Aalto University

Controlled LTI processes, with an observer

CHEM-E7190 (was E7140), 2023

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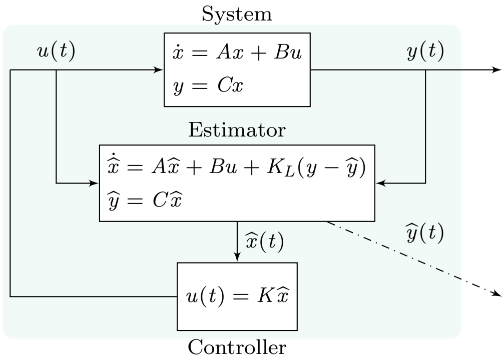
Putting things together

LTI systems

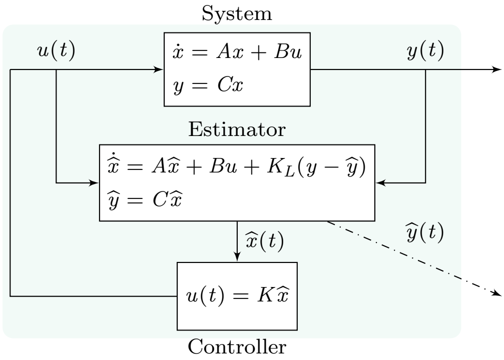
Putting things together

The state-feedback controller when the state is not-measurable requires a state-observer

- This is possible, if and only if the system is observable
- Asymptotic state observer, the Luenberger observer



Putting things together (cont.)



We can determine the controller gain K, by placing the eigenvalues of $A - BK$

- Possible, iff the system if controllable

$$\dot{x}(t) = (A - KB)\, x(t)$$

We can determine the observer gain K_L, by placing the eigenvalues of $A - K_L C$

$$\dot{e}(t) = (A - K_L C)\, e(t)$$

Putting things together (cont.)

Theorem

Consider the linear time-invariant system in $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

Consider the state-feedback control law

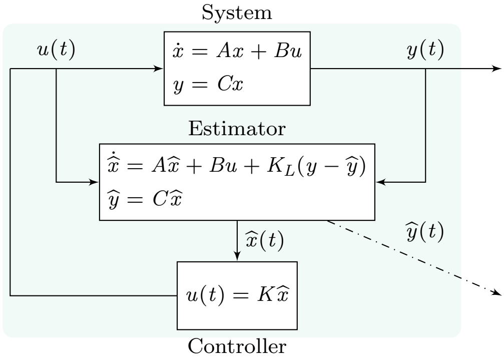
$$u(t) = -K\widehat{x}(t)$$

$\widehat{x}(t) \in \mathcal{R}^{N_x}$ denotes the estimate of state $x(t)$ obtained with the Luenberger observer

$$\begin{aligned}\dot{\widehat{x}}(t) &= A\widehat{x}(t) + Bu(t) + K_L(y(t) - \widehat{y}(t))\\ \widehat{y}(t) &= C\widehat{x}(t)\end{aligned}$$

The resulting closed-loop system is a dynamical system of order $2 \times N_x$ whose eigenvalues are the union of the N_x eigenvalues of $A - KB$ and the N_x eigenvalues of $A - K_LC$

Putting things together (cont.)



$$\begin{bmatrix} \dot{x}(t) \\ \dot{\widehat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix}}_{A_{CLL}} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix}$$

Putting things together (cont.)

