

CHEM-E7190  
2023

Process systems

System modelling  
and identification

Analysis, control  
and optimisation

System validation  
and diagnosis

Classification of  
systems/models

Controlled  
process systems

System/model  
representation

Input-output  
representation

State-space  
representation



Aalto University

# Process system analysis and control

CHEM-E7190 (was E7140) | 2023

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# Process system analysis and control

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# Process system analysis and control

There is a wide spectrum of topics around **process system analysis and control**

## A (process) system

A **(process) system** can be defined as a set of elements (or components) that cooperate in order to perform a specific functionality which would be otherwise impossible to attain for the individual components alone

This definition is very fine, but it does not highlight one important element

- There is no notion of the **dynamical behaviour** of the system

A central paradigm will be that systems are subjected to external stimuli

↪ Stimuli influence the temporal evolution of the system itself

## A (process) system, reloaded

A (process) system is a physical entity, typically consisting of different interacting elements (or components), that responds to external stimuli according to some determined, or specific, dynamical behaviour

# Process system analysis and control (cont.)

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- ↪ System **modelling** and identification
- ↪ System **analysis** and **control**
- ↪ System **optimisation**
- ↪ System verification
- ↪ System diagnosis

Process systems theory

We study how to analyse and control, mathematically, a variety of process systems

Our scope is to understand their dynamical behaviour

- ↪ We want to operate them appropriately
- ↪ We want to design control devices

A methodological approach, both formal and system (process) independent

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What sort of systems and what sort of elements/components?

- Examples from chemical process engineering
- Modern examples, as natural extensions

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# System modelling and identification

## Process system analysis and control

## Modelling

To study a(ny) system, the availability of a **mathematical model** is a fundamental point

↪ Models provide a quantitative description of the behaviour of the system

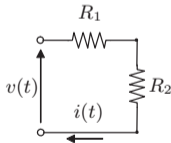
The model is often constructed on the knowledge of the component devices

- Some knowledge of the laws the system obeys to must be available

### Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow  $i(t)$ [A] through system depends on tension  $v(t)$ [V]
- This dependence is assumed to be valid for any point  $t$  in time



- $R_1 = 1[\Omega]$
- $R_2 = 3[\Omega]$

Both resistors will follow Ohm's law

↪  $v(t) = (R_1 + R_2)i(t) = 4i(t)$

↪ The dependence is linear

↪ (Assumptions!)

The potential difference ('voltage') across an ideal conductor is proportional to the current that flows through it, the proportionality constant is known as 'resistance'

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# Identification

Oftentimes, we only have an incomplete knowledge about the system's components

- The model must be constructed from observations
  - By using observations of the system behaviour
- 

**Case A)** We have a knowledge on the type/number of component devices

- Not all of their parameters are known
- System observations are available

↪ **White-box identification**

**Case B)** We have no knowledge on the components and their parameters

- Observations of the system are available

↪ **Black-box identification**

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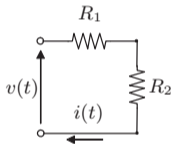
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## Example

Consider an electric circuit consisting of two resistors that are serially arranged

- Current flow  $i(t)$ [A] through system depends on tension  $v(t)$ [V]



- $R_1 = ?$
- $R_2 = ?$

$$\rightsquigarrow v(t) = (R_1 + R_2)i(t) = Ri(t)$$

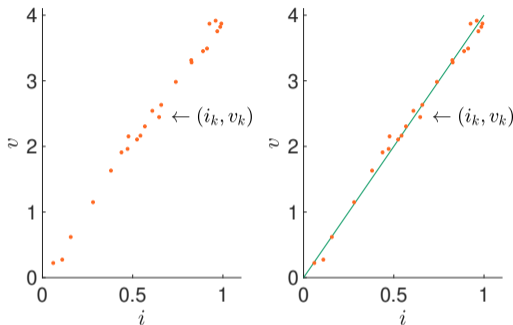
Both resistors can still be assumed to follow Ohm's laws

- $R = R_1 + R_2$  is now an unknown model parameter
- $R$  can/should/must be identified from data



## Identification (cont.)

We can observe the system by collecting  $K$  pairs of measurements  $\{(v_k, i_k)\}_{k=1}^K$



$$v_k = R i_k + \varepsilon_k$$

Often (always), such points will not be perfectly aligned along a line of slope  $R$

↪ **Disturbances** alter the behaviour of the system

↪ **Measurement errors** are always present

We choose  $R$  corresponding to the line that *best* approximates the measurements

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# System analysis, control and optimisation

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## Example

The marine ecosystem is described through the time evolution of its fauna and flora

- Birth-growth-dead processes

They *recently* spoke about reducing CO<sub>2</sub> emissions by injecting it into the sea

- CO<sub>2</sub> dissolves in sea water

The behaviour of the ecosystem is influenced by a large number of factors

- Climate, food availability, human predators, pollutants, ...

The lack of a valid model limits our understanding of the system

- We do not know the response of the ecosystem

**Systems analysis** is understanding the system and forecasting its future behaviour

↪ Autonomously and in response to the external stimuli it is subjected to

The availability of a mathematical model of the system is fundamental

- It is needed to approach the problem in a quantitative manner

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The objective of **control** is about imposing a desired behaviour to a system

We need to explicitly formulate what we mean by ‘desired behaviour’

↪ The **specifications** that such behaviour must satisfy

We need to design a device for implementing this task, a **controller**

↪ The scope of a controller is to stimulate the system

↪ Drive its evolution toward the desired behaviour

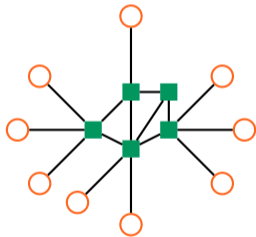
## Example

Consider a conventional network for the distribution of drinking water in urban setups

- Water pressure must be kept constant throughout the network

We can measure the pressure at various network locations

- Locations have nominal (target) pressure values



Specs suggest that instantaneous pressure variations should be kept at  $\pm 10\%$  of nominal value

Two stimuli act on the system (and affect it)

- ↪ The flow-rate of water that is withdrawn
- ↪ The pressure imposed by the pumps

We cannot control water withdrawals, they are understood as disturbances

Pump pressures can/must be **manipulated** to meet the specifications

- The adjustment of the pumps is performed by a controller

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We want to achieve a certain system's behaviour, while optimising a performance index

- **Optimisation** can be understood as a special case of control

We impose a desired behaviour to the system, while optimising a **performance index**

- The index measures the quality of the behaviour of the system
- (In economic, environmental and/or operational terms)

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## Example

Consider the suspension system of a vehicle for human locomotion, a conventional car

These systems are designed to satisfy two different needs

- ↪ An appropriate level of passengers' comfort
- ↪ Good handling in all types of conditions

Modern cars have suspensions based on 'semi-active' technology (fancy springs)

- A controller (dynamically, in real-time) changes the damping factor
- These actions guarantees (a compromise between) the two needs

The optimiser/controller accounts for cabin and wheel oscillations



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# System validation and diagnosis

## Process system analysis and control



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## Example

Consider an elevator, the system is controlled to guarantee correct responses to requests

Formal verification can be used to guarantee the correct functioning

- The controller is a so called abstract machine
- Programmable logic controller (PLC)

Suppose that a model of the system under study is available (someone derived it)

- Suppose that a set of desired properties can be formally expressed
- **Validation** checks whether a model satisfies such properties

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# Fault diagnosis

## Example

We understand the human body as a complex system, it is subjected to many faults

- We conventionally call these faults diseases

Consider the presence of fever, or another anomalous condition

- Symptoms reveal the presence of a disease

A doctor, once identified the pathology, prescribes a therapy

Systems deviate from nominal behaviour because of occurrence of faults

- ↪ We need to detect the presence of an anomaly
- ↪ We need to identify the typology of fault
- ↪ We need to devise a corrective action

## Fault diagnosis

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The diversity of systems leads to a number of methodological (modelling) approaches

- Each approach pertains a particular class of models

Conventional methodological approaches and dynamical model/system classification

## Models, by general typology

### ↪ Time-evolving systems

- Discrete-event systems
- Hybrid systems

## Models, by representation

### ↪ State-space models

- Input-output models

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# Time-evolving systems

## Time-evolving systems

The system/model behaviour is described with functions

- The independent variable is time ( $t$  or  $k$ )
- The dependent variable varies (duh!)

