p (machine takes one part) can only occur when the depot is not empty

$$p \rightsquigarrow \begin{cases} L_i \rightarrow L_{i-1} \end{cases}$$

Event g and r determine the switches $L_i \rightarrow G_i$ and $G_i \rightarrow L_i$, respectively

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Hybrid models can combine time-evolving dynamics and discrete-event dynamics

~> They are the most general class of dynamical systems

Process systems

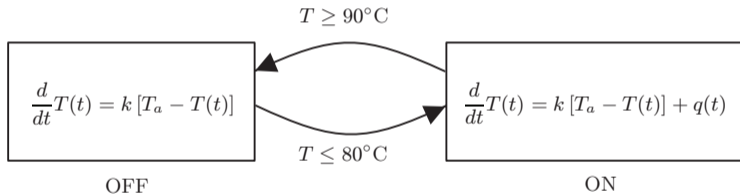
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Example

Hybrid systems

Consider a modern but mild sauna, a cabin where the temperature is regulated

- A thermostat controls a stove used as heat generator
- Keep the temperature between 80°C and 90°C



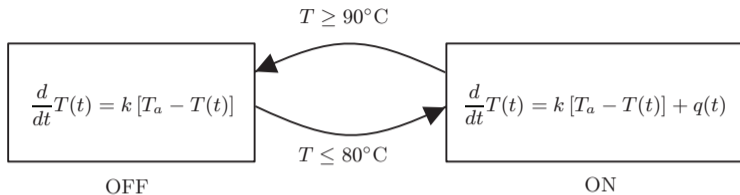
The thermostat can be represented using a discrete-event model

- Switch $\{\text{ON}, \text{OFF}\}$

The cabin can be represented using a time-evolving model

- Temperature $T(t)$

Hybrid systems(cont.)



Suppose that the state of the thermostat is OFF, $T(t)$ in the cabin decreases

- Heat is exchanged with the outside [$T_a < T(t)$]

$$\rightsquigarrow \frac{d}{dt}T(t) = k [T_a - T(t)], \quad \text{with } k > 0$$

Suppose that the state of the thermostat is ON, $T(t)$ in the cabin increases

- Heat is exchanged with the outside [$T_a < T(t)$]
- Heat is generated by the stove $q(t)$

$$\rightsquigarrow \frac{d}{dt}T(t) = k [T_a - T(t)] + q(t)$$

Hybrid systems(cont.)

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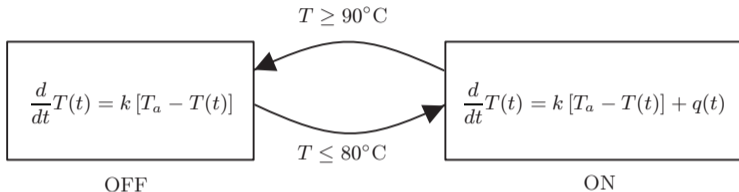
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The state of the system is $x = (l, T)$

- A logical variable $l \in \{\text{ON}, \text{OFF}\}$, representing the discrete state
- A real function $T(t) \in \mathcal{R}^+$, representing the continuous state



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A process is a set of units (reactors, distillation columns, pumps, compressors, ...)

- These units constitute the main plant elements
- (Auxiliary and complementary elements)

Objectives of the plant/process

- ↪ Receive raw materials, and use sources of energy to produce products
- ↪ In the most economic and, sustainable, environmentally aware way

Plant/process requirements

- Safety (people and the environment)
- Operation constraints (mass, energy capacities)
- Production specification (desired product quality and quantity)

Controlled process systems (cont.)

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The satisfaction of the objectives and requirements requires external intervention

↪ Generally, the **process automation system**

↪ Specifically, the **process control system**

The process control system is designed to fulfil some basic and yet critical tasks

- Reduce the influence of **external disturbances** on the process
- Ensure the **stability** and **performances** of the process

Controlled process systems (cont.)

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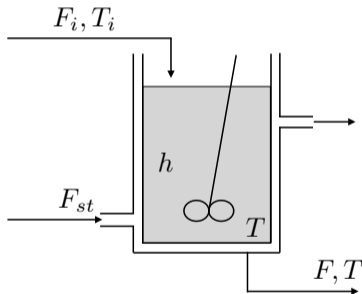
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Example

Heating tank

Consider a perfectly mixed tank in which some liquid is heated using steam circulation



- Input liquid flowrate, $F_i(t)$
- Input liquid temperature, $T_i(t)$
- Output liquid flowrate, $F(t)$
- Output liquid temperature, $T(t)$
- Liquid level in the tank, $h(t)$
- Steam flowrate, $F_{st}(t)$

The objective of the process is to maintain the liquid temperature at desired value, T_d

- Another objective is to maintain the liquid level at some desired value h_d

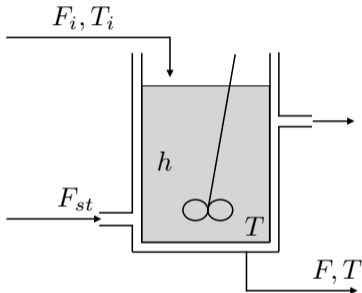
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Controlled process systems (cont.)

To operate such a system, first we need to go through a predefined startup procedure

- The startup procedure brings the system to some steady-state (*SS*) conditions
- In steady-state, all the variables remain constant, stationary, over time (t)



In steady-state conditions, we have

$$\rightsquigarrow T(t) = \text{constant}$$

$$\rightsquigarrow h(t) = \text{constant}$$

$$\rightsquigarrow \dots$$

Suppose that there are no changes in inflow and steam (F_i , T_i and F_{st} are constant)

- \rightsquigarrow Then, the system will remain in steady-state conditions
- \rightsquigarrow The temperature T will stay stationary
- \rightsquigarrow (The level h , and thus also F will)

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Controlled process systems (cont.)

