Aalto University

Linear time-invariant processes: Control

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State-feedback control

LTI systems - Control

State-feedback control

We studied solutions to homogeneous linear and time-invariant systems $\dot{x}(t) = Ax(t)$

$\rightsquigarrow$ The force-free response, from initial condition $x(0) \neq 0$

$$\underbrace{x(t) = e^{At}x(0)}_{\text{force-free response } x_u(t)} \qquad (u(t) = 0, \text{ for } t \geq 0)$$

- The stability from the eigenvalues of state matrix A

$$\det(\lambda I - A) = 0$$

- Eigenvalues and eigenvectors of A for diagonalisation

$$\dot{z}(t) = Dz(t)$$

The forced response, from initial condition $x(0) = 0$ and some input $u(t) \neq 0$ for $t \geq 0$

- A weighted sum of the input $u(t)$, with weighting function $e^{A(t-\tau)}B$

$$\underbrace{x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{\text{forced response } x_f(t)}$$

State-feedback control (cont.)

System

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}} \longrightarrow y(t)$$

We can compute the complete response, from $x(0) \neq 0$ and with $u(t) \neq 0$ for $t \geq 0$

- Because of linearity, by superposition of the force-free and forced response

$$x(t) = x_u(t) + x_f(t)$$

We have the exact time-evolution of the state variables

$$x(t) = \underbrace{e^{At}x(0)}_{x_u(t)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{x_f(t)}$$

We also have the time-evolution for the measurements

$$y(t) = C\underbrace{\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau\right)}_{x(t)} + Du(t)$$

State-feedback control (cont.)

System

$$u(t) \longrightarrow \boxed{\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}} \longrightarrow y(t)$$

Controlling a process consists of designing a device, the **controller**, that computes a function $u(t)$, the temporal sequence of **control actions**, capable to steer the system

- $\rightsquigarrow$ From any initial state $x(t_0)$, at time $t_0 = 0$
- $\rightsquigarrow$ To any final state $x(t_f)$, at time t_f
- $\rightsquigarrow$ In a finite time interval, t_f

To proceed, we need to verify whether the system under study is actually controllable

- We must check whether it is always possible to determine function $u(t)$
- It is required that such function exist for any pair $\big(x(t_0), x(t_f)\big)$

State-feedback control (cont.)

Definition

One, formal, definition of controllability

A linear and time-invariant system $\dot{x}(t) = Ax(t) + Bu(t)$ is said to be controllable if and only if, it is possible to transfer it from any arbitrary initial state $x(t_0)$ to any final state $x(t_f)$, in finite time ($t_f < \infty$), by choosing an appropriate control $u(t)$

In this definition, the understanding is that the sequence of control actions $u(t)$ is capable of influencing the evolution of all the state variables, through the integral

$$\int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau$$

Remember the general form of the forced response, the first Lagrange equation,

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau$$

Let $t_0 = 0$ and $t = t_f$ and assume without loss of generality that $x(t_0) = 0$,

$$\rightsquigarrow \quad x(t_f) = \int_{0}^{t_f} e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau$$

State-feedback control (cont.)

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

From the forced response, we see that controllability must depend only on A and B

We make some simplifying assumptions that will help us focusing on control

- The assumptions have no implications on the general results

We assume that we can measure all state variables and no feedthrough

- That is, we have $C = I$ and $D = 0$

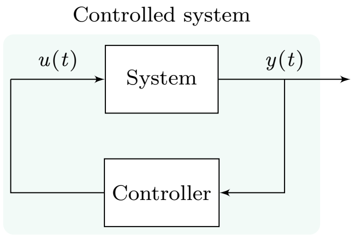
$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{C=I} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{D=0} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)}$$

Incidentally, think about the practical meaning of having $D = 0$ (a common situation)

State feedback (cont.)

One particular thing that we are interested in is to *re-shape* the system's dynamics

- In particular, we want the (controlled) system to be stable, if it is not
- We want the (controlled) system to be fast/slower, if already stable



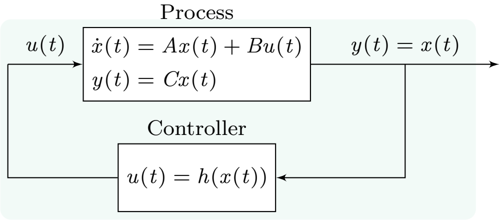
How to manipulate (control) the system, through the design of control actions $u(t)$?

State-feedback control (cont.)

The idea of state feedback control is to design a $u(t)$ which depends on the state $x(t)$

State-feedback control

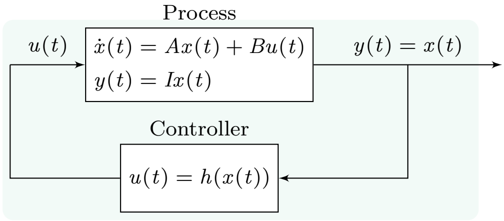
$$\rightsquigarrow \quad u(t) = h\left(x(t)\right)$$



The control $u(t)$ used to manipulate the system is a function of state $x(t)$, we think of the controller as a device that transforms the state and feeds it back into the system

- Function $h(\cdot)$ transforms knowledge about the state of the system $x(t)$
- Function $h(\cdot)$ converts the state into an appropriate control action $u(t)$
- This operation is repeated at each time point t

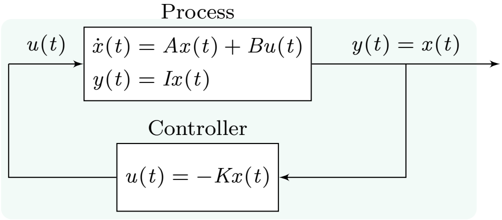
State-feedback control (cont.)