Functions of time are also called signals

---

## Continuous time-evolving systems

↪ The time variable varies continuously

## Discrete time-evolving systems

↪ The time variable takes discrete values

A particular case of (continuous or discrete) time-evolving systems

↪ The signal that can only take values in a discrete set

↪ **Digital time-evolving systems**

# Time-evolving systems (cont.)

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The evolution of any dynamical models is completely based on the passage of time

Signals associated to model behaviour satisfy **differential/difference equations**

- These equations specify a relation between functions and their derivatives

## Process systems

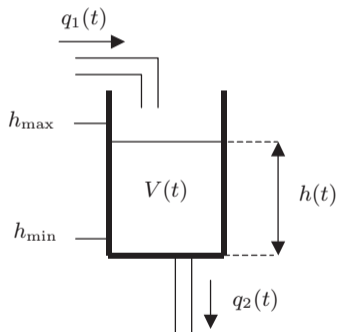
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## Example

## Continuous time-evolving systems

Consider a surge tank in which the volume of liquid  $V(t)$  [ $\text{m}^3$ ] varies over time

- This variation is only due to input and output flows,  $q_1(t)$  and  $q_2(t)$
- (Inflow and outflow with externally operated pumps)



(The tank cannot be emptied/overflowed)

$\rightsquigarrow$  Output flow-rate  $q_2(t) \geq 0$  [ $\text{m}^3\text{s}^{-1}$ ]

$\rightsquigarrow$  Input flow-rate  $q_1(t) \geq 0$  [ $\text{m}^3\text{s}^{-1}$ ]

$$\rightsquigarrow \frac{dV(t)}{dt} = q_1(t) - q_2(t)$$

We are interested in the evolution of  $V$

- Function  $V(t)$

The **differential equation** relates continuous-time functions  $V(t)$ ,  $q_1(t)$ , and  $q_2(t)$

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## Time-evolving systems(cont.)

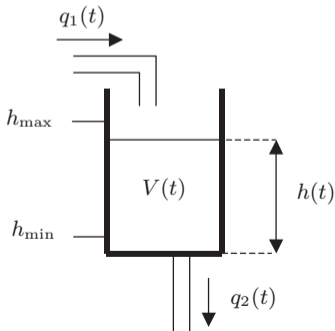
## Example

## Discrete time-evolving systems

Consider a surge tank in which the volume of liquid  $V(t)$  [ $\text{m}^3$ ] varies over time

- Suppose that measurements are not continuously available
- Sensor acquisitions only at  $\Delta t$ -apart units of time

We are still interested in the evolution of  $V$ , at times  $\{0, \Delta t, 2\Delta t, \dots, k\Delta t, \dots\}$



We can consider discrete-time functions

For  $k = 0, 1, 2, \dots$ , we define

$$\rightsquigarrow V(k) = V(k\Delta t)$$

$$\rightsquigarrow q_1(k) = q_1(k\Delta t)$$

$$\rightsquigarrow q_2(k) = q_2(k\Delta t)$$



# Time-evolving systems(cont.)

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We can approximate the derivative in the balance equation with the difference quotient

$$\frac{dV(t)}{dt} \approx \frac{\Delta V}{\Delta t} = \frac{V(k+1) - V(k)}{\Delta t} = q_1(k) - q_2(k)$$


Multiply both sides by  $\Delta t$

$$V(k+1) - V(k) = [q_1(k) - q_2(k)]\Delta t$$

Or, equivalently

$$\rightsquigarrow V(k+1) = V(k) + [q_1(k) - q_2(k)]\Delta t$$

The **difference equation** relates discrete-time functions  $V(k)$ ,  $q_1(k)$ , and  $q_2(k)$



# Time-evolving systems(cont.)

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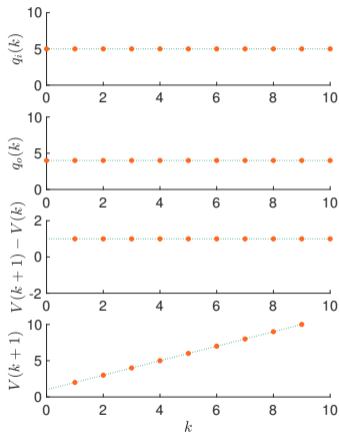
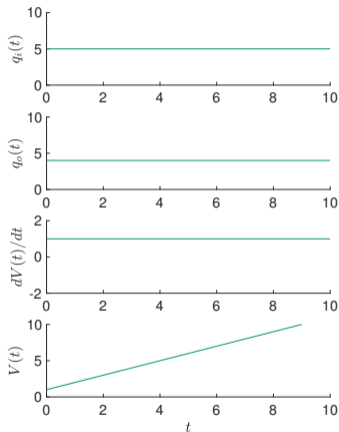
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# Discrete-event systems

## Discrete-event systems

These are systems whose *state* variables take logical or symbolic values (not numeric)

Their dynamic behaviour is characterised by the occurrence of instantaneous events

↪ Events occur at irregular (perhaps unknown beforehand) times

↪ The occurrence of events triggers the evolution in time

The behaviour of such systems is represented (modelled) in terms of **states** and **events**

# Discrete-event systems (cont.)

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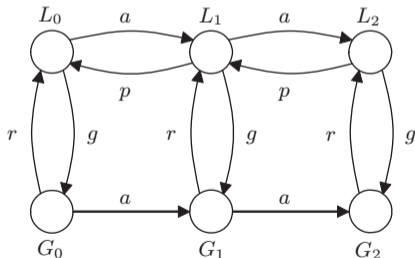
State-space representation

## Example

### Discrete-event systems

Consider a depot where mechanical parts are awaiting to be processed by a machine

- The number of parts awaiting to be processed cannot be larger than 2
- The machine can be either healthy (working) or faulty (stopped)

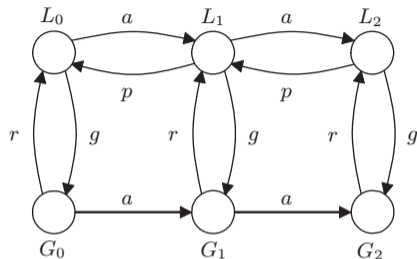


The complete state of the system

$$(\{0, 1, 2\} \times \{H, F\})$$

- Number of awaiting parts  
 $\{0, 1, 2\}$
- Status of the machine  
 $\{H, F\}$

## Discrete-event systems (cont.)



Six possible states (nodes)

- $L_0$ ,  $L_1$  and  $L_2$
  - $G_0$ ,  $G_1$  and  $G_2$
- 
- $L_0$ , the machine is working and the depot is empty
  - $L_1$ , the machine is working and there is one part in the depot
  - $L_2$ , the machine is working and there are two parts in the depot
- 
- $G_0$ , the machine is not working and the depot is empty
  - $G_1$ , the machine is not working and there is one part in the depot
  - $G_2$ , the machine is not working and there are two parts in the depot

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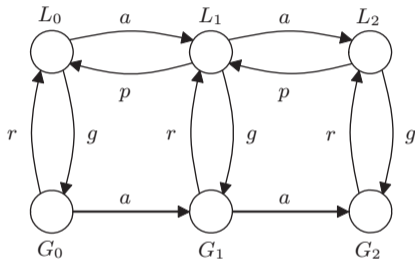
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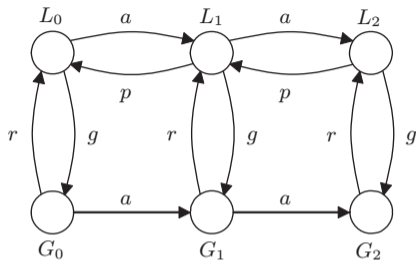
The events the system can be subjected to are all possible causes of changes in state



Four possible events (transitions)

- $a$  and  $p$
  - $g$  and  $r$
- $a$ , a new part arrives to the depot
  - $p$ , the machines takes one part from the depot
  - $g$ , the machine gets faulty
  - $r$ , the machine gets fixed

## Discrete-event systems (cont.)



Event  $a$  (new part arrives) can only occur when the depot does not have two parts

$$a \rightsquigarrow \begin{cases} L_i \rightarrow L_{i+1} \\ G_i \rightarrow G_{i+1} \end{cases}$$

Event  $p$  (machine takes one part) can only occur when the depot is not empty

$$p \rightsquigarrow \begin{cases} L_i \rightarrow L_{i-1} \end{cases}$$

Event  $g$  and  $r$  determine the switches  $L_i \rightarrow G_i$  and  $G_i \rightarrow L_i$ , respectively