Consider the dynamic equation for the system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) - Bu(t) \\ &= Ax(t) - BK\widehat{x}(t)\end{aligned}$$

Consider the dynamic equation for the Luenberger observer,

$$\begin{aligned}\dot{x}(t) &= A\widehat{x}(t) + Bu(t) + K_L\left(y(t) - \widehat{y}(t)\right) \\ &= A\widehat{x}(t) + Bu(t) + K_L\left(Cx(t) - C\widehat{x}(t)\right) \\ &= A\widehat{x}(t) + Bu(t) + K_L Cx(t) - K_L C\widehat{x}(t) \\ &= \left(A - K_L C\right)\widehat{x}(t) + Bu(t) + K_L Cx(t) \\ &= \left(A - K_L C\right)\widehat{x}(t) - BK\widehat{x}(t) + K_L Cx(t) \\ &= K_L Cx(t) + \left(A - BK - K_L C\right)\widehat{x}(t)\end{aligned}$$

Therefore, we have the two dynamic equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) - BK\widehat{x}(t) \\ \dot{\widehat{x}}(t) &= K_L Cx(t) + \left(A - BK - K_L C\right)\widehat{x}(t)\end{aligned}$$

Putting things together (cont.)

$$\dot{x}(t) = Ax(t) - BK\widehat{x}(t)$$
$$\dot{\widehat{x}}(t) = K_L Cx(t) + (A - BK - K_L C)\,\widehat{x}(t)$$

In the more compact matrix notation, we have the closed-loop dynamics with observer

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\widehat{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix}}_{A_{CLL}} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix}$$

We want to know about the eigenvalues of A_{CLL}, the dynamics of closed-loop system

- Are they the union of the eigenvalues of $(A - BK)$ and $(A - K_L C)$?

Putting things together (cont.)

Consider the following similarity transformation P of the (augmented) state vector

$$\begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix} = P \begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix}$$

We consider a specific similarity transformation P such that

$$P = \underbrace{\begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix}}_{(N_x \times N_x)} = P^{-1}$$

We have the relation between the (augmented) state vectors

$$\begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix} = \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix}$$

- $x(t) = z(t)$
- $\widehat{x}(t) = z(t) - \widehat{z}(t)$

Putting things together (cont.)

Because the similarity transformation is invertible, for the transformed state we have

$$\begin{aligned}\begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix} &= P^{-1} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix} \\ &= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix}\end{aligned}$$

The augmented state satisfies the new state-space representation with state matrix A'

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\widehat{z}}(t) \end{bmatrix} = A'_{CLL} \begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix}$$

⇝ Where we have that $A'_{CLL} = P^{-1} A_{CLL} P$

For a similarity transformation $x(t) = Pz(t)$,

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= A'x(t) + B'u(t) \\ y(t) &= C'x(t) + D'u(t) \end{aligned}} \longrightarrow y(t)$$

System

$$\begin{cases} \dot{z}(t) = A'z(t) + B'u(t) \\ y(t) = C'z(t) + D'u(t) \end{cases}$$

⇝ $A' = P^{-1} A P$

⇝ $B' = P^{-1} B$

⇝ $C' = CP$

⇝ $D' = D$

Putting things together (cont.)

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\widehat{z}}(t) \end{bmatrix} = A'_{CLL} \begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix}$$

For the state matrix A'_{CLL}, we have

$$\begin{aligned}
A'_{CLL} &= P^{-1} A_{CLL} P \\
&= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix} \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \\
&= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A - BK & BK \\ \cancel{K_L C} + A - BK - \cancel{K_L C} & -A + BK + K_L C \end{bmatrix} \\
&= \begin{bmatrix} I_{N_x} & 0_{N_x} \\ I_{N_x} & -I_{N_x} \end{bmatrix} \begin{bmatrix} A - BK & BK \\ A - BK & -A + BK + K_L C \end{bmatrix} \\
&= \begin{bmatrix} A - BK & BK \\ \cancel{A} - \cancel{BK} - \cancel{A} - \cancel{BK} & \cancel{BK} + A - \cancel{BK} - K_L C \end{bmatrix} \\
&= \begin{bmatrix} A - BK & BK \\ 0 & A - K_L C \end{bmatrix}
\end{aligned}$$

Putting things together (cont.)