The pair process-controller defines a system that is autonomous, no external inputs

- Function A, B and C (I) are known, may coming from linearisation
- Function h must be determined, the objective of control design

State-feedback control (cont.)

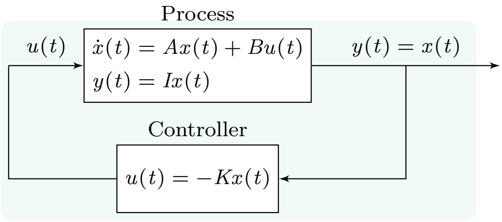


For linear time-invariant system, function $h(x(t)) = -Kx(t)$ is optimal, in some sense

$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)} = - \underbrace{\underbrace{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,N_x} \\ k_{21} & k_{22} & \cdots & k_{2,N_x} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N_u,1} & k_{N_u,2} & \cdots & k_{N_u,N_x} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)}}_{h(x(t))}$$

- For a system in state $x(t)$, the optimal control action is $u(t) = -Kx(t)$
- (We will briefly also discuss the underlaying optimality criterion)

State-feedback control (cont.)



Among all possible functions $h(\cdot)$ that can be used to transform the state $x(t)$ of the system into an optimal control action $u(t)$, a matrix K, size $N_u \times N_x$, is all is needed

- ⇝ When applied to the state, $h(\cdot)$ will generate the best control action
- ⇝ To drive the system to zero state (stabilisation/regulation task)
- Only one requirement, the system must be controllable

Matrix K is called the **closed-loop gain matrix**

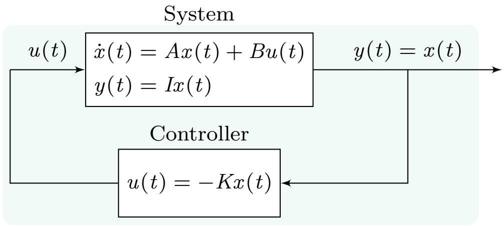
- In general, $K = K(t)$, a function of time

State feedback (cont.))

We have perfect measurement (observation) variables $y(t)$ (from the system's sensors)

- We assume that $y(t)$ returns all state variables $x(t)$, $y(t) = Ix(t)$

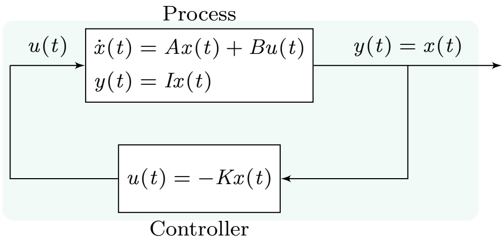
We have system $\dot{x}(t) = Ax(t) + Bu(t)$, we can perfectly measure its state $x(t) = y(t)$



We design controllers that define an optimal control action $u(t)$, given the state $x(t)$

$$\rightsquigarrow \quad u(t) = -Kx(t)$$

State-feedback control (cont.)



There exist several procedures that can be used to determine gain K, we discuss two

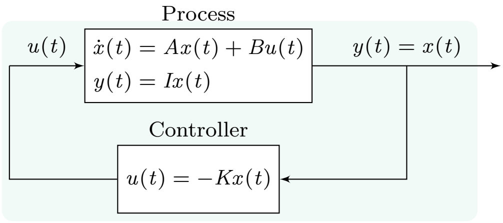
- In general, note that the correct answer depends on the specific control task
- Also, note that we will derive only solutions that do not enforce constraints[1]

[1] In process control, there are always control and state constraints that must be satisfied. The control constraints are imposed by the technological limits on the actuators. The state constrains are physical limits or desirables. They are important, we cover those in CHEM-E7225.

State-feedback control (cont.)

We could choose gain K that impose predetermined dynamics to the closed-loop system

- Remember, the (open-loop) dynamics of the system are given by matrix A
- (Specifically, its 'stability' properties are determined by its eigenvalues)



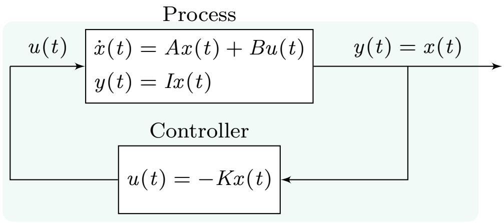
The resulting closed-loop system is homogeneous

- We already know how to treat it

In the controlled system, process and controller operate together as a new system

- The closed-loop system has its own dynamics
- We derive its state-space representation

State-feedback control (cont.)



We have the state and measurement equations or the open-loop system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Ix(t) \end{cases}$$

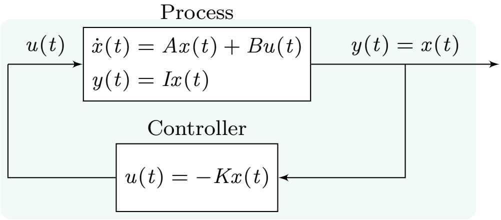
We know the optimal controller equation

$$u(t) = -Kx(t)$$

State-feedback control (cont.)