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Hybrid models can combine time-evolving dynamics and discrete-event dynamics

~> They are the most general class of dynamical systems



## Process systems

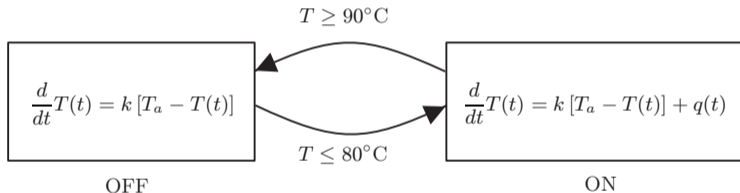
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## Example

## Hybrid systems

Consider a modern but mild sauna, a cabin where the temperature is regulated

- A thermostat controls a stove used as heat generator
- Keep the temperature between  $80^{\circ}\text{C}$  and  $90^{\circ}\text{C}$



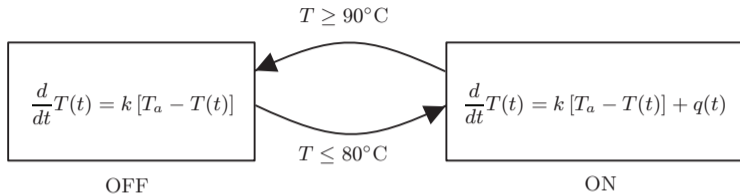
The thermostat can be represented using a discrete-event model

- Switch  $\{\text{ON}, \text{OFF}\}$

The cabin can be represented using a time-evolving model

- Temperature  $T(t)$

## Hybrid systems(cont.)



Suppose that the state of the thermostat is OFF,  $T(t)$  in the cabin decreases

- Heat is exchanged with the outside [ $T_a < T(t)$ ]

$$\rightsquigarrow \frac{d}{dt}T(t) = k [T_a - T(t)], \quad \text{with } k > 0$$

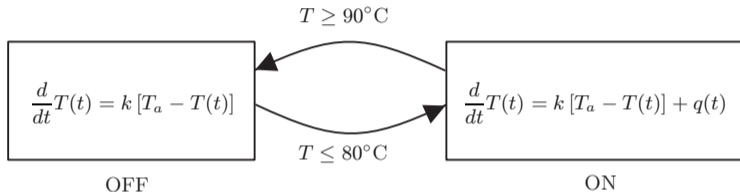
Suppose that the state of the thermostat is ON,  $T(t)$  in the cabin increases

- Heat is exchanged with the outside [ $T_a < T(t)$ ]
- Heat is generated by the stove  $q(t)$

$$\rightsquigarrow \frac{d}{dt}T(t) = k [T_a - T(t)] + q(t)$$

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The state of the system is  $x = (l, T)$

- A logical variable  $l \in \{\text{ON}, \text{OFF}\}$ , representing the discrete state
- A real function  $T(t) \in \mathcal{R}^+$ , representing the continuous state



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# Controlled process systems

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A process is a set of units (reactors, distillation columns, pumps, compressors, ...)

- These units constitute the main plant elements
- (Auxiliary and complementary elements)

Objectives of the plant/process

- ↪ Receive raw materials, and use sources of energy to produce products
- ↪ In the most economic and, sustainable, environmentally aware way

Plant/process requirements

- Safety (people and the environment)
- Operation constraints (mass, energy capacities)
- Production specification (desired product quality and quantity)

# Controlled process systems (cont.)

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The satisfaction of the objectives and requirements requires external intervention

↪ Generally, the **process automation system**

↪ Specifically, the **process control system**

The process control system is designed to fulfil some basic and yet critical tasks

- Reduce the influence of **external disturbances** on the process
- Ensure the **stability** and **performances** of the process

# Controlled process systems (cont.)

## Process systems

System modelling  
and identification

Analysis, control  
and optimisation

System validation  
and diagnosis

## Classification of systems/models

## Controlled process systems

## System/model representation

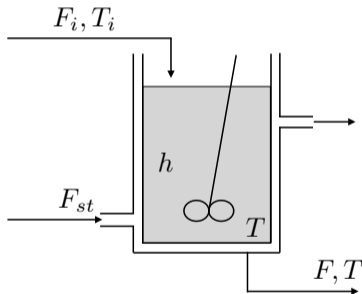
Input-output  
representation

State-space  
representation

## Example

### Heating tank

Consider a perfectly mixed tank in which some liquid is heated using steam circulation



- Input liquid flowrate,  $F_i(t)$
- Input liquid temperature,  $T_i(t)$
- Output liquid flowrate,  $F(t)$
- Output liquid temperature,  $T(t)$
- Liquid level in the tank,  $h(t)$
- Steam flowrate,  $F_{st}(t)$

The objective of the process is to maintain the liquid temperature at desired value,  $T_d$

- Another objective is to maintain the liquid level at some desired value  $h_d$

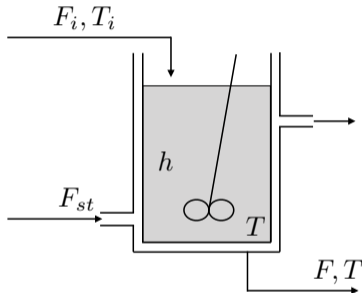
## Process systems

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## Controlled process systems (cont.)

To operate such a system, first we need to go through a predefined startup procedure

- The startup procedure brings the system to some steady-state (*SS*) conditions
- In steady-state, all the variables remain constant, stationary, over time ( $t$ )



In steady-state conditions, we have

$$\rightsquigarrow T(t) = \text{constant}$$

$$\rightsquigarrow h(t) = \text{constant}$$

$$\rightsquigarrow \dots$$

Suppose that there are no changes in inflow and steam ( $F_i$ ,  $T_i$  and  $F_{st}$  are constant)

- $\rightsquigarrow$  Then, the system will remain in steady-state conditions
- $\rightsquigarrow$  The temperature  $T$  will stay stationary
- $\rightsquigarrow$  (The level  $h$ , and thus also  $F$  will)



Process systems

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## Controlled process systems (cont.)

In this ideal (unrealistic) situation, this means that there is no need of a control system

- Given that the steady-state corresponds to the desired value(s) of  $T$  (and  $h$ )
- 

This scenario is implausible as the inflow and steam variables will necessarily change

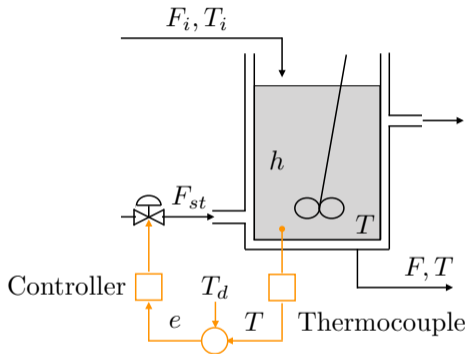
- We do not have any control on the inflow flow-rate and temperature
- The value of these variables depends on upstream processes
- Even in the most optimistic cases, they slightly change

As a consequence, the system variables may drift away from these desired values

- We need to intervene on the system to bring it back
- A controller is the device designed for this task

## Controlled process systems (cont.)

Consider the problem of controlling the temperature  $T$  of the liquid in the tank, at  $T_d$



- 1 Read the temperature of the liquid in the tank

$$T(t)$$

- 2 Compare this value with some desired value  $T_d$
- ↪ (Compute a difference)

$$e(t) = T_d - T(t)$$

- 3 The error is used to compute the control action
- 4 Control action is implemented in the steam valve

Suppose that the error is positive,  $e(t) = T_d - T(t) > 0$ , the controller opens the valve

- We need to steer the system's temperature  $T(t)$  towards  $T_s$
- The controller will increase the steam flow-rate  $F_{st}(t)$

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## Process systems

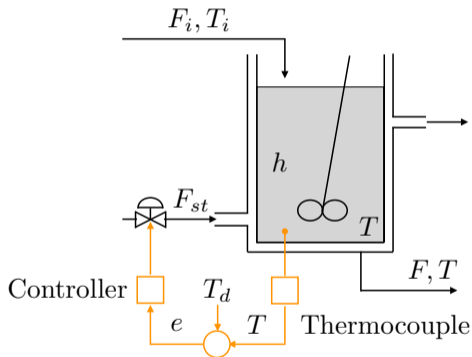
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## Controlled process systems (cont.)