In this ideal (unrealistic) situation, this means that there is no need of a control system

- Given that the steady-state corresponds to the desired value(s) of T (and h)
-

This scenario is implausible as the inflow and steam variables will necessarily change

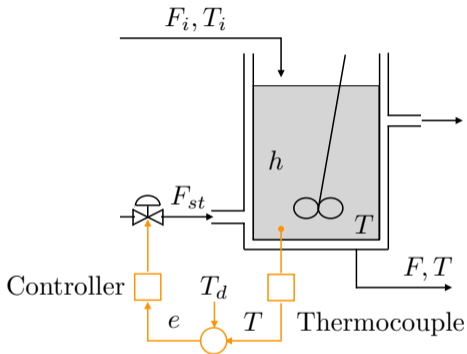
- We do not have any control on the inflow flow-rate and temperature
- The value of these variables depends on upstream processes
- Even in the most optimistic cases, they slightly change

As a consequence, the system variables may drift away from these desired values

- We need to intervene on the system to bring it back
- A controller is the device designed for this task

Controlled process systems (cont.)

Consider the problem of controlling the temperature T of the liquid in the tank, at T_d



- 1 Read the temperature of the liquid in the tank

$$T(t)$$

- 2 Compare this value with some desired value T_d
- ↪ (Compute a difference)

$$e(t) = T_d - T(t)$$

- 3 The error is used to compute the control action
- 4 Control action is implemented in the steam valve

Suppose that the error is positive, $e(t) = T_d - T(t) > 0$, the controller opens the valve

- We need to steer the system's temperature $T(t)$ towards T_s
- The controller will increase the steam flow-rate $F_{st}(t)$

Process systems

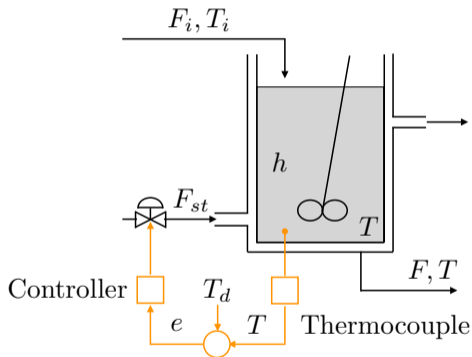
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Controlled process systems (cont.)

Consider a system at steady-state and suppose that an increase of inflow occurs ($F_i \uparrow$)

↪ Other variables being constant, the temperature of the liquid decreases ($T \downarrow$)

↪ Comparison with the desired value gives a positive error ($e = T_d - T > 0$)



↪ The control action is to request for more steam by increasing its flow-rate

↪ This is again practically implemented by opening the steam valve ($F_{st} \uparrow$)

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Stability

Consider the time evolution of a (set of) variable(s) of system originally at steady-state

- At some point in time, the system is perturbed (some change occurs)
- ↪ The system will respond to the perturbation (move away from SS)
- ↪ (Its variables will start varying, changing their value)

A system is stable if its variable(s) return autonomously to their steady-state value(s)

- A stable process is also said to be a self-regulating process
- A stable process would not need a controller, in general
- (If the steady-state condition is the desired state)
- (And, if we have an infinite amount of time)

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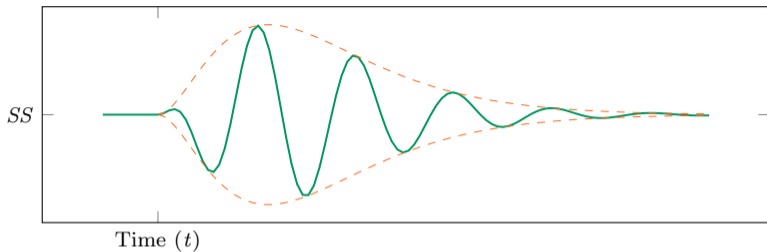
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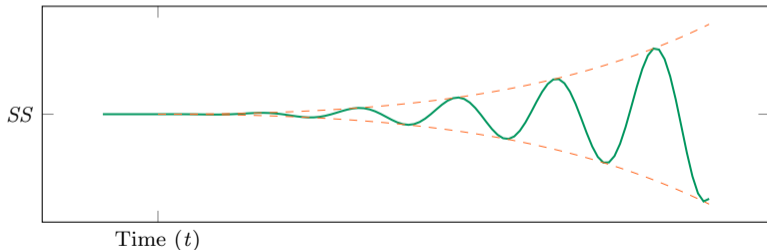
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Stable



Unstable



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Controlled process systems (cont.)

Performance

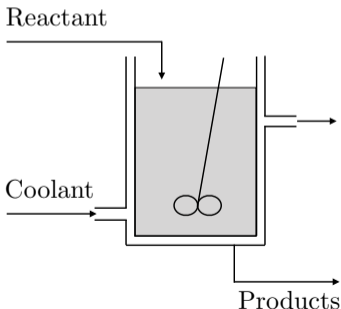
Consider a process for which operational safety and production specifications are met

- The next important objective to be satisfied is (profit) optimisation

Example

Continuously stirred-tank reactor

Consider a jacketed continuous stirred tank reactor, reaction sequence $A \rightarrow B \rightarrow C$



The reaction develops heat, exothermic

↪ To be removed with some coolant

- Reactant A enters the process
- Products leave the process
- B is the desired product
- C is undesired

Interest to maximise profit over time

$$\varphi = \int_0^t f[\text{profit}(B), \text{cost}(A + \text{coolant})] dt$$

Process systems

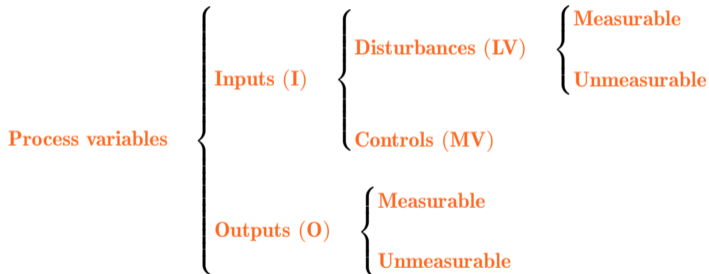
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Controlled process systems (cont.)