We can substitute $u(t)$, to get

$$\begin{cases} \dot{x}(t) = Ax(t) + B\left(\underbrace{-Kx(t)}_{u(t)}\right) \\ y(t) = x(t) \end{cases}$$



We can rearrange terms, to get the dynamics of the autonomous (controlled) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BKx(t) \\ &= \underbrace{\left(\underbrace{A}_{N_x \times N_x} - \underbrace{B}_{N_x \times N_u} \underbrace{K}_{N_u \times N_x}\right)}_{A_{CL}} x(t) \\ &= \underbrace{A_{\mathrm{CL}}}_{N_x \times N_x} x(t) \end{aligned}$$

State-feedback control (cont.)

The dynamics of the closed-loop system are then represented by matrix $A_{\mathrm{CL}} = A - BK$

The measurements are not changed

- ⇝ $u(t)$ does not affect $y(t)$
- ⇝ $y(t) = Ix(t)$

Controlled process

$$\dot{x}(t) = \underbrace{(A - BK)}_{A_{CL}} x(t)$$

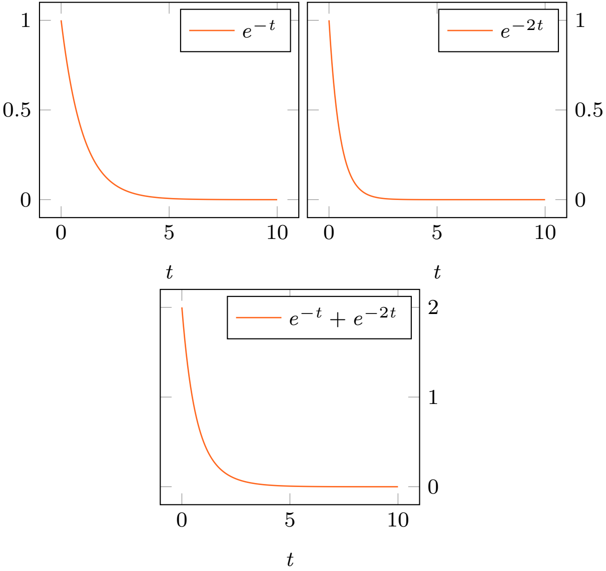
$$y(t) = Ix(t)$$

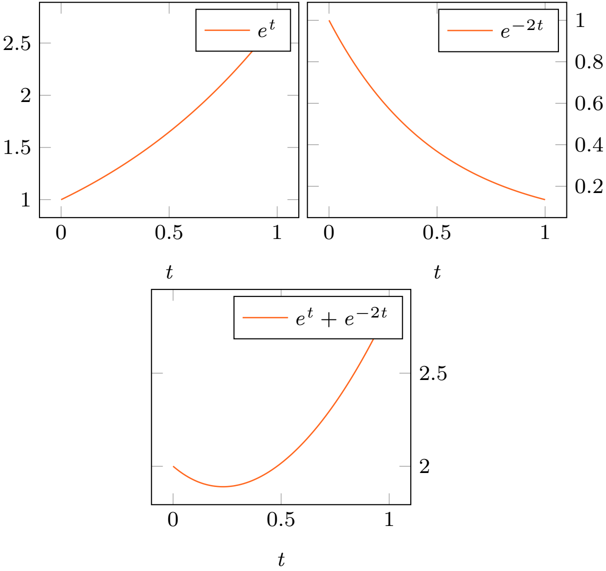
$$y(t) = x(t)$$

Matrix A and B are given by the process model that we are interested to control

- The dynamics of the closed-loop are given by $A_{\mathrm{CL}} = (A - BK)$
- Only matrix K must be chosen, in some sensible way
- Different choices of K will affect A_{CL}, the dynamics

We will focus our attention on what happens to the eigenvalues of A_{CL} (stability)


e^{-t}
e^{-2t}
$e^{-t} + e^{-2t}$
1
0.5
0
2
1
0
0
5
10
t


e^t
2.5
2
1.5
1
0
0.5
1
t
e^{-2t}
1
0.8
0.6
0.4
0.2
0
0.5
1
t
$e^t + e^{-2t}$
2.5
2
0
0.5
1
t

State-feedback control (cont.)

Two major cases can be considered, they are both based on the original dynamics, A

- If A is an unstable matrix, then we could choose K that renders A_{CL} stable
- If A is a stable matrix, then we choose K such that A_{CL} remains stable

Controlled process

$$\dot{x}(t) = A_{CL}x(t)$$
$$y(t) = Ix(t)$$

$$y(t) = x(t)$$

'To choose K' means to place the eigenvalues of $A_{\text{CL}} = A - BK$ at desirable locations

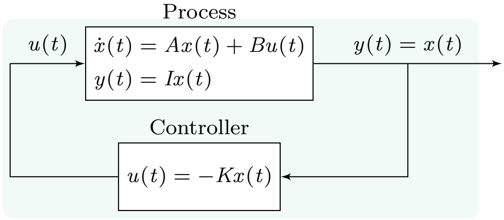
- This operation can performed if and only if the system is **controllable**

State-feedback control (cont.)

