

CHEM-E7190
2020-2021

State feedback
and estimation

Duality



Aalto University

Controlled LTI processes, with an observer

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Francesco Corona

Chemical and Metallurgical Engineering
School of Chemical Engineering

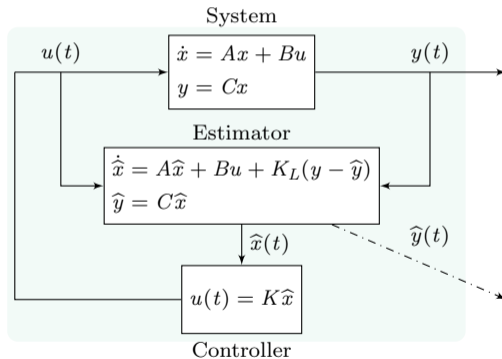
Putting things together

LTI systems

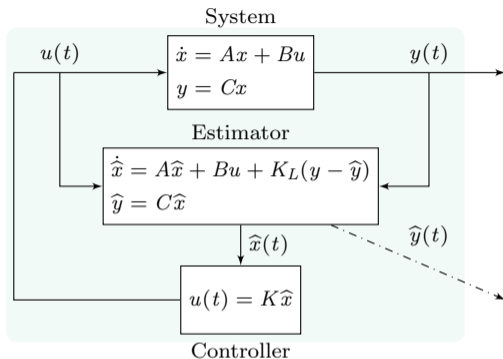
Putting things together

The state-feedback controller when the state is not-measurable requires a state-observer

- This is possible, if and only if the system is observable
- Asymptotic state observer, the Luenberger observer



Putting things together (cont.)



We can determine the controller gain K , by placing the eigenvalues of $A - BK$

- Possible, iff the system is controllable

$$\dot{x}(t) = (A - KB)x(t)$$

We can determine the observer gain K_L , by placing the eigenvalues of $A - K_L C$

$$\dot{e}(t) = (A - K_L C)e(t)$$

Putting things together (cont.)

Theorem

Consider the linear time-invariant system in $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Consider the state-feedback control law

$$u(t) = -K\hat{x}(t)$$

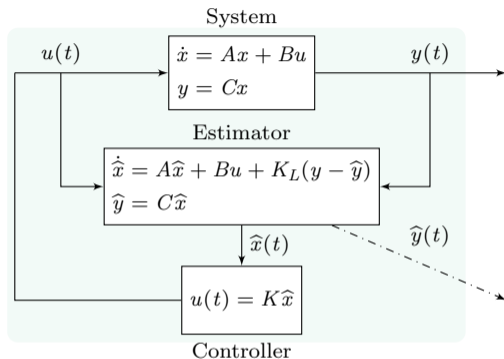
$\hat{x}(t) \in \mathcal{R}^{N_x}$ denotes the estimate of state $x(t)$ obtained with the Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t))$$

$$\hat{y}(t) = C\hat{x}(t)$$

The resulting closed-loop system is a dynamical system of order $2 \times N_x$ whose eigenvalues are the union of the N_x eigenvalues of $A - KB$ and the N_x eigenvalues of $A - K_L C$

Putting things together (cont.)



$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix}}_{A_{CLL}} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

Putting things together (cont.)

Consider the dynamic equation for the system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) - Bu(t) \\ &= Ax(t) - BK\hat{x}(t)\end{aligned}$$

Consider the dynamic equation for the Luenberger observer,

$$\begin{aligned}\dot{x}(t) &= A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t)) \\ &= A\hat{x}(t) + Bu(t) + K_L(Cx(t) - C\hat{x}(t)) \\ &= A\hat{x}(t) + Bu(t) + K_L Cx(t) - K_L C\hat{x}(t) \\ &= (A - K_L C)\hat{x}(t) + Bu(t) + K_L Cx(t) \\ &= (A - K_L C)\hat{x}(t) - BK\hat{x}(t) + K_L Cx(t) \\ &= K_L Cx(t) + (A - BK - K_L C)\hat{x}(t)\end{aligned}$$

Therefore, we have the two dynamic equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BK\hat{x}(t) \\ \dot{\hat{x}}(t) &= K_L Cx(t) + (A - BK - K_L C)\hat{x}(t)\end{aligned}$$

Putting things together (cont.)

$$\dot{x}(t) = Ax(t) - BK\hat{x}(t)$$

$$\dot{\hat{x}}(t) = K_L Cx(t) + (A - BK - K_L C)\hat{x}(t)$$

In the more compact matrix notation, we have the closed-loop dynamics with observer

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix}}_{A_{CLL}} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

We want to know about the eigenvalues of A_{CLL} , the dynamics of closed-loop system

- Are they the union of the eigenvalues of $(A - BK)$ and $(A - K_L C)$?

Putting things together (cont.)

Consider the following similarity transformation P of the (augmented) state vector

$$\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = P \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

We consider a specific similarity transformation P such that

$$P = \underbrace{\begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix}}_{(N_x \times N_x)} = P^{-1}$$

We have the relation between the (augmented) state vectors

$$\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

- $x(t) = z(t)$
- $\hat{x}(t) = z(t) - \hat{z}(t)$

Putting things together (cont.)