Classification of (process) variables

We considered two types of process variables, **input variables** and **output variables**

- ↪ Inputs are understood as entering (as in ‘stimulating’) the system
- ↪ Outputs are understood as exiting the system (as in ‘responses’)



The **controlled variables (CV)** are the third type of variables involved in control

- ↪ They are those variables that we would want to maintain at a desired value
- ↪ They often, but not necessarily, correspond to the measured outputs

Controlled process systems (cont.)

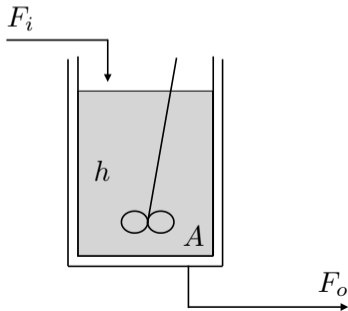
Example

Liquid tank

Consider a cylindrical tank used to store some desired volume of liquid (that is, $h = h_d$)

- Liquid enters with flow-rate F_i and the outflow has flow-rate, F_o
- The cross-sectional area A of the tank is constant

The liquid level h is the controlled variable (CV), what are the I and O variables?



A single input variable (I)

- F_i , often measurable

A single output variable (O)

- h , measurable

F_o is also often measurable

- It can also be an input
- It can be an output

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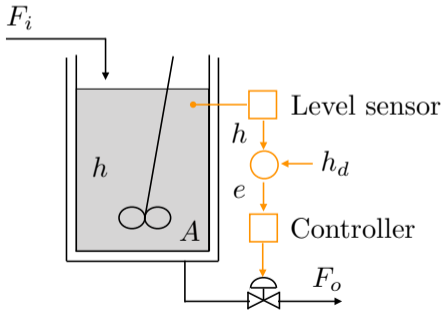
We measure the CV using a level sensor, then we compare its value with a target value

↪ This generates an error $e = h_d - h$ which is passed to the controller

Case 1

One possible control variable (MV) is the outflow flow-rate

↪ The control action is implement in its control valve



↪ $h(t)$, controlled variable (CV)

↪ $F_o(t)$, control variable (MV)

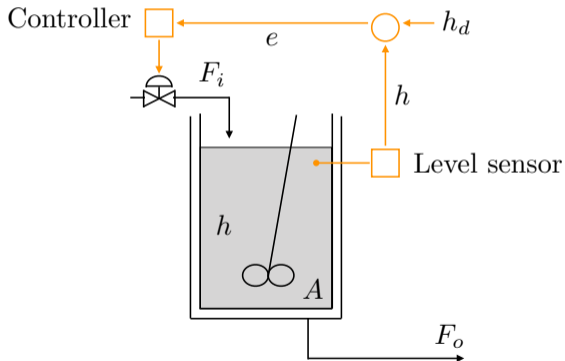
↪ $F_i(t)$, disturbance (LV)

Controlled process systems (cont.)

Case 2

One alternative control variable (MV) is the inflow flow-rate

- The control action is implemented in its control valve



↪ h , controlled variable

↪ F_i , control variable

↪ F_o , disturbance?

- $(F_o, \text{ now input})$

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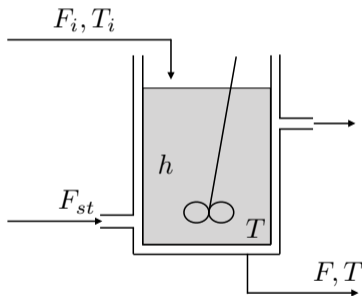
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Example

Heating tank

Consider a perfectly mixed tank in which some liquid is heated using steam circulation



- Input liquid flowrate, $F_i(t)$
- Input liquid temperature, $T_i(t)$
- Output liquid flowrate, $F(t)$
- Output liquid temperature, $T(t)$
- Liquid level in the tank, $h(t)$
- Steam flowrate, $F_{st}(t)$

The objective of the process is to keep the liquid temperature T at desired value, T_d

- Another objective is to maintain the liquid level h at some desired value h_d

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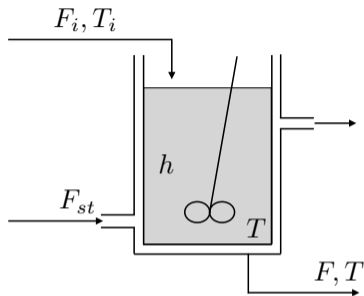
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Controlled process systems (cont.)



The objective of the process is to keep the liquid temperature T at target value, T_d

- Adjust the steam flow-rate (F_{st} , MV), F_i and T_i are disturbances (LV)
- Are there alternative control structures usable for the task?

Another important objective is to maintain the liquid level h at a desired value h_d

- Adjust the outflow flow-rate (F , MV), F_i is a disturbance (LV)
- Adjust the inflow flow-rate (F_i , MV), F may be a disturbance (LV)

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We provided fundamental concepts for the analysis of time-evolving systems/models

- Evolution from the passing of time, focus on continuous-time models

A fundamental step to use formal techniques to study time-evolving systems/models

↪ We describe the system/model behaviour in terms of functions

For given input functions, we are interested in studying how the system evolves in time

- This can be done by analysing the system's representation

Model representations (cont.)