Consider a system at steady-state and suppose that an increase of inflow occurs ( $F_i \uparrow$ )

↪ Other variables being constant, the temperature of the liquid decreases ( $T \downarrow$ )

↪ Comparison with the desired value gives a positive error ( $e = T_d - T > 0$ )



↪ The control action is to request for more steam by increasing its flow-rate

↪ This is again practically implemented by opening the steam valve ( $F_{st} \uparrow$ )

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# Controlled process systems (cont.)

## Stability

Consider the time evolution of a (set of) variable(s) of system originally at steady-state

- At some point in time, the system is perturbed (some change occurs)
- ↪ The system will respond to the perturbation (move away from SS)
- ↪ (Its variables will start varying, changing their value)

---

A system is stable if its variable(s) return autonomously to their steady-state value(s)

- A stable process is also said to be a self-regulating process
- A stable process would not need a controller, in general
- (If the steady-state condition is the desired state)
- (And, if we have an infinite amount of time)

Process systems

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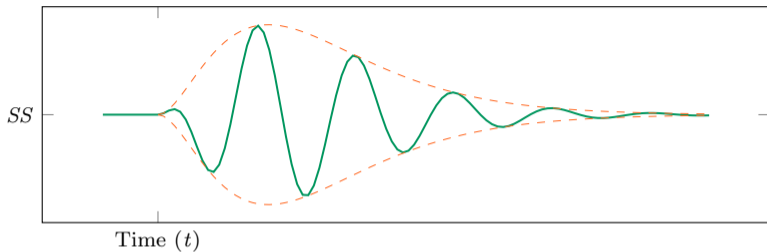
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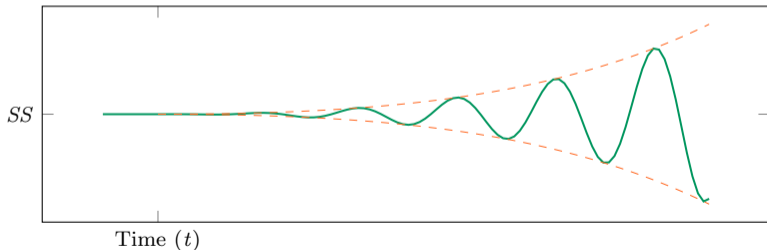
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Stable



Unstable



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# Controlled process systems (cont.)

## Performance

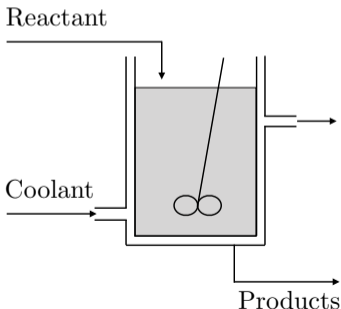
Consider a process for which operational safety and production specifications are met

- The next important objective to be satisfied is (profit) optimisation

## Example

### Continuously stirred-tank reactor

Consider a jacketed continuous stirred tank reactor, reaction sequence  $A \rightarrow B \rightarrow C$



The reaction develops heat, exothermic

↪ To be removed with some coolant

- Reactant  $A$  enters the process
- Products leave the process
- $B$  is the desired product
- $C$  is undesired

Optimisation of an objective over time

$$\varphi = \int_0^t f[\text{profit}(B), \text{cost}(A + \text{coolant})] dt$$

## Process systems

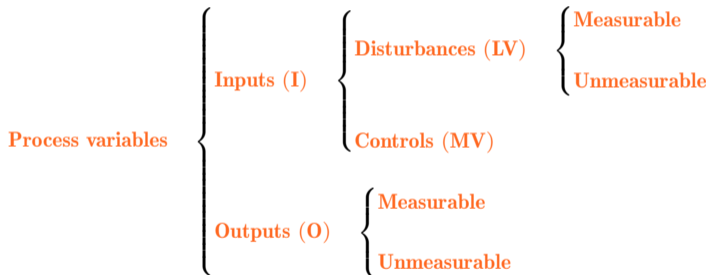
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# Controlled process systems (cont.)

## Classification of (process/model) variables

We considered two types of process variables, **input variables** and **output variables**

- ↪ Inputs are understood as entering (as in ‘stimulating’) the system
- ↪ Outputs are understood as exiting the system (as in ‘responses’)



The **controlled variables (CV)** are the third type of variables involved in control

- ↪ They are those variables that we would want to maintain at a desired value
- ↪ They often, but not necessarily, correspond to the measured outputs

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# Controlled process systems (cont.)

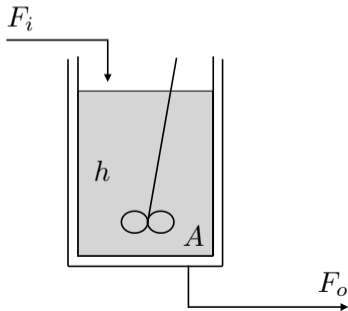
## Example

### Liquid tank

Consider a cylindrical tank used to store some desired volume of liquid (that is,  $h = h_d$ )

- Liquid enters with flow-rate  $F_i$  and the outflow has flow-rate,  $F_o$
- The cross-sectional area  $A$  of the tank is constant

The liquid level  $h$  is the controlled variable (CV), what are the I and O variables?



A single input variable (I)

- $F_i$ , often measurable

A single output variable (O)

- $h$ , measurable

$F_o$  is also often measurable

- It can also be an input
- It can be an output



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## Controlled process systems (cont.)

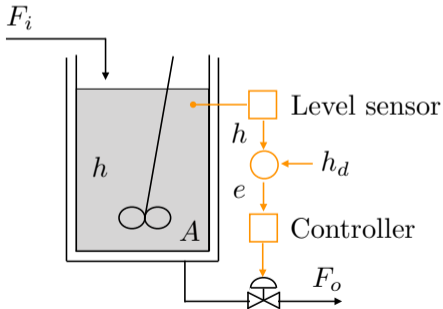
We measure the CV using a level sensor, then we compare its value with a target value

↪ This generates an error  $e = h_d - h$  which is passed to the controller

### Case 1

One possible control variable (MV) is the outflow flow-rate

↪ The control action is implement in its control valve



↪  $h(t)$ , controlled variable (CV)

↪  $F_o(t)$ , control variable (MV)

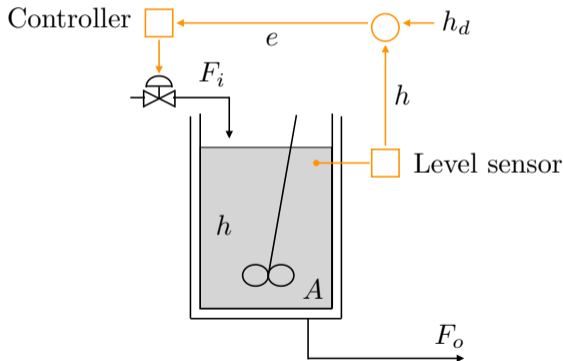
↪  $F_i(t)$ , disturbance (LV)

## Controlled process systems (cont.)

### Case 2

One alternative control variable (MV) is the inflow flow-rate

- The control action is implemented in its control valve



↪  $h$ , controlled variable

↪  $F_i$ , control variable

↪  $F_o$ , disturbance?

- ( $F_o$ , now input)

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## System/model representation

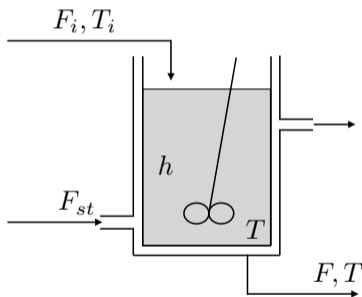
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## Example

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Consider a perfectly mixed tank in which some liquid is heated using steam circulation



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- Output liquid flowrate,  $F(t)$
- Output liquid temperature,  $T(t)$
- Liquid level in the tank,  $h(t)$
- Steam flowrate,  $F_{st}(t)$

The objective of the process is to keep the liquid temperature  $T$  at desired value,  $T_d$

- Another objective is to maintain the liquid level  $h$  at some desired value  $h_d$

Process systems

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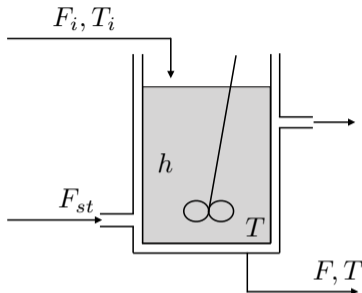
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## Controlled process systems (cont.)