In practical terms, controllability means that we are able to choose any gain matrix K

- In such a way that the eigenvalues of $A_{\text{CL}} = A - BK$ can be anywhere

When can we claim that some system is controllable? Can we test for controllability?



We discovered that the controllability of a system only depends on the pair (A, B)

- On the dynamics of the homogeneous system (its stability properties)
- On how the inputs (the choice of actuators) affect the state variables

Matrix K does not affect controllability, when chosen it defines the control strategy

State-feedback control (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = \begin{bmatrix} -0.4 & 0 \\ 0.2 & -0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.2 \\ -0.5 & 0 \end{bmatrix} u(t)$$

- Determine stability of matrix A and its eigenvalues

Assuming that the process is controllable, suggest a place for the eigenvalues for A_{CL}

- What is the control objective that you wanted to pursue?

State-feedback control (cont.)

Example

Consider two linear and time-invariant linear systems with pairs (A_1, B_1) and (A_2, B_2)

$$\left(A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$\left(A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

- Determine stability of matrices A and their eigenvalues

Assuming that the process is controllable, suggest a place for the eigenvalues for A_{CL}

- What are the control objectives that you wanted to pursue?

State-feedback control (cont.)

Example

Consider the linear and time-invariant systems (A, B), $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

- State matrix A is not a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

- State matrix A is a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Determine stability of matrices A and their eigenvalues

Assuming that the process is controllable, suggest a place for the eigenvalues for A_{CL}

- What are the control objectives that you wanted to pursue?

State-feedback control (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & +1 \\ 0.1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Determine stability of matrix A and its eigenvalues

Assuming that the process is controllable, suggest a place for the eigenvalues for A_{CL}

- What is the control objective that you wanted to pursue?

State-feedback control (cont.)

The notion of state feedback is valid whatever the complexity of the process model

- ⇝ The solution however can be computationally demanding
- ⇝ Some classes of problems have simple solutions
- ⇝ Linear and time-invariant dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Another condition for simplicity is quadratic cost functions in state vars and inputs

$$\underset{u(\cdot)}{\text{minimise}} \int_{t_0}^{\infty} \Big(\underbrace{x'(t)Qx(t)}_{\textit{Distance of } x(t) \text{ from zero}} + \underbrace{u'(t)Ru(t)}_{\textit{Magnitude of } u(t)} \Big) \mathrm{d}t,$$

The sequence of controls $u(t_0 \rightsquigarrow \infty)$ that would drive all state variables $x(t)$ to zero

- As quickly as possible, over an infinite-horizon, and with the smallest effort

Q and R are user-defined matrices of size $(N_x \times N_x)$ and $(N_u \times N_u)$, respectively

- They are understood as tuning parameters
- They must satisfy certain properties
- ($Q \geq 0$ and $R > 0$)

State-feedback control (cont.)

$$\underset{u(\cdot)}{\text{minimise}} \int_{t_0}^{\infty} \Big(\underbrace{x'(t)Qx(t)}_{\textit{Distance of } x(t) \text{ from zero}} + \underbrace{u'(t)Ru(t)}_{\textit{Magnitude of } u(t)} \Big)\mathrm{d}t,$$

Consider the integrand, cost function $l(x(t), u(t))$ at time t is the sum of two terms

$$L(x(\cdot), u(\cdot)) = \int_{t_0}^{\infty} \Big(\underbrace{x'(t)Qx(t) + u'(t)Ru(t)}_{l(x(t),u(t))} \Big)\mathrm{d}t$$

The two terms are conventional numbers, they are added inside the integral

- The integral, then repeats this summation along time

State-feedback control (cont.)

$$L(x(\cdot), u(\cdot)) = \int_{t_0}^{\infty} \Big(\underbrace{x'(t) Q x(t)} + u'(t) R u(t) \Big) \mathrm{d}t$$

- The first term is the (squared) distance between current state $x(t)$ and zero

$$\underbrace{x'(t) Q x(t)}_{\geq 0} = (x(t) - 0)' \, Q \, (x(t) - 0)$$

$$= \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_{N_x}(t) \end{bmatrix} Q \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}$$

- Matrix Q is used to define what state variables are more important

State-feedback control (cont.)

$$
\begin{aligned}
x'(t)Qx(t) &= (x(t)-0)'\,Q\,(x(t)-0) \\
&= \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_{N_x} \end{bmatrix} Q \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} \\
& \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_{N_x}(t) \end{bmatrix} \begin{bmatrix} q_{1,1} & 0 & \cdots & 0 \\ 0 & q_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{N_x,N_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix} \\
&= x_1(t)q_{1,1}x_1(t) + x_2(t)q_{2,2}x_2(t) + \cdots + x_{N_x}(t)q_{N_x,N_x}x_{N_x}(t) \\
&= \sum_{n_x=1}^{N_x} q_{n_x,n_x} x_{n_x}^2
\end{aligned}
$$

In general, the farthest the state is from zero, the largest is the cost term at time t

State-feedback control (cont.)

$$L(x(\cdot), u(\cdot)) = \int_{t_0}^{\infty} \Big(x'(t) Q x(t) + \underbrace{u'(t) R u(t)} \Big) \mathrm{d}t$$

- Second term is the (squared) distance between input $u(t)$ and zero input

$$\underbrace{u'(t) R u(t)}_{\geq 0} = (u(t) - 0)' \, R \, (u(t) - 0)$$

$$= \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_{N_u}(t) \end{bmatrix} R \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}$$