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We introduce the two main forms that are used for describing such systems/models

- **Input-output (IO)** representation
- ↔ **State-space (SS)** representation

The mathematic formulations and examples specific for continuous-space systems

- Yet another classification based on properties of the representation

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Consider the quantities involved in the input-output (IO) representation of a system

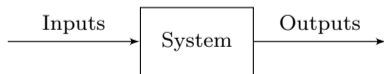
Causes

- ↪ Quantities that are generated outside the system
- Their evolution influences the system behaviour
- Not influenced by the system behaviour

Effects

- ↪ Quantities whose behaviour is influenced by the causes
- Their evolution depends on the nature of the system

By convention,

$$\left\{ \begin{array}{ll} \text{Causes} & \rightsquigarrow \text{Inputs} \\ \text{Effects} & \rightsquigarrow \text{Outputs} \end{array} \right.$$


Input-output representation (cont.)

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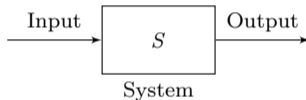
System/model representation

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A (process) system

The system/model S can be seen as an operator or a processing/computing unit



- The system assigns a specific evolution to the output variables (effects)
- One for each possible evolution of the input variables (causes)

Input-output representation (cont.)

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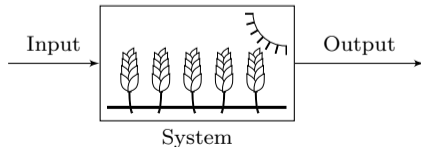
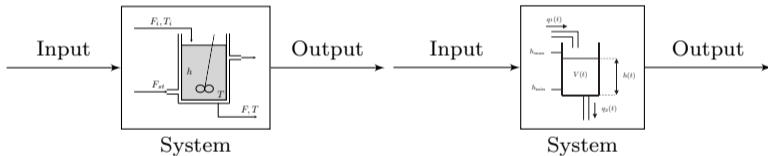
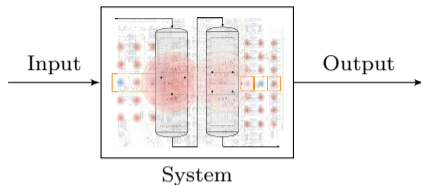
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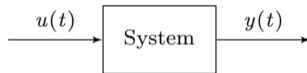
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Input-output representation (cont.)

A system/model can have more than one (N_u) input and more than one (N_y) output

- Both inputs and outputs will assumed to be measurable



$\rightsquigarrow N_u$ **inputs** $u(t)$, in \mathcal{R}^{N_u}

$$u(t) = [u_1(t) \cdots u_{N_u}(t)]'$$

$\rightsquigarrow N_y$ **outputs** $y(t)$, in \mathcal{R}^{N_y}

$$y(t) = [y_1(t) \cdots y_{N_y}(t)]'$$

Manipulable inputs

- They can be used for control

Non-manipulable inputs

- The disturbances

Input-output representation (cont.)

Example

A car (IO representation)

Let the position and speed of a car be the output variables, $y(t) \in \mathcal{R}^{N_y=2}$

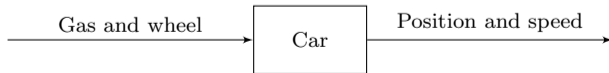
- They are both measurable

As input variables, we can consider wheel and gas position, $u(t) \in \mathcal{R}^{N_u=2}$

- They are both measurable
- They are both manipulable

By acting on the input variables, we influence the behaviour of the output

- How the outputs change depend on the specific system (car)
- (More precisely, on the system's dynamics)



Input-output representation (cont.)

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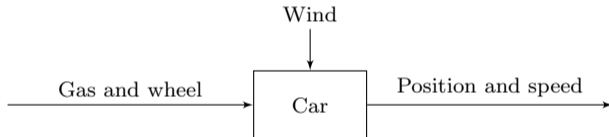
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Wind speed could be considered as an additional input variable to the car system

- It may be measurable, but it is hardly manipulable
- We treat it as non-manipulable input, disturbance



In summary, we have $N_u = 2 + 1 = 3$ inputs and $N_y = 2$ outputs

↪ A Multiple-Input-Multiple-Output (MIMO) system

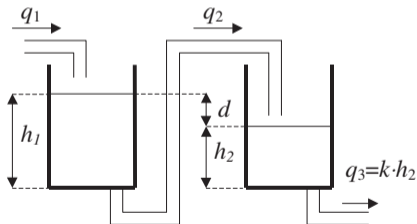


Example

Two tanks (IO representation)

Consider a system consisting of two cylindric liquid tanks, same cross section B [m^2]

- A main inflow to tank 1, a main outflow from tank 2
- The outflow from tank 1 is the inflow to tank 2



First liquid tank

- Inflow, rate q_1 [m^3s^{-1}]
- Outflow, rate q_2 [m^3s^{-1}]
- h_1 is the liquid level [m]

Second liquid tank

- Inflow, rate q_2 [m^3s^{-1}]
- Outflow, rate q_3 [m^3s^{-1}]
- h_2 is the liquid level [m]

We can characterise the involved system variables using standard process notation

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Input-output representation (cont.)