The objective of the process is to keep the liquid temperature  $T$  at target value,  $T_d$

- Adjust the steam flow-rate ( $F_{st}$ , MV),  $F_i$  and  $T_i$  are disturbances (LV)
- Are there alternative control structures usable for the task?

Another important objective is to maintain the liquid level  $h$  at a desired value  $h_d$

- Adjust the outflow flow-rate ( $F$ , MV),  $F_i$  is a disturbance (LV)
- Adjust the inflow flow-rate ( $F_i$ , MV),  $F$  may be a disturbance (LV)

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# Representation of systems/models

# Model representations

## Process systems

System modelling  
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## Classification of systems/models

## Controlled process systems

## System/model representation

Input-output  
representation

State-space  
representation

We provided fundamental concepts for the analysis of time-evolving systems/models

- Evolution from the passing of time, focus on continuous-time models

A fundamental step to use formal techniques to study time-evolving systems/models

↪ We describe the system/model behaviour in terms of functions

---

For given input functions, we are interested in studying how the system evolves in time

- This can be done by analysing the system's representation

## Model representations (cont.)

### Process systems

System modelling  
and identification

Analysis, control  
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System validation  
and diagnosis

### Classification of systems/models

### Controlled process systems

### System/model representation

Input-output  
representation

State-space  
representation

We introduce the two main forms that are used for describing such systems/models

- **Input-output (IO)** representation
- ↔ **State-space (SS)** representation

The mathematic formulations and examples specific for continuous-space systems

- Yet another classification based on properties of the representation

Process systems

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# Input-output representation

## Model representations



# Input-output representation

Consider the quantities involved in the input-output (IO) representation of a system

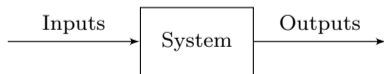
## Causes

- ↪ Quantities that are generated outside the system
- Their evolution influences the system behaviour
  - Not influenced by the system behaviour

## Effects

- ↪ Quantities whose behaviour is influenced by the causes
- Their evolution depends on the nature of the system

By convention,

$$\left\{ \begin{array}{ll} \text{Causes} & \rightsquigarrow \text{Inputs} \\ \text{Effects} & \rightsquigarrow \text{Outputs} \end{array} \right.$$


# Input-output representation (cont.)

## Process systems

System modelling  
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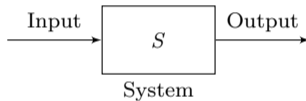
## System/model representation

### Input-output representation

State-space  
representation

## A (process) system

The system/model  $S$  can be seen as an operator or a processing/computing unit



- The system assigns a specific evolution to the output variables (effects)
- One for each possible evolution of the input variables (causes)

# Input-output representation (cont.)

## Process systems

System modelling  
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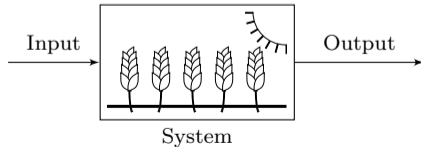
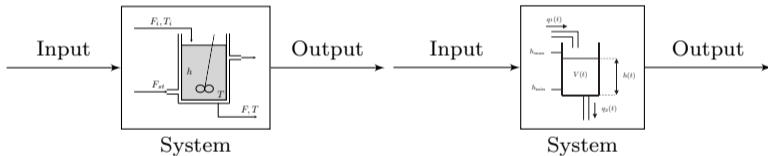
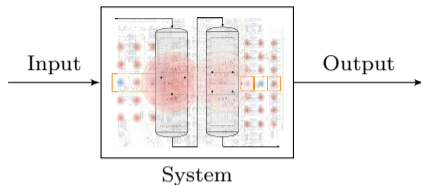
## Classification of systems/models

## Controlled process systems

## System/model representation

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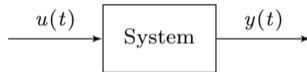
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# Input-output representation (cont.)

A system/model can have more than one ( $N_u$ ) input and more than one ( $N_y$ ) output

- Both inputs and outputs will assumed to be measurable



$\rightsquigarrow N_u$  **inputs**  $u(t)$ , in  $\mathcal{R}^{N_u}$

$$u(t) = [u_1(t) \cdots u_{N_u}(t)]'$$

$\rightsquigarrow N_y$  **outputs**  $y(t)$ , in  $\mathcal{R}^{N_y}$

$$y(t) = [y_1(t) \cdots y_{N_y}(t)]'$$

## Manipulable inputs

- They can be used for control

## Non-manipulable inputs

- The disturbances

# Input-output representation (cont.)

## Example

### A car (IO representation)

Let the position and speed of a car be the output variables,  $y(t) \in \mathcal{R}^{N_y=2}$

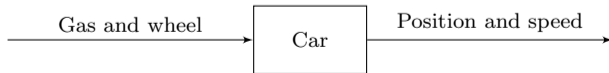
- They are both measurable

As input variables, we can consider wheel and gas position,  $u(t) \in \mathcal{R}^{N_u=2}$

- They are both measurable
- They are both manipulable

By acting on the input variables, we influence the behaviour of the output

- How the outputs change depend on the specific system (car)
- (More precisely, on the system's dynamics)



## Input-output representation (cont.)

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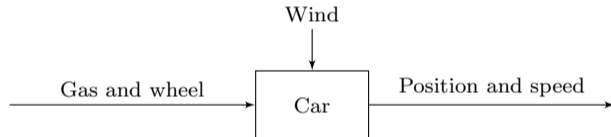
System/model  
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Input-output  
representation

State-space  
representation

Wind speed could be considered as an additional input variable to the car system

- It may be measurable, but it is hardly manipulable
- We treat it as non-manipulable input, disturbance



In summary, we have  $N_u = 2 + 1 = 3$  inputs and  $N_y = 2$  outputs

↪ A Multiple-Input-Multiple-Output (MIMO) system



Process systems

System modelling and identification

Analysis, control and optimisation

System validation and diagnosis

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Input-output representation

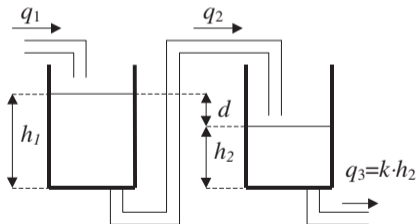
State-space representation

## Example

### Two tanks (IO representation)

Consider a system consisting of two cylindric liquid tanks, same cross section  $B$  [m<sup>2</sup>]

- A main inflow to tank 1, a main outflow from tank 2
- The outflow from tank 1 is the inflow to tank 2



#### First liquid tank

- Inflow, rate  $q_1$  [m<sup>3</sup>s<sup>-1</sup>]
- Outflow, rate  $q_2$  [m<sup>3</sup>s<sup>-1</sup>]
- $h_1$  is the liquid level [m]

#### Second liquid tank

- Inflow, rate  $q_2$  [m<sup>3</sup>s<sup>-1</sup>]
- Outflow, rate  $q_3$  [m<sup>3</sup>s<sup>-1</sup>]
- $h_2$  is the liquid level [m]

We can characterise the involved system variables using standard process notation

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## Input-output representation (cont.)

Suppose that flow-rates  $q_1(t)$  and  $q_2(t)$  can be set to desired values (pumps), at each  $t$

Also, suppose that  $q_3(t)$  depends linearly on the liquid level in the tank,  $h_2(t)$

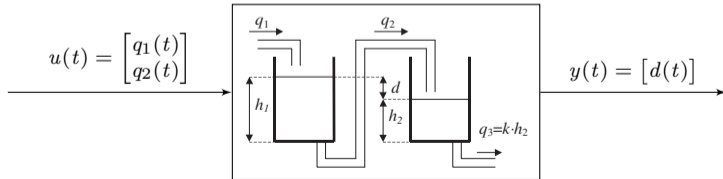
- $q_3(t) = k \cdot h_2(t)$  [ $\text{m}^3\text{s}^{-1}$ ], with  $k$  [ $\text{m}^2\text{s}^{-1}$ ] some appropriate constant

**Inputs**,  $q_1$  and  $q_2$ 

- ↪ Measurable and manipulable
- ↪ They influence the liquid levels in the tanks

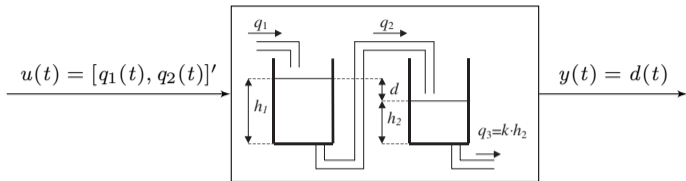
**Output**,  $d = h_1 - h_2$ 

- ↪ Measurable but it cannot be manipulated
- ↪ But, it is influenced by the inputs