- Matrix R is used to define what input variables are more important

State-feedback control (cont.)

$$
\begin{aligned}
u'(t)Ru(t) &= (u(t)-0)' \, R \, (u(t)-0) \\
&= \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_{N_u} \end{bmatrix} R \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix} \\
&= \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_{N_u} \end{bmatrix} \begin{bmatrix} r_{1,1} & 0 & \cdots & 0 \\ 0 & r_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{N_u,N_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdots \\ u_{N_u}(t) \end{bmatrix} \\
&= u_1(t) r_{1,1} u_1(t) + u_2(t) r_{2,2} u_2(t) + \cdots + u_{N_u}(t) r_{N_u,N_u} u_{N_u}(t) \\
&\rightsquigarrow \sum_{n_u=1}^{N_u} r_{n_u,n_u} u_{n_u}^2(t)
\end{aligned}
$$

In general, the farthest the input is from zero, the largest is the cost term at time t

State-feedback control (cont.)

How to determine function $h(\cdot)$ such that $u(t) = h\left(x(t)\right)$ is optimal for the process?

$$\underset{u(\cdot)}{\text{minimise}} \int_{t_0}^{\infty} \left(x'(t)Qx(t) + u'(t)Ru(t)\right) \mathrm{d}t$$

To develop an intuition on how to design such a controller, switch to discrete-time

$$\underset{u(0),u(1),\dots,u(\infty)}{\text{minimise}} \sum_{k=0}^{\infty} \left(x'(k)Qx(k) + u'(k)Ru(k)\right)$$

Then, consider a finite-horizon of length K (rather than an infinitely long one)

$$\underset{u(0),u(1),\dots,u(\infty)}{\text{minimise}} \sum_{k=0}^{K} \left(x'(k)Qx(k) + u'(k)Ru(k)\right)$$

State-feedback control (cont.)

$$\underset{u(0),u(1),\dots,u(\infty)}{\text{minimise}} \sum_{k=0}^{K} \left(x'(k) Q x(k) + u'(k) R u(k) \right)$$

Single out the last time step, when time is up and we cannot apply a control nomore

$$\underset{u(0),u(1),\dots,u(K-1)}{\text{minimise}} x'(K) Q_f x(K) + \sum_{k=0}^{K-1} \left(x'(k) Q x(k) + u'(k) R u(k) \right)$$

The last terms measures how far we are from zero, when the time is over

- Matrix Q_f is used to define what state variables are more important
- At the final time, in general it could be that $Q_f = Q$

State-feedback control (cont.)

$$\underset{u(0),u(1),\dots,u(K-1)}{\text{minimise}} \; x'(K)Q_f x(K) + \sum_{k=0}^{K-1} \big(x'(k)Qx(k) + u'(k)Ru(k) \big)$$

We are explicitly looking for a specific sequence of control actions $u(1), u(2), \dots, u(K)$

- One that drives the system from an initial state $x(0)$ to the zero state
- Such that the cost function given as sum of terms is the smallest
- Given the we know the dynamics of the process

$$x(k+1) = Ax(k) + Bu(k)$$

We are also assuming that the initial state $x(0)$ is known (that is, we measured it)

How to solve this optimisation problem?

State-feedback control (cont.)

Example

Consider a linear and time-invariant process with single state variable and single input

The system dynamics, in discrete-time

$$x(k+1) = ax(k) + bu(k), \quad \text{with } x(k), u(k) \in \mathcal{R}$$

The control problem, in discrete-time

$$\underset{u(0),u(1),\ldots,u(K-1)}{\text{minimise}} \; x'(K)q_f x(K) + \sum_{k=0}^{K-1} \big(x'(k)qx(k) + u'(k)ru(k)\big)$$

Consider a finite-horizon of length one ($K = 1$)

$$\underset{u(0)}{\text{minimise}} \; x'(1)q_f x(1) + \sum_{k=0}^{0} \big(x'(k)qx(k) + u'(k)ru(k)\big)$$

We have,

$$\underset{u(0)}{\text{minimise}} \; x'(1)q_f x(1) + x'(0)qx(0) + u'(0)ru(0)$$

State-feedback control (cont.)

In this simple case, we only need to (optimise to) find a single control action, $u(0)$

- Under the constraint that $x(1) = ax(0) + bu(0)$
- The initial state $x(0)$ is known

We have,

$$\underset{u(0)}{\text{minimise}} \quad \underbrace{x'(1)}_{ax(0)+bu(0)} \quad q_f \quad \underbrace{x(1)}_{ax(0)+bu(0)} \quad +x'(0)qx(0) + u'(0)ru(0)$$

All the terms in the cost function are known, with the exception of $u(0)$

- It is the decision variable, it is a scalar

State-feedback control (cont.)

$$\underset{u(0)}{\text{minimise}} \quad \underbrace{x'(1)}_{ax(0)+bu(0)} \quad q_f \quad \underbrace{x(1)}_{ax(0)+bu(0)} \quad + x'(0)qx(0) + u'(0)ru(0)$$

Substituting and rearranging, we have a quadratic equation $u(0)$

$$\underset{u(0)}{\text{minimise}} \; \underbrace{qx^2(0) + ru^2(0) + q_f(ax(0) + bu(0))^2}_{f(u(0))}$$

- We are interested in value $u(0)$ that minimise this function

---

After some algebra, we see that the cost function is a parabola

$$\begin{aligned} f(u(0)) &= qx^2(0) + ru^2(0) + q_f(ax(0) + bu(0)) \\ &= (q + a^2 q_f)x^2(0) + 2(b\,a\,q_f\,x(0))u(0) + (b^2 q_f + r)u^2(0) \end{aligned}$$

We know where the minimum of parabola (its vertex) is ...