Because the similarity transformation is invertible, for the transformed state we have

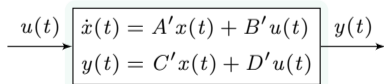
$$\begin{aligned} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} &= P^{-1} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \\ &= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ 0_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \end{aligned}$$

The augmented state satisfies the new state-space representation with state matrix A'

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = A'_{CLL} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

\rightsquigarrow Where we have that $A'_{CLL} = P^{-1} A_{CLL} P$

For a similarity transformation $x(t) = Pz(t)$,



System

$$\begin{cases} \dot{z}(t) = A'z(t) + B'u(t) \\ y(t) = C'z(t) + D'u(t) \end{cases}$$

$$\rightsquigarrow A' = P^{-1}AP$$

$$\rightsquigarrow B' = P^{-1}B$$

$$\rightsquigarrow C' = CP$$

$$\rightsquigarrow D' = D$$

Putting things together (cont.)

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = A'_{CLL} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

For the state matrix A'_{CLL} , we have

$$\begin{aligned} A'_{CLL} &= P^{-1} A_{CLL} P \\ &= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix} \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \\ &= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A - BK & BK \\ \cancel{K_L C} + A - BK - \cancel{K_L C} & -A + BK + K_L C \end{bmatrix} \\ &= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A - BK & BK \\ A - BK & -A + BK + K_L C \end{bmatrix} \\ &= \begin{bmatrix} A - BK & BK \\ \cancel{A - BK} - \cancel{A - BK} & BK + A - \cancel{BK} - K_L C \end{bmatrix} \\ &= \begin{bmatrix} A - BK & BK \\ 0 & A - K_L C \end{bmatrix} \end{aligned}$$

Putting things together (cont.)

The transformed dynamics A'_{CLL} are represented by an upper-block-triangular matrix

- Its eigenvalues are the eigenvalues of the blocks along the diagonal
- The controller $A - BK$ and the observer $A - K_L C$ state matrix

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_L C \end{bmatrix} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} \begin{matrix} \xrightarrow{y(t)} \\ \searrow \hat{y}(t) \end{matrix}$$

Similarity transformations do not modify the eigenvalues of the original state matrix

↪ The eigenvalues of A_{CLL} are equal to those of A'_{CLL}

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \begin{matrix} \xrightarrow{y(t)} \\ \searrow \hat{y}(t) \end{matrix}$$

Duality controllability-observability

LTI systems

Duality controllability-observability

Consider the linear and time invariant state-space model (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

- Dimensions $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

Consider the linear time invariant state-space model (A^T, B^T, C^T)

$$\dot{z}(t) = A^T z(t) + C^T v(t)$$

$$s(t) = B^T z(t)$$

- Dimensions $z(t) \in \mathcal{R}^{N_x}$, $v(t) \in \mathcal{R}^{N_u}$, and $s(t) \in \mathcal{R}^{N_y}$

Duality controllability-observability (cont.)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$\dot{z}(t) = A^T z(t) + C^T v(t)$$

$$s(t) = B^T z(t)$$

The dimension of all the system(s) matrices

$$A \in \mathcal{R}^{N_x \times N_x}$$

$$B \in \mathcal{R}^{N_x \times N_u}$$

$$C \in \mathcal{R}^{N_y \times N_x}$$

$$A^T \in \mathcal{R}^{N_x \times N_x}$$

$$B^T \in \mathcal{R}^{N_u \times N_x}$$

$$C^T \in \mathcal{R}^{N_x \times N_y}$$

Duality controllability-observability (cont.)

Theorem

We have the linear and time invariant system

$$\mathcal{S}_1 \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

We have the linear and time-invariant system

$$\mathcal{S}_2 \quad \begin{cases} \dot{z}(t) = A^T z(t) + C^T v(t) \\ s(t) = B^T z(t) \end{cases}$$

We have the following result,

- ↪ System \mathcal{S}_1 is controllable if and only if system \mathcal{S}_2 is observable
- ↪ System \mathcal{S}_1 is observable if and only if system \mathcal{S}_2 is controllable

Duality controllability-observability (cont.)

Proof

Let \mathcal{C}_i and \mathcal{O}_i with $i = 1, 2$ be the controllability and the observability matrices of \mathcal{S}_i

We have,

$$\begin{aligned}\mathcal{C}_1 &= [B \quad AB \quad A^2B \quad \dots \quad A^{N_x-1}B] \\ &= \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^{N_x-1} \end{bmatrix} \\ &= \mathcal{O}_2^T\end{aligned}$$

Similarly, we have

$$\mathcal{O}_1 = \mathcal{C}_2^T$$



Duality controllability-observability (cont.)

Example

Consider the linear and time-invariant dynamical system

$$\dot{x}(t) = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 0 \end{bmatrix} x(t)$$

The dual system,

$$\dot{z}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} z(t) + \begin{bmatrix} 3 \\ 0 \end{bmatrix} v(t)$$

$$s(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} z(t)$$

$$C_1 = \begin{bmatrix} 2 & 4 \\ 3 & 11 \end{bmatrix} = O_2^T$$

$$O_1 = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} = C_2^T$$