## Input-output representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \frac{dV_1(t)}{dt} = q_1(t) - q_2(t) \\ \frac{dV_2(t)}{dt} = q_2(t) - q_3(t) = q_2(t) - kh_2(t) \end{cases}$$

We can set  $h_1 = V_1/B$ ,  $h_2 = V_2/B$ , and  $q_3 = kh_2$

$$\rightsquigarrow \begin{cases} \dot{h}_1(t) = \frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \\ \dot{h}_2(t) = \frac{1}{B}q_2(t) - \frac{1}{B}q_3(t) = \frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \end{cases}$$

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## Input-output representation (cont.)

$$\rightsquigarrow \begin{cases} \dot{h}_1(t) = \frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \\ \dot{h}_2(t) = \frac{1}{B}q_2(t) - \frac{1}{B}q_3(t) = \frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \end{cases}$$

By taking the first derivative of  $y(t) = h_1(t) - h_2(t)$ , we have

$$\begin{aligned} \dot{y}(t) &= \dot{h}_1(t) - \dot{h}_2(t) \\ &= \underbrace{\left[ \frac{1}{B}q_1(t) - \frac{1}{B}q_2(t) \right]}_{\dot{h}_1(t)} - \underbrace{\left[ \frac{1}{B}q_2(t) - \frac{k}{B}h_2(t) \right]}_{\dot{h}_2(t)} \\ &= \frac{1}{B} \underbrace{q_1(t)}_{u_1(t)} - 2 \frac{1}{B} \underbrace{q_2(t)}_{u_2(t)} + \frac{k}{B}h_2(t) \\ &= \frac{1}{B}u_1(t) - 2 \frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)] \end{aligned}$$

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## Input-output representation (cont.)

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$$\dot{y}(t) = \frac{1}{B}u_1(t) - 2\frac{1}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)]$$

By taking the second derivative of  $y(t)$ , we have

$$\begin{aligned}\ddot{y}(t) &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B}\dot{h}_1(t) - \frac{k}{B}\dot{y}(t) \\ &= \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \underbrace{\frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t)}_{\frac{k}{B}\dot{h}_1(t)} - \frac{k}{B}\dot{y}(t)\end{aligned}$$

We used  $\dot{h}_1(t) = (u_1(t) - u_2(t)) / B$

## Input-output representation (cont.)

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$$\ddot{y}(t) = \frac{1}{B}\dot{u}_1(t) - 2\frac{1}{B}\dot{u}_2(t) + \frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t) - \frac{k}{B}\dot{y}(t)$$

Rearranging terms, the IO system's representation is an ordinary differential equation

$$\rightsquigarrow \underbrace{\ddot{y}(t) + \frac{k}{B}\dot{y}(t)}_{y \text{ and its derivatives}} - \underbrace{\left(\frac{1}{B}\dot{u}_1(t) + \frac{2}{B}\dot{u}_2(t) - \frac{k}{B^2}u_1(t) + \frac{k}{B^2}u_2(t)\right)}_{u \text{ and its derivatives}} = 0$$

The system model we have derived for the two-tank process is in the general IO form



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## Input-output representation (cont.)

The IO model of a system is a relationship between the system output  $y(t) \in \mathcal{R}^{N_y}$  and its derivatives, the system input  $u(t) \in \mathcal{R}^{N_u}$  and its derivatives, a differential equation

The IO model of a Single-Input Single-Output (SISO,  $N_y = 1, N_u = 1$ ) system

$$h \left[ \underbrace{y(t), \dot{y}(t), \dots, y^{(n)}(t)}_{\text{output}}, \underbrace{u(t), \dot{u}(t), \dots, u^{(m)}(t)}_{\text{input}}, \underbrace{t}_{\text{time}} \right] = 0$$

$h$  is a multi-parametric function that depends on the system

- $n$  is the maximum order of derivation of the output
- $m$  is the maximum order of derivation of the input

The **order of the system (model)** is  $n$

- $\dot{y}(t) = \frac{dy(t)}{dt}$ ,  $\ddot{y}(t) = \frac{d^2y(t)}{dt^2}$  and  $y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$
- $\dot{u}(t) = \frac{du(t)}{dt}$ ,  $\ddot{u}(t) = \frac{d^2u(t)}{dt^2}$  and  $u^{(m)}(t) = \frac{d^m u(t)}{dt^m}$

## Input-output representation (cont.)

## Linear and linear time-invariant differential equation

Consider the differential equation

$$h \left[ y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t), t \right] = 0$$

The equation is linear if and only if function  $h$  is a linear combination of the output and its derivatives  $y(t), \dot{y}(t), \dots, y^{(n)}(t)$ , and of the input and its derivatives  $u(t), \dot{u}(t), \dots, u^{(m)}(t)$

$$\alpha_0(t)y(t) + \alpha_1(t)\dot{y}(t) + \dots + \alpha_n(t)y^{(n)}(t) + \beta_0(t)u(t) + \beta_1(t)\dot{u}(t) + \dots + \beta_m(t)u^{(m)}(t) = 0$$

↪ A zero-sum weighted sum of inputs, outputs, and respective derivatives

The equation is linear and time-invariant if and only if the function  $h$  is a time-independent linear combination of the output, the input and their derivatives

$$\alpha_0 y(t) + \alpha_1 \dot{y}(t) + \dots + \alpha_n y^{(n)}(t) + \beta_0 u(t) + \beta_1 \dot{u}(t) + \dots + \beta_m u^{(m)}(t) = 0$$

↪ A zero-sum weighted sum of inputs, outputs, and derivatives

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## Input-output representation (cont.)

The IO model of a Multiple-Input Multiple-Output (MIMO,  $N_y > 1, N_u > 1$ ) system

$$\begin{cases}
 h_1 \left[ \underbrace{y_1(t), \dot{y}_1(t), \dots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{1,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{1,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0 \\
 h_2 \left[ \underbrace{y_2(t), \dot{y}_2(t), \dots, y_2^{(n_2)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{1,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{1,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0 \\
 \vdots \\
 h_{N_y} \left[ \underbrace{y_{N_y}(t), \dot{y}_{N_y}(t), \dots, y_{N_y}^{(n_{N_y})}(t)}_{\text{output } N_y}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{N_y,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_{N_u}(t), \dots, u_{N_u}^{(m_{N_y,N_u})}(t)}_{\text{input } N_u}, t \right] \\
 = 0
 \end{cases}$$

Each  $h_i$  ( $i = 1, \dots, N_y$ ) is a multi-parametric function depending on the system

- $n_i$ , max order of derivation of the  $i$ -th component of output  $y_i(t)$
- $m_i$ , max order of derivation of the  $i$ -th component of input  $u_i(t)$

A total of  $N_y$  differential equations

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# State-space representation

## Model representations



# State-space representation

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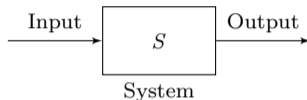
## Classification of systems/models

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Input-output  
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State-space  
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For a given behaviour of the inputs, system  $S$  defines the behaviour of the outputs

- ↪ The system's output at time  $t$  is not only dependent on the input at time  $t$
- ↪ It also depends on the past of the system, through its current state

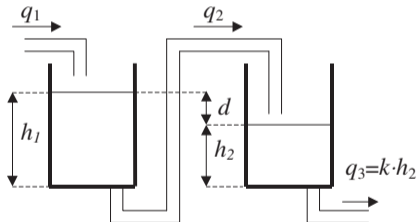
## Example

## Two tanks (SS representation)

Consider a system consisting of two cylindric liquid tanks, same cross-section  $B$  [m<sup>2</sup>]

Let  $d_0 = h_{1,0} - h_{2,0} > 0$  be some positive value of the output at time  $t_0$

- (Equivalent to let  $h_{1,0}$  and  $h_{2,0}$  be different liquid levels at  $t_0$ )



Suppose that all input variables ( $q_1$  and  $q_2$ ) are zero at time  $t_0$

- $q_{1,0} = 0$
- $q_{2,0} = 0$

Output  $d(t)$  at any time  $t > t_0$  does not depend only on input values  $q_1(t)$  and  $q_2(t)$

- Yet  $y(t)$  will vary over the entire interval  $[t_0, t]$

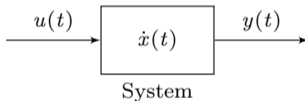
↪ ... regardless of  $u(t)$

## State-space representation (cont.)

We can take this observation into account by introducing an *intermediate* variable

This system variable can be understood to *exist* between inputs and outputs

- ↪ The **state** variable of the system
- ↪ We use  $x(t)$  to denote it



- $N_u$  **inputs**  $u(t)$ , in  $\mathcal{R}^{N_u}$

$$u(t) = [u_1(t) \cdots u_{N_u}(t)]'$$

- $N_y$  **outputs**  $y(t)$ , in  $\mathcal{R}^{N_y}$

$$y(t) = [y_1(t) \cdots y_{N_y}(t)]'$$

- $N_x$  **states**  $x(t)$ , in  $\mathcal{R}^{N_x}$

$$x(t) = [x_1(t) \cdots x_{N_x}(t)]'$$

The state variable condenses information about the past and present of the system

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## State-space representation (cont.)