State-feedback control (cont.)

$$f(u(0)) = (q + a^2 q_f) x^2(0) + 2(b\, a\, q_f\, x(0)) u(0) + (b^2 q_f + r) u^2(0)$$

$f(u(0))$ is a parabola and it is smallest at the value $u(0)$ that makes its derivative zero

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u(0)} f(u(0)) &= 2 b\, q_f\, a\, x(0) + 2(b^2 q_f + r) u(0) \\ &= 0\end{aligned}$$

We have the solution to the optimisation/control problem

$$\begin{aligned}u(0) &= -\underbrace{\frac{2 b\, q_f\, a}{2(b^2 q_f + r)}}_{k} x(0) \\ &= -k x(0)\end{aligned}$$

(Remember the requirement $R > 0$?)

For systems with multiple state variables and multiple inputs, the structure is identical

$$u(0) = -\underbrace{\left(B' Q_f B + R\right)^{-1} B' Q_f A}_{K}\, x(0)$$

Controllability

LTI systems - Control

Controllability

Controllability refers to the possibility for the system to reach a specified final state

- Given an arbitrary value of the initial time and of the initial state

$$u(t) \longrightarrow \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}} \xrightarrow{\;y(t)\;}$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Controllability for linear and time-invariant systems depends only on the pair (A, B)

$$\rightsquigarrow \quad \dot{x}(t) = Ax(t) + Bu(t)$$

We present a formal definition of controllability for linear time-invariant systems

- Necessary and sufficient conditions and invariance under similarity

Controllability (cont.)

Definition

Controllability

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$x(t) = Ax(t) + Bu(t)$$

The system is said to be **controllable**, if and only if it is possible to transfer the state of the system from any initial value $x_0 = x(0)$ to any other final value $x_f = x(t_f)$

- ..., only by manipulating the input $u(t)$
- ..., in some finite time $t_f \geq 0$

The final state x_f is called the **zero-state** or the **target-state**

Controllability (cont.)

We analyse the controllability of a linear time-invariant system by using three criteria

- **Controllability gramian**
- **Controllability matrix**
- (**Popov-Belevich test**)

All these criteria are complementary, as for their practical usefulness

Controllability (cont.)

Definition

Controllability gramian

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$x(t) = Ax(t) + Bu(t)$$

The system's **controllability gramian** is a $(N_x \times N_x)$ matrix, real and symmetric

$$W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} \mathrm{d}\tau$$

Theorem

Controllability test (I)

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$x(t) = Ax(t) + Bu(t)$$

Let $W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} \mathrm{d}\tau$ be the controllability gramian of the system

- The system is controllable iff $W_c(t)$ is non-singular, for all $t > 0$

Controllability (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^2$ and $u(t) \in \mathcal{R}$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

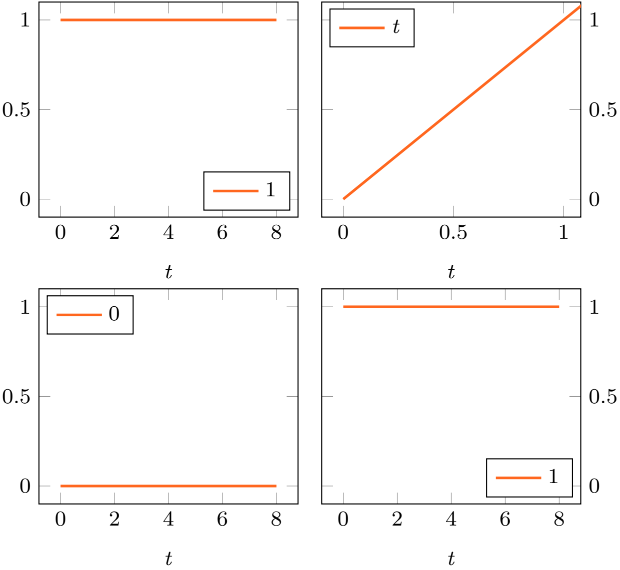
Let $x(0) = (0, 0)'$, we are interested in verifying the controllability of the system

- Firstly, we need to compute its controllability gramian
- Then, we must determine whether its invertible

To compute the controllability gramian, we need the state transition matrix

$$\begin{aligned} e^{A\tau} &= e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\tau} = e^{\begin{bmatrix} 0 & \tau \\ 0 & 0 \end{bmatrix}} \\ &= \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Controllability (cont.)