Suppose that flow-rates $q_1(t)$ and $q_2(t)$ can be set to desired values (pumps), at each t

Also, suppose that $q_3(t)$ depends linearly on the liquid level in the tank, $h_2(t)$

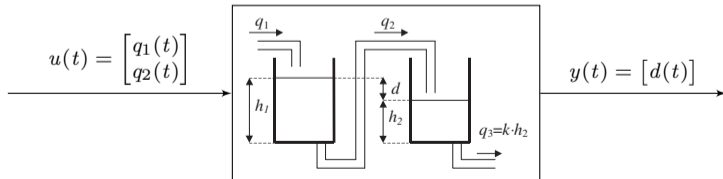
- $q_3(t) = k \cdot h_2(t)$ [m^3s^{-1}], with k [m^2s^{-1}] some appropriate constant

Inputs, q_1 and q_2

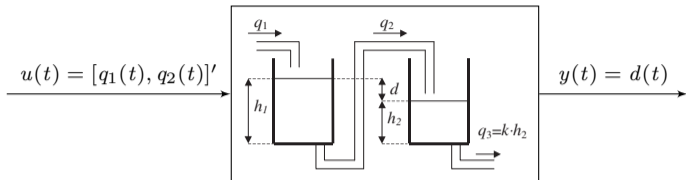
- ↪ Measurable and manipulable
- ↪ They influence the liquid levels in the tanks

Output, $d = h_1 - h_2$

- ↪ Measurable but it cannot be manipulated
- ↪ But, it is influenced by the inputs



Input-output representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \frac{dV_1(t)}{dt} = q_1(t) - q_2(t) \\ \frac{dV_2(t)}{dt} = q_2(t) - q_3(t) = q_2(t) - kh_2(t) \end{cases}$$

We can set $h_1 = V_1/B$, $h_2 = V_2/B$, and $q_3 = kh_2$

$$\rightsquigarrow \begin{cases} \dot{h}_1(t) = \frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \\ \dot{h}_2(t) = \frac{1}{B}q_2(t) - \frac{1}{B}q_3(t) = \frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \end{cases}$$

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Input-output representation (cont.)

$$\rightsquigarrow \begin{cases} \dot{h}_1(t) = \frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \\ \dot{h}_2(t) = \frac{1}{B}q_2(t) - \frac{1}{B}q_3(t) = \frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \end{cases}$$

By taking the first derivative of $y(t) = h_1(t) - h_2(t)$, we have

$$\begin{aligned} \dot{y}(t) &= \dot{h}_1(t) - \dot{h}_2(t) \\ &= \underbrace{\left[\frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \right]}_{\dot{h}_1(t)} - \underbrace{\left[\frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \right]}_{\dot{h}_2(t)} \\ &= \frac{1}{B}q_1(t) - 2\frac{1}{B}q_2(t) + \frac{k}{B}h_2(t) \\ &= \frac{1}{B}u_1(t) - 2\frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)] \end{aligned}$$

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Input-output representation (cont.)

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$$\dot{y}(t) = \frac{1}{B}u_1(t) - 2\frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)]$$

By taking the second derivative of $y(t)$, we have

$$\begin{aligned}\ddot{y}(t) &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B}\dot{h}_1(t) - \frac{k}{B}\dot{y}(t) \\ &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \underbrace{\frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t)}_{\frac{k}{B}\dot{h}_1(t)} - \frac{k}{B}\dot{y}(t)\end{aligned}$$

We used $\dot{h}_1(t) = (u_1(t) - u_2(t)) / B$

Input-output representation (cont.)

$$\ddot{y}(t) = \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t) - \frac{k}{B}\dot{y}(t)$$

Rearranging terms, the IO system's representation is an ordinary differential equation

$$\rightsquigarrow \underbrace{\ddot{y}(t) + \frac{k}{B}\dot{y}(t)}_{y \text{ and its derivatives}} - \underbrace{\frac{1}{B}\dot{u}_1(t) + \frac{2}{B}\dot{u}_2(t) - \frac{k}{B^2}u_1(t) + \frac{k}{B^2}u_2(t)}_{u \text{ and its derivatives}} = 0$$

The system model we have derived for the two-tank process is in the general IO form



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Input-output representation (cont.)

The IO model of a system is a relationship between the system output $y(t) \in \mathcal{R}^{N_y}$ and its derivatives, the system input $u(t) \in \mathcal{R}^{N_u}$ and its derivatives, a differential equation

The IO model of a Single-Input Single-Output (SISO, $N_y = 1, N_u = 1$) system

$$h \left[\underbrace{y(t), \dot{y}(t), \dots, y^{(n)}(t)}_{\text{output}}, \underbrace{u(t), \dot{u}(t), \dots, u^{(m)}(t)}_{\text{input}}, \underbrace{t}_{\text{time}} \right] = 0$$

h is a multi-parametric function that depends on the system

- n is the maximum order of derivation of the output
- m is the maximum order of derivation of the input

The **order of the system (model)** is n

- $\dot{y}(t) = \frac{dy(t)}{dt}$, $\ddot{y}(t) = \frac{d^2y(t)}{dt^2}$ and $y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$
- $\dot{u}(t) = \frac{du(t)}{dt}$, $\ddot{u}(t) = \frac{d^2u(t)}{dt^2}$ and $u^{(m)}(t) = \frac{d^m u(t)}{dt^m}$

Input-output representation (cont.)

Linear and linear time-invariant differential equation

Consider the differential equation

$$h \left[y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t), t \right] = 0$$

The equation is linear if and only if function h is a linear combination of the output and its derivatives $y(t), \dot{y}(t), \dots, y^{(n)}(t)$, and of the input and its derivatives $u(t), \dot{u}(t), \dots, u^{(m)}(t)$

$$\alpha_0(t)y(t) + \alpha_1(t)\dot{y}(t) + \dots + \alpha_n(t)y^{(n)}(t) + \beta_0(t)u(t) + \beta_1(t)\dot{u}(t) + \dots + \beta_m(t)u^{(m)}(t) = 0$$

↪ A zero-sum weighted sum of inputs, outputs, and respective derivatives

The equation is linear and time-invariant if and only if the function h is a time-independent linear combination of the output, the input and their derivatives

$$\alpha_0 y(t) + \alpha_1 \dot{y}(t) + \dots + \alpha_n y^{(n)}(t) + \beta_0 u(t) + \beta_1 \dot{u}(t) + \dots + \beta_m u^{(m)}(t) = 0$$

↪ A zero-sum weighted sum of inputs, outputs, and derivatives

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Input-output representation (cont.)