### Definition

#### State variable

The **state variable** of a system/model at time  $t_0$  is a variable that contains the necessary information to univocally determine the behaviour of output  $y(t)$  for  $t \geq t_0$

- 1 Given the behaviour of input  $u(t)$ , for  $t \geq t_0$
- 2 Given the state itself at  $t_0$ ,  $x(t_0)$

The **state**  $x(t) = [x_1(t) \cdots x_{N_x}(t)]^T$  is a vector (a point in space) with  $N_x$  components

- ↪ We say that  $N_x$  is the order of the system/model
- (In the state-space representation)

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representation**

In general, it is possible to use different physical/non-physical entities as state variables

- The state variable is neither univocally defined, nor it is determined
- It is anything that can be seen as an *internal cause* of evolution
- (Again, in general)

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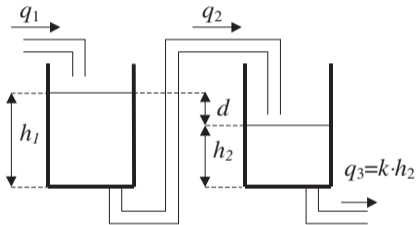
System/model representation

Input-output representation

State-space representation

## Example

### Two tanks (SS representation)



#### First tank

- Inflow, rate  $q_1$  [ $\text{m}^3\text{s}^{-1}$ ]
- Outflow, rate  $q_2$  [ $\text{m}^3\text{s}^{-1}$ ]
- $h_1$  is the liquid level [m]

#### Second tank

- Inflow, rate  $q_2$  [ $\text{m}^3\text{s}^{-1}$ ]
- Outflow, rate  $q_3$  [ $\text{m}^3\text{s}^{-1}$ ]
- $h_2$  is the liquid level [m]

Suppose that flow-rates  $q_1(t)$  and  $q_2(t)$  can be set to some desired value (pumps)

Also, suppose that  $q_3(t)$  depends linearly on the liquid level in the tank,  $h_2(t)$

- $q_3(t) = k \cdot h_2(t)$  [ $\text{m}^3\text{s}^{-1}$ ], with  $k$  [ $\text{m}^2\text{s}^{-1}$ ] some appropriate constant

## State-space representation (cont.)

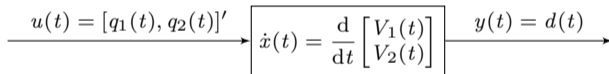
**Inputs**,  $q_1$  and  $q_2$

- ↪ Measurable and manipulable
- ↪ They influence the liquid levels in the tanks

**Output**,  $d = h_1 - h_2$

- ↪ Measurable but it cannot be manipulated
- ↪ But, it is influenced by the inputs

As for the state variables, we can select the liquid volume in the tanks,  $V_1(t)$  and  $V_2(t)$

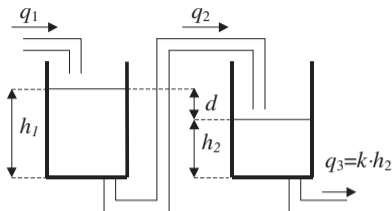


**States**,  $x_1 = V_1$  and  $x_2 = V_2$

- ↪ Measurable but cannot be manipulated
- ↪ They are influenced by the inputs

$$\rightsquigarrow x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

## State-space representation (cont.)



For an incompressible fluid, by mass conservation

$$\begin{cases} \frac{dV_1(t)}{dt} = q_1(t) - q_2(t) \\ \frac{dV_2(t)}{dt} = q_2(t) - q_3(t) = q_2(t) - k \underbrace{\frac{V_2(t)}{B}}_{h_2(t)} \end{cases}$$

By the definition of the output,

$$\begin{aligned} d(t) &= h_1(t) - h_2(t) \\ &= \frac{V_1(t)}{B} - \frac{V_2(t)}{B} \end{aligned}$$

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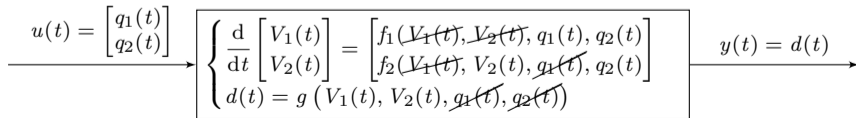


## State-space representation (cont.)

Summarising, we have

$$\left\{ \begin{array}{l} \frac{dV_1(t)}{dt} = \underbrace{q_1(t) - q_2(t)}_{f_1(V_1(t), V_2(t), q_1(t), q_2(t))} \\ \frac{dV_2(t)}{dt} = \underbrace{q_2(t) - k \frac{V_2(t)}{B}}_{f_2(V_1(t), V_2(t), q_1(t), q_2(t))} \\ d(t) = \underbrace{\frac{V_1(t)}{B} - \frac{V_2(t)}{B}}_{g(V_1(t), V_2(t), q_1(t), q_2(t))} \end{array} \right.$$

Diagrammatically,



## State-space representation (cont.)

Rearranging terms, the state-space representation of the two-tank system

$$\rightsquigarrow \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t) \\ \dot{x}_2(t) = -k/Bx_2(t) + u_2(t) \\ y(t) = x_1(t)/B - x_2(t)/B \end{cases}$$

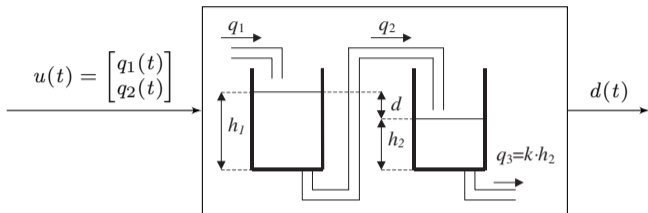
The model is set of ordinary differential equations and an algebraic equation

- State variables,  $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$
- Input (control) variables  $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$
- Output (measurement) variables =  $u(t) = [y(t)] = [d(t)]$

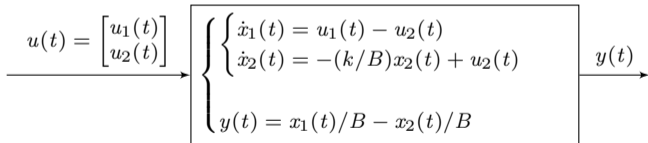
Quantity  $k$  and  $B$  are constant (do not change w/ time), system parameters

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}V_1(t) \\ \frac{d}{dt}V_2(t) \end{bmatrix}$$

## State-space representation (cont.)



Equivalently,



## State-space representation (cont.)

The SS model of a system describes how the evolution (the change in time)  $\dot{x}(t) \in \mathcal{R}^{N_x}$  of the system state depends on the state  $x(t) \in \mathcal{R}^{N_x}$  itself and on the input  $u(t) \in \mathcal{R}^{N_u}$

- The **state equation**
- A set of differential equations

$$\begin{cases} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

The SS model of a system also describes how the system output  $y(t) \in \mathcal{R}^{N_y}$  depends on system state  $x(t) \in \mathcal{R}^{N_x}$  and on system input  $u(t) \in \mathcal{R}^{N_u}$

- The **output transformation**
- A set of algebraic equations

$$\begin{cases} y_1(t) = g_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ y_2(t) = g_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ y_{N_y}(t) = g_{N_y}[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

---

For compactness, we used  $u(t) = [u_1(t), u_2(t), \dots, u_{N_u}(t)]$

## State-space representation (cont.)

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The state equation is a set of  $N_x$  first-order ordinary differential equations

- Regardless of the fact that the system is SISO or MIMO

The output transformation is a scalar or vectorial algebraic equation

- Depending on the number  $p$  of output variables

## State-space representation (cont.)

The SS model of a SISO ( $y(t) \in \mathcal{R}^{N_y=1}$  and  $u(t) \in \mathcal{R}^{N_u=1}$ ) system with  $N_x$  states

$$\begin{cases} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_1(t), \dots, x_{N_x}(t), u(t), t] \\ y(t) = g[x_1(t), \dots, x_{N_x}(t), u(t), t] \end{cases}$$

Let  $\dot{x}(t) \in \mathcal{R}^{N_x}$  be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \rightsquigarrow \begin{cases} \dot{x}(t) = f[x(t), u(t), t] \\ y(t) = g[x(t), u(t), t] \end{cases}$$

$f$  is a multi-parametric vectorial function with  $i$ -th component  $f_i$ ,  $i = 1, \dots, N_x$

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## State-space representation (cont.)