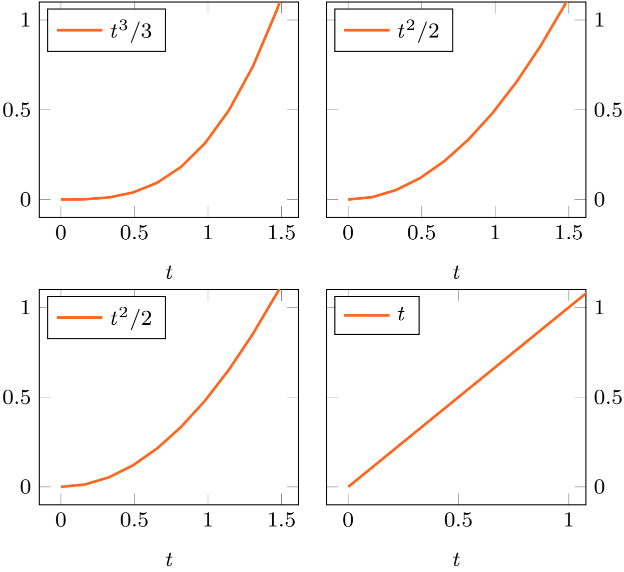


Controllability (cont.)

We can compute the controllability gramian of the system, by applying the definition

$$
\begin{aligned}
W_c(t) &= \int_0^t e^{A\tau} B B^T e^{A^T \tau} \mathrm{d}\tau \\
&= \int_0^t \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \mathrm{d}\tau \\
&= \int_0^t \begin{bmatrix} \tau^2 & \tau \\ \tau & 1 \end{bmatrix} \mathrm{d}\tau \\
&= \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix}
\end{aligned}
$$

Controllability (cont.)



Controllability (cont.)

$$W_c(t) = \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix}$$

To verify whether the controllability gramian $W_c(t)$ is singular, check determinant

- We need to check whether it is zero or it is non-zero
- Whatever the value of t (that is, at any time)

$$\begin{aligned} \det\left(W_c(t)\right) &= t^4/3 - t^4/4 \\ &= t^4/12 \\ &> 0 \quad (\forall t > 0) \end{aligned}$$

Since $\det\left(W_c(t)\right) \neq 0$ for all $t > 0$, we can conclude that the system is controllable

Controllability (cont.)

Theorem

Controllability matrix and controllability test (II)

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

We define the $(N_x \times (N_u \times N_x))$ **controllability matrix**

$$\mathcal{C} = \begin{bmatrix} B & | & AB & | & A^2B & | & \cdots & | & A^{N_x-1}B \end{bmatrix}$$

Necessary and sufficient condition for controllability

$$\text{rank}(\mathcal{C}) = N_x$$

Controllability (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix} u(t)$$

We are interested in verifying its controllability using the controllability matrix

The controllability matrix has dimensions $(N_x = 3 \times (N_u = 2 \times N_x = 3)) = (3 \times 6)$

$$\mathcal{C} = \begin{bmatrix} B & | & AB & | & A^2B \end{bmatrix}$$

We know B, we need to compute AB and A^2B,

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1.5 \\ 0 & 1.5 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

Controllability (cont.)

$$
\begin{aligned}
A^2B &= A(AB) \\
&= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1.5 \\ 0 & 1.5 \\ 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 12 \\ 0 & 9 \\ 0 & 12 \end{bmatrix}
\end{aligned}
$$

Thus, we have the controllability matrix

$$
\mathcal{C} = \begin{bmatrix} 1 & 0 & 2 & 1.5 & 4 & 12 \\ 0 & 0 & 0 & 1.5 & 0 & 9 \\ 0 & 3 & 0 & 6 & 0 & 12 \end{bmatrix}
$$

We check controllability from its rank,

$$
\begin{aligned}
\operatorname{rank}(\mathcal{C}) &= 3 \\
&= N_x
\end{aligned}
$$

Controllability (cont.)

```
>> help ctrb                                    % CTRB computes the controllability matrix
                                                % of pair (A,B)
                                                % Read about it and how to use it

>> A = [?];                                     % Define state matrix A
>> B = [?];                                     % Define control matrix B

>> [Nx,Nu] = size(B);                           % Nx and Nu

>> Cmat = ctrb(A,B)                             % Controllability matrix

>> rnkCmat = rank(Cmat)                         % Rank of the controllability matrix

>> rnkCmat == Nx                                % Return 0/1 for controllabilty
```

Controllability (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = \begin{bmatrix} -0.4 & 0 \\ 0.2 & -0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0.2 \\ -0.5 & 0 \end{bmatrix} u(t)$$

Determine their controllability, by checking the rank of the controllability matrix $\mathcal{C}$

Example

Consider two linear and time-invariant linear systems with pairs (A_1, B_1) and (A_2, B_2)

$$\left(A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
$$\left(A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Determine their controllability, by checking the rank of the controllability matrix $\mathcal{C}$

Controllability (cont.)

Example

Consider the linear and time-invariant systems (A, B), $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

- State matrix A is not a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

- State matrix A is a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Determine their controllability, by checking the rank of the controllability matrix $\mathcal{C}$

Controllability (cont.)