The IO model of a Multiple-Input Multiple-Output (MIMO, $N_y > 1, N_u > 1$) system

$$\left. \begin{array}{l}
 h_1 \left[\underbrace{y_1(t), \dot{y}_1(t), \dots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{1,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{1,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0 \\
 h_2 \left[\underbrace{y_2(t), \dot{y}_2(t), \dots, y_2^{(n_2)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{1,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{1,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0 \\
 \vdots \\
 h_{N_y} \left[\underbrace{y_{N_y}(t), \dot{y}_{N_y}(t), \dots, y_{N_y}^{(n_{N_y})}(t)}_{\text{output } N_y}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{N_y,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{N_y,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0
 \end{array} \right\}$$

Each h_i ($i = 1, \dots, N_y$) is a multi-parametric function depending on the system

- n_i , max order of derivation of the i -th component of output $y_i(t)$
- m_i , max order of derivation of the i -th component of input $u_i(t)$

A total of N_y differential equations

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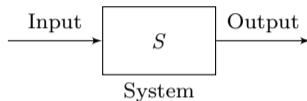
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For a given behaviour of the inputs, system S defines the behaviour of the outputs

- ↪ The system's output at time t is not only dependent on the input at time t
- ↪ It also depends on the past of the system, through its current state

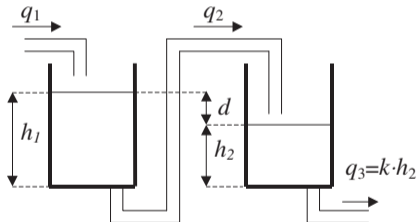
Example

Two tanks (SS representation)

Consider a system consisting of two cylindric liquid tanks, same cross-section B [m²]

Let $d_0 = h_{1,0} - h_{2,0} > 0$ be some positive value of the output at time t_0

- (Equivalent to let $h_{1,0}$ and $h_{2,0}$ be different liquid levels at t_0)



Suppose that all input variables (q_1 and q_2) are zero at time t_0

- $q_{1,0} = 0$
- $q_{2,0} = 0$

Output $d(t)$ at any time $t > t_0$ does not depend only on input values $q_1(t)$ and $q_2(t)$

- Yet $y(t)$ will vary over the entire interval $[t_0, t]$

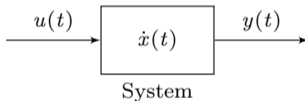
↪ ... regardless of $u(t)$

State-space representation (cont.)

We can take this observation into account by introducing an *intermediate* variable

This system variable can be understood to *exist* between inputs and outputs

- ↪ The **state** variable of the system
- ↪ We use $x(t)$ to denote it



- N_u **inputs** $u(t)$, in \mathcal{R}^{N_u}

$$u(t) = [u_1(t) \cdots u_{N_u}(t)]'$$

- N_y **outputs** $y(t)$, in \mathcal{R}^{N_y}

$$y(t) = [y_1(t) \cdots y_{N_y}(t)]'$$

- N_x **states** $x(t)$, in \mathcal{R}^{N_x}

$$x(t) = [x_1(t) \cdots x_{N_x}(t)]'$$

The state variable condenses information about the past and present of the system

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State-space representation (cont.)

Definition

State variable

The **state variable** of a system/model at time t_0 is a variable that contains the necessary information to univocally determine the behaviour of output $y(t)$ for $t \geq t_0$

- ➊ Given the behaviour of input $u(t)$, for $t \geq t_0$
- ➋ Given the state itself at t_0 , $x(t_0)$

The **state** $x(t) = [x_1(t) \cdots x_{N_x}(t)]^T$ is a vector (a point in space) with N_x components

- ↪ We say that N_x is the order of the system/model
- (In the state-space representation)

State-space representation (cont.)

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In general, it is possible to use different physical/non-physical entities as state variables

- The state variable is neither univocally defined, nor it is determined
- It is anything that can be seen as an *internal cause* of evolution
- (Again, in general)

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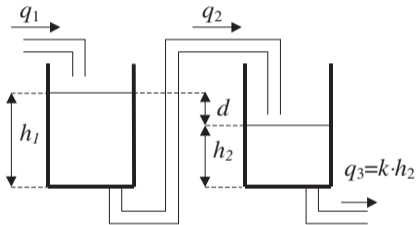
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Example

Two tanks (SS representation)



First tank

- Inflow, rate q_1 [m^3s^{-1}]
- Outflow, rate q_2 [m^3s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Inflow, rate q_2 [m^3s^{-1}]
- Outflow, rate q_3 [m^3s^{-1}]
- h_2 is the liquid level [m]

Suppose that flow-rates $q_1(t)$ and $q_2(t)$ can be set to some desired value (pumps)

Also, suppose that $q_3(t)$ depends linearly on the liquid level in the tank, $h_2(t)$

- $q_3(t) = k \cdot h_2(t)$ [m^3s^{-1}], with k [m^2s^{-1}] some appropriate constant

State-space representation (cont.)