The SS model of a MIMO ( $y(t) \in \mathcal{R}^{N_y \neq 1}$  and  $u(t) \in \mathcal{R}^{N_u \neq 1}$ ) system with  $N_x$  states

$$\left\{ \begin{array}{l} \dot{x}_1(t) = f_1[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \dot{x}_2(t) = f_2[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \vdots \\ \dot{x}_{N_x}(t) = f_{N_x}[x_{N_x}(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \\ y_1(t) = g_1[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ y_2(t) = g_2[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \\ \vdots \\ y_{N_y}(t) = g_{N_y}[x_1(t), \dots, x_{N_x}(t), u_1(t), \dots, u_{N_u}(t), t] \end{array} \right.$$

Let  $\dot{x}(t) \in \mathcal{R}^{N_x}$  be the vector whose components are the derivatives of the state

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} \rightsquigarrow \begin{cases} \dot{x}(t) = f[x(t), u(t), t] \\ y(t) = g[x(t), u(t), t] \end{cases}$$

$f$  and  $g$  are multi-parametric vectorial functions depending on the system

- $f_i$  with  $i = 1, \dots, N_x$  and  $g_i$  with  $i = 1, \dots, N_y$

# State-space representation (cont.)

## Linear and linear time-invariant SS representation

A necessary and sufficient condition for a system to be linear is that state equation and output transformation in the SS model are linear equations

$$\left\{ \begin{array}{l}
 \dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,N_x}(t)x_{N_x}(t) + b_{1,1}(t)u_1(t) + \cdots + b_{1,N_u}(t)u_{N_u}(t) \\
 \dot{x}_2(t) = a_{2,1}(t)x_1(t) + \cdots + a_{2,N_x}(t)x_{N_x}(t) + b_{2,1}(t)u_1(t) + \cdots + b_{2,N_u}(t)u_{N_u}(t) \\
 \vdots \\
 \dot{x}_{N_x}(t) = \\
 \quad a_{N_x,1}(t)x_1(t) + \cdots + a_{N_x,N_x}(t)x_{N_x}(t) + b_{N_x,1}(t)u_1(t) + \cdots + b_{N_x,N_u}(t)u_{N_u}(t) \\
 \\
 y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,N_x}(t)x_{N_x}(t) + d_{1,1}(t)u_1(t) + \cdots + d_{1,N_u}(t)u_{N_u}(t) \\
 y_2(t) = c_{2,1}(t)x_1(t) + \cdots + c_{2,N_x}(t)x_{N_x}(t) + d_{2,1}(t)u_1(t) + \cdots + d_{2,N_u}(t)u_{N_u}(t) \\
 \vdots \\
 y_{N_y}(t) = \\
 \quad c_{N_y,1}(t)x_1(t) + \cdots + c_{N_y,N_x}(t)x_{N_x}(t) + d_{N_y,1}(t)u_1(t) + \cdots + d_{N_y,N_u}(t)u_{N_u}(t)
 \end{array} \right.$$

Process systems

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$$\rightsquigarrow \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

$$\rightsquigarrow A(t) = \{a_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_x}$$

$$\rightsquigarrow B(t) = \{b_{i,j}(t)\} \in \mathcal{R}^{N_x \times N_u}$$

$$\rightsquigarrow C(t) = \{c_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_x}$$

$$\rightsquigarrow D(t) = \{d_{i,j}(t)\} \in \mathcal{R}^{N_y \times N_u}$$

Coefficient matrices  $A(t)$ ,  $B(t)$ ,  $C(t)$  and  $D(t)$  are time dependent (varying)

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} &= \begin{bmatrix} a_{1,1}(t) & \cdots & a_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ a_{N_x,1}(t) & \cdots & a_{N_x,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} \\ &+ \begin{bmatrix} b_{1,1}(t) & \cdots & b_{1,N_y}(t) \\ \vdots & \ddots & \vdots \\ b_{N_x,1}(t) & \cdots & b_{N_x,N_y}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} &= \begin{bmatrix} c_{1,1}(t) & \cdots & c_{1,N_x}(t) \\ \vdots & \ddots & \vdots \\ c_{N_y,1}(t) & \cdots & c_{N_y,N_x}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} \\ &+ \begin{bmatrix} d_{1,1}(t) & \cdots & d_{1,N_u}(t) \\ \vdots & \ddots & \vdots \\ d_{N_y,1}(t) & \cdots & d_{N_y,N_u}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix} \end{aligned}$$

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representation

$$\rightsquigarrow \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$\rightsquigarrow A = \{a_{i,j}\} \in \mathcal{R}^{N_x \times N_x}$$

$$\rightsquigarrow B = \{b_{i,j}\} \in \mathcal{R}^{N_x \times N_u}$$

$$\rightsquigarrow C = \{c_{i,j}\} \in \mathcal{R}^{N_y \times N_x}$$

$$\rightsquigarrow D = \{d_{i,j}\} \in \mathcal{R}^{N_y \times N_u}$$

Coefficient matrices  $A$ ,  $B$ ,  $C$  and  $D$  are time independent (fixed)

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{N_x}(t) \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,N_u} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,N_x} \\ \vdots & \ddots & \vdots \\ c_{N_y,1} & \cdots & c_{N_y,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} + \begin{bmatrix} d_{1,1} & \cdots & d_{1,N_u} \\ \vdots & \ddots & \vdots \\ d_{N_y,1} & \cdots & d_{N_y,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

## State-space representation (cont.)

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Common to choose as state those variables that characterise energy within the system

Consider a system in which there is energy stored, its state is not zero

- The system will evolve even in the absence of external inputs

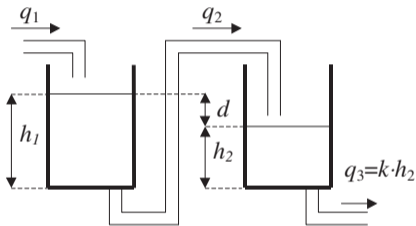
The state can be understood as a possible (internal) cause of evolution

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- For a cylindrical tank of base  $B$  and liquid level  $h(t)$ , the potential energy at time  $t$  is  $E_p(t) = 1/2\rho gV^2(t)/B$ , with  $\rho$  the density of the liquid and  $V(t) = Bh(t)$ .  $V(t)$  or equivalently  $h(t)$  can be used as state variable
- For a spring with elastic constant  $k$ , the potential energy at time  $t$  is  $E_k(t) = 1/2kz^2(t)$  with  $z(t)$  the spring deformation with respect to an equilibrium position.  $z(t)$  can be used as state variable
- For a mass  $m$  moving with speed  $v(t)$  on a plane, the kinetic energy at time  $t$  is  $E_m(t) = 1/2mv^2(t)$ .  $v(t)$  can be used as state of the system

## Example

## Two tanks (SS representation, reloaded)



## First tank

- Inflow, rate  $q_1$  [ $\text{m}^3\text{s}^{-1}$ ]
- Outflow, rate  $q_2$  [ $\text{m}^3\text{s}^{-1}$ ]
- $h_1$  is the liquid level [m]

## Second tank

- Inflow, rate  $q_2$  [ $\text{m}^3\text{s}^{-1}$ ]
- Outflow, rate  $q_3$  [ $\text{m}^3\text{s}^{-1}$ ]
- $h_2$  is the liquid level [m]

Each of the tanks can store a certain amount of potential energy

- The amount of energy depends on the liquid volumes

The complete (two-tank) system has order  $N_x = 2$