The transformed dynamics A'_{CLL} are represented by an upper-block-triangular matrix

- Its eigenvalues are the eigenvalues of the blocks along the diagonal
- The controller $A - BK$ and the observer $A - K_L C$ state matrix

$$\boxed{\begin{bmatrix} \dot{z}(t) \\ \dot{\widehat{z}}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_L C \end{bmatrix} \begin{bmatrix} z(t) \\ \widehat{z}(t) \end{bmatrix}}$$

$y(t)$

$\widehat{y}(t)$

Similarity transformations do not modify the eigenvalues of the original state matrix

⇝ The eigenvalues of A_{CLL} are equal to those of A'_{CLL}

$$\boxed{\begin{bmatrix} \dot{x}(t) \\ \dot{\widehat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ K_L C & A - BK - K_L C \end{bmatrix} \begin{bmatrix} x(t) \\ \widehat{x}(t) \end{bmatrix}}$$

$y(t)$

$\widehat{y}(t)$

Duality controllability-observability

LTI systems

Duality controllability-observability

Consider the linear and time invariant state-space model (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Dimensions $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

Consider the linear time invariant state-space model (A^T, B^T, C^T)

$$\dot{z}(t) = A^T z(t) + C^T v(t)$$
$$s(t) = B^T z(t)$$

- Dimensions $z(t) \in \mathcal{R}^{N_x}$, $v(t) \in \mathcal{R}^{N_u}$, and $s(t) \in \mathcal{R}^{N_y}$

Duality controllability-observability (cont.)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$$\dot{z}(t) = A^T z(t) + C^T v(t)$$
$$s(t) = B^T z(t)$$

The dimension of all the system(s) matrices

$$A \in \mathcal{R}^{N_x \times N_x}$$
$$B \in \mathcal{R}^{N_x \times N_u}$$
$$C \in \mathcal{R}^{N_y \times N_x}$$

$$A^T \in \mathcal{R}^{N_x \times N_x}$$
$$B^T \in \mathcal{R}^{N_u \times N_x}$$
$$C^T \in \mathcal{R}^{N_x \times N_y}$$

Duality controllability-observability (cont.)

Theorem

We have the linear and time invariant system

$$\mathcal{S}_1 \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

We have the linear and time-invariant system

$$\mathcal{S}_2 \quad \begin{cases} \dot{z}(t) = A^T z(t) + C^T v(t) \\ s(t) = B^T z(t) \end{cases}$$

We have the follwing result,

- ⇝ System $\mathcal{S}_1$ is controllable if and only if system $\mathcal{S}_2$ is observable
- ⇝ System $\mathcal{S}_1$ is observable if and only if system $\mathcal{S}_2$ is controllable

Duality controllability-observability (cont.)

Proof

Let $\mathcal{C}_i$ and $\mathcal{O}_i$ with $i = 1, 2$ be the controllability and the observability matrices of $\mathcal{S}_i$

We have,

$$\begin{aligned}
\mathcal{C}_1 &= \begin{bmatrix} B & AB & A^2B & \cdots & A^{N_x-1}B \end{bmatrix} \\
&= \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^{N_x-1} \end{bmatrix} \\
&= \mathcal{O}_2^T
\end{aligned}$$

Similarly, we have

$$\mathcal{O}_1 = \mathcal{C}_2^T$$

■

Duality controllability-observability (cont.)

Example

Consider the linear and time-invariant dynamical system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 3 & 0 \end{bmatrix} x(t) \end{aligned}$$

The dual system,

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} z(t) + \begin{bmatrix} 3 \\ 0 \end{bmatrix} v(t) \\ s(t) &= \begin{bmatrix} 2 & 3 \end{bmatrix} z(t) \end{aligned}$$

$$\mathcal{C}_1 = \begin{bmatrix} 2 & 4 \\ 3 & 11 \end{bmatrix} = \mathcal{O}_2^T$$

$$\mathcal{O}_1 = \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} = \mathcal{C}_2^T$$