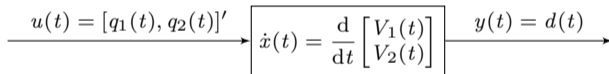
Inputs, q_1 and q_2

- ↪ Measurable and manipulable
- ↪ They influence the liquid levels in the tanks

Output, $d = h_1 - h_2$

- ↪ Measurable but it cannot be manipulated
- ↪ But, it is influenced by the inputs

As for the state variables, we can select the liquid volume in the tanks, $V_1(t)$ and $V_2(t)$

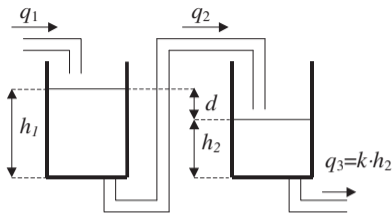


States, $x_1 = V_1$ and $x_2 = V_2$

- ↪ Measurable but cannot be manipulated
- ↪ They are influenced by the inputs

$$\rightsquigarrow x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

State-space representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \frac{dV_1(t)}{dt} = q_1(t) - q_2(t) \\ \frac{dV_2(t)}{dt} = q_2(t) - q_3(t) = q_2(t) - k \underbrace{\frac{V_2(t)}{B}}_{h_2(t)} \end{cases}$$

By the definition of the output,

$$\begin{aligned} d(t) &= h_1(t) - h_2(t) \\ &= \frac{V_1(t)}{B} - \frac{V_2(t)}{B} \end{aligned}$$

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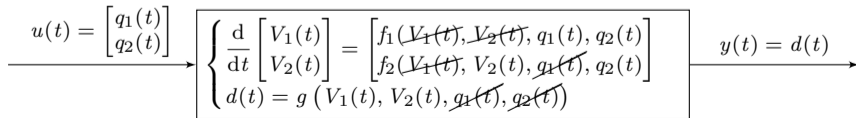
State-space
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State-space representation (cont.)

Summarising, we have

$$\left\{ \begin{array}{l} \frac{dV_1(t)}{dt} = \underbrace{q_1(t) - q_2(t)}_{f_1(V_1(t), V_2(t), q_1(t), q_2(t))} \\ \frac{dV_2(t)}{dt} = \underbrace{q_2(t) - k \frac{V_2(t)}{B}}_{f_2(V_1(t), V_2(t), q_1(t), q_2(t))} \\ d(t) = \underbrace{\frac{V_1(t)}{B} - \frac{V_2(t)}{B}}_{g(V_1(t), V_2(t), q_1(t), q_2(t))} \end{array} \right.$$

Diagrammatically,



State-space representation (cont.)

Rearranging terms, the state-space representation of the two-tank system

$$\rightsquigarrow \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t) \\ \dot{x}_2(t) = -k/Bx_2(t) + u_2(t) \\ y(t) = x_1(t)/B - x_2(t)/B \end{cases}$$

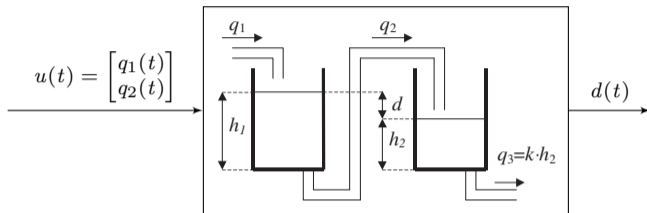
The model is set of ordinary differential equations and an algebraic equation

- State variables, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$
- Input (control) variables $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$
- Output (measurement) variables = $u(t) = [y(t)] = [d(t)]$

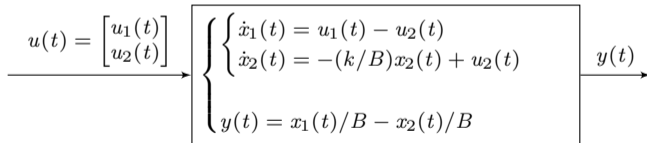
Quantity k and B are constant (do not change w/ time), system parameters

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}V_1(t) \\ \frac{d}{dt}V_2(t) \end{bmatrix}$$

State-space representation (cont.)



Equivalently,



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State-space representation (cont.)

The SS model of a system describes how the evolution (the change in time) $\dot{x}(t) \in \mathcal{R}^{N_x}$ of the system state depends on the state $x(t) \in \mathcal{R}^{N_x}$ itself and on the input $u(t) \in \mathcal{R}^{N_u}$

- The **state equation**
- A set of differential equations

$$\begin{cases} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

The SS model of a system also describes how the system output $y(t) \in \mathcal{R}^{N_y}$ depends on system state $x(t) \in \mathcal{R}^{N_x}$ and on system input $u(t) \in \mathcal{R}^{N_u}$

- The **output transformation**
- A set of algebraic equations

$$\begin{cases} y_1(t) = g_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ y_2(t) = g_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ y_{N_y}(t) = g_{N_y}[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

For compactness, we used $u(t) = [u_1(t), u_2(t), \dots, u_{N_u}(t)]$

State-space representation (cont.)

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The state equation is a set of N_x first-order ordinary differential equations

- Regardless of the fact that the system is SISO or MIMO

The output transformation is a scalar or vectorial algebraic equation

- Depending on the number p of output variables

State-space representation (cont.)

The SS model of a SISO ($y(t) \in \mathcal{R}^{N_y=1}$ and $u(t) \in \mathcal{R}^{N_u=1}$) system with N_x states

$$\begin{cases} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ y(t) = g[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

Let $\dot{x}(t) \in \mathcal{R}^{N_x}$ be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \rightsquigarrow \begin{cases} \dot{x}(t) = f[x(t), u(t), t] \\ y(t) = g[x(t), u(t), t] \end{cases}$$

f is a multi-parametric vectorial function with i -th component f_i , $i = 1, \dots, N_x$

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State-space representation (cont.)

The SS model of a MIMO ($y(t) \in \mathcal{R}^{N_y \neq 1}$ and $u(t) \in \mathcal{R}^{N_u \neq 1}$) system with N_x states

$$\left\{ \begin{array}{l} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_{N_x}(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \\ y_1(t) = g_1[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ y_2(t) = g_2[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \vdots \\ y_{N_y}(t) = g_{N_y}[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \end{array} \right.$$

Let $\dot{x}(t) \in \mathcal{R}^{N_x}$ be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \rightsquigarrow \begin{cases} \dot{x}(t) = f[x(t), u(t), t] \\ y(t) = g[x(t), u(t), t] \end{cases}$$

f and g are multi-parametric vectorial functions depending on the system

- f_i with $i = 1, \dots, N_x$ and g_i with $i = 1, \dots, N_y$

State-space representation (cont.)

Linear and linear time-invariant SS representation

A necessary and sufficient condition for a system to be linear is that state equation and output transformation in the SS model are linear equations

$$\left\{ \begin{array}{l}
 \dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,N_x}(t)x_{N_x}(t) + b_{1,1}(t)u_1(t) + \cdots + b_{1,N_u}(t)u_{N_u}(t) \\
 \dot{x}_2(t) = a_{2,1}(t)x_1(t) + \cdots + a_{2,N_x}(t)x_{N_x}(t) + b_{2,1}(t)u_1(t) + \cdots + b_{2,N_u}(t)u_{N_u}(t) \\
 \vdots \\
 \dot{x}_{N_x}(t) = \\
 \quad a_{N_x,1}(t)x_1(t) + \cdots + a_{N_x,N_x}(t)x_{N_x}(t) + b_{N_x,1}(t)u_1(t) + \cdots + b_{N_x,N_u}(t)u_{N_u}(t) \\
 \\
 y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,N_x}(t)x_{N_x}(t) + d_{1,1}(t)u_1(t) + \cdots + d_{1,N_u}(t)u_{N_u}(t) \\
 y_2(t) = c_{2,1}(t)x_1(t) + \cdots + c_{2,N_x}(t)x_{N_x}(t) + d_{2,1}(t)u_1(t) + \cdots + d_{2,N_u}(t)u_{N_u}(t) \\
 \vdots \\
 y_{N_y}(t) = \\
 \quad c_{N_y,1}(t)x_1(t) + \cdots + c_{N_y,N_x}(t)x_{N_x}(t) + d_{N_y,1}(t)u_1(t) + \cdots + d_{N_y,N_u}(t)u_{N_u}(t)
 \end{array} \right.$$

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$$\rightsquigarrow \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\rightsquigarrow A(t) = \{a_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_x}$$

$$\rightsquigarrow B(t) = \{b_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_u}$$

$$\rightsquigarrow C(t) = \{c_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_x}$$

$$\rightsquigarrow D(t) = \{d_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_u}$$

Coefficient matrices $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are time dependent (varying)

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ a_{N_x,1}(t) & \cdots & a_{N_x,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} b_{1,1}(t) & \cdots & b_{1,N_y}(t) \\ \vdots & \ddots & \vdots \\ b_{N_x,1}(t) & \cdots & b_{N_x,N_y}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} = \begin{bmatrix} c_{1,1}(t) & \cdots & c_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ c_{N_y,1}(t) & \cdots & c_{N_y,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} d_{1,1}(t) & \cdots & d_{1,N_u}(t) \\ \vdots & \ddots & \vdots \\ d_{N_y,1}(t) & \cdots & d_{N_y,N_u}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

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$$\rightsquigarrow \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$\rightsquigarrow A = \{a_{i,j}\} \in \mathcal{R}^{N_x \times N_x}$$

$$\rightsquigarrow B = \{b_{i,j}\} \in \mathcal{R}^{N_x \times N_u}$$

$$\rightsquigarrow C = \{c_{i,j}\} \in \mathcal{R}^{N_y \times N_x}$$

$$\rightsquigarrow D = \{d_{i,j}\} \in \mathcal{R}^{N_y \times N_u}$$

Coefficient matrices A , B , C and D are time independent (fixed)

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,N_u} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,N_x} \\ \vdots & \ddots & \vdots \\ c_{N_y,1} & \cdots & c_{N_y,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} d_{1,1} & \cdots & d_{1,N_u} \\ \vdots & \ddots & \vdots \\ d_{N_y,1} & \cdots & d_{N_y,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

State-space representation (cont.)

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and diagnosis

Classification of systems/models

Controlled process systems

System/model representation

Input-output
representation

State-space
representation

Common to choose as state those variables that characterise energy within the system

Consider a system in which there is energy stored, its state is not zero

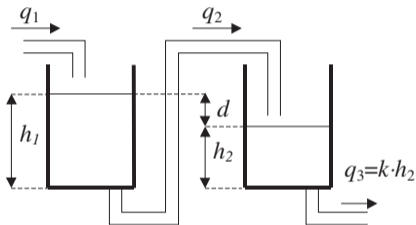
- The system will evolve even in the absence of external inputs

The state can be understood as a possible (internal) cause of evolution

- For a cylindrical tank of base B and liquid level $h(t)$, the potential energy at time t is $E_p(t) = 1/2\rho g V^2(t)/B$, with ρ the density of the liquid and $V(t) = Bh(t)$. $V(t)$ or equivalently $h(t)$ can be used as state variable
- For a spring with elastic constant k , the potential energy at time t is $E_k(t) = 1/2kz^2(t)$ with $z(t)$ the spring deformation with respect to an equilibrium position. $z(t)$ can be used as state variable
- For a mass m moving with speed $v(t)$ on a plane, the kinetic energy at time t is $E_m(t) = 1/2mv^2(t)$. $v(t)$ can be used as state of the system

Example

Two tanks (SS representation, reloaded)



First tank

- Inflow, rate q_1 [m^3s^{-1}]
- Outflow, rate q_2 [m^3s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Inflow, rate q_2 [m^3s^{-1}]
- Outflow, rate q_3 [m^3s^{-1}]
- h_2 is the liquid level [m]

Each of the tanks can store a certain amount of potential energy

- The amount of energy depends on the liquid volumes

The complete (two-tank) system has order $N_x = 2$