Example

Consider the linear and time-invariant systems (A, B), $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

- State matrix A is not a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- State matrix A is a stable matrix

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Determine their controllability, by checking the rank of the controllability matrix $\mathcal{C}$

Controllability (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & +1 \\ 0.1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Determine the stability of matrix A and the system controllability from the pair (A, B)

Controllability (cont.)

$$\mathcal{C} = \underbrace{\begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{N_x-1}B \end{bmatrix}}_{N_x \times (N_u \times N_x)}$$

The rank controllability test states that, for controllability, $\mathcal{C}$ need be full column-rank

- The controllability matrix $\mathcal{C}$ must have N_x independent columns
- The columns of $\mathcal{C}$ must span the entire state-space, $\mathcal{R}^{N_x}$

Conversely, if $\text{rank}(\mathcal{C}) < N_x$ then there exist directions in $\mathcal{R}^{N_x}$ that cannot be reached

$\rightsquigarrow$ Hence, the uncontrollability of the system

---

That is, a system is said to be controllable if and only if $\mathcal{C}$ is full-rank, $\text{rank}(\mathcal{C}) = N_x$

- It is a simple notion, and it is binary (only 'Yes/No' information)
- Controllability is not a concept that can be quantified

Controllability tests only reports on whether a system is controllable or not-controllable

Controllability (cont.)

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{N_x-1}B \end{bmatrix}$$

To build an intuition on what the controllability matrix is, we resort to discrete time

Consider a linear and time-invariant system with dynamics in discrete-time,

$$x(k+1) = Ax(k) + Bu(k), \quad \text{with } x(k) \in \mathcal{R}^{N_x}, u(k) \in \mathcal{R}$$

- At time $k = 0$, system is at state $x(0) = 0$ and we apply an input $u(0) = 1$

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ &= Ax(0) + B \\ &= B \end{aligned}$$

- At time $k = 1$, system is at state $x(1) = B$ and we apply input $u(1) = 0$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= Ax(1) \\ &= A(B) \end{aligned}$$

Controllability (cont.)

- At time $k = 2$, system is at state $x(2) = AB$ and we apply input $u(2) = 0$

$$\begin{aligned} x(3) &= Ax(2) + Bu(2) \\ &= Ax(2) \\ &= A(AB) \end{aligned}$$

- At time $k = 3$, system is at state $x(3) = A^2B$ and we apply input $u(3) = 0$

$$\begin{aligned} x(4) &= Ax(3) + Bu(3) \\ &= Ax(3) \\ &= A(A^2B) \end{aligned}$$

- ...
- At $k = N_x - 2$, system is in $x(N_x - 2) = A^{N_x-3}B$, we apply $u(N_x - 2) = 0$

$$\begin{aligned} x(N_x - 1) &= Ax(N_x - 2) + Bu(N_x - 2) \\ &= Ax(N_x - 2) \\ &= A(A^{N_x-2}B) \\ &= A^{N_x-1}B \end{aligned}$$

Controllability (cont.)

The system started from an initial condition corresponding to the origin $x(0) = 0$

Then, it started evolving in this coordinate system subjected to a unitary input

- Firstly, it moved along direction B,
- Secondly, along direction AB,
- Thirdly, direction A^2B
- $\ldots$

If the system moves along all these directions and they are independent of each other, then $\mathcal{C}$ is full-rank, or rank$\mathcal{C} = N_x$, showing that it can reach any point (state) in $\mathcal{R}^{N_x}$

- That is, we can make it visit any place in the N_x-dimensional state-space
- If this condition is verified, we can claim that the system is controllable

Controllability (cont.)

There exist system's realisations for which the controllability analysis can be simplified

- For example, a system whose matrix A is diagonal, with distinct eigenvalues

Theorem

Controllability for diagonal representations

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N_x} \end{bmatrix} x(t) + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,N_u} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,N_u} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N_x,1} & b_{N_x,2} & \cdots & b_{N_x,N_u} \end{bmatrix} u(t)$$

Matrix A is diagonal and suppose that all of its eigenvalues are distinct

$$\lambda_i \neq \lambda_j, \quad \text{for all } i \neq j$$

Necessary and sufficient condition for controllability of the system (A, B)

- Matrix B must not have any row whose elements are all zero

Controllability (cont.)

Example

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^{N_x}$ and $u(t0) \in \mathcal{R}^{N_u}$

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

The state matrix A is diagonal and its eigenvalues are all real and distrinct

- $\lambda_1 = 1$
- $\lambda_2 = 2$
- $\lambda_3 = 3$

The third row of the input matrix B is equal zero, system is not controllable

Controllability (cont.)

Controllability of a linear and time-invariant system is not specific to the realisation

- Controllability is invariant with respect to any similarity transformation

Theorem

Consider two realisations (A, B) and (A', B') of a linear and time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$\dot{z}(t) = A'z(t) + B'u(t)$$

- $x(t) = Pz(t) \in \mathcal{R}^{N_x}$
- $u(t) \in \mathcal{R}^{N_u}$

(We assume that the similarity transformation matrix $P \in \mathcal{R}^{N_x \times N_x}$ is non-singular)

The first realisation is controllable if and only if the second one is controllable

Controllability (cont.)

Proof

Consider the controllability matrix $\mathcal{C}'$ associated to the second realisation

$$
\begin{aligned}
\mathcal{C}' &= \left[B' | A'B' | \cdots | A'^{n-1}B'\right] \\
&= [P^{-1}B | P^{-1}AP \cdot P^{-1}B | \cdots | \cdots \\
&\qquad \cdots | \overbrace{P^{-1}AP^{-1} \cdot P^{-1}AP \cdots P^{-1}AP}^{(n-1)\text{ times}} \cdot P^{-1}B] \\
&= \left[P^{-1}B | P^{-1}AB | \cdots | P^{-1}A^{n-1}B\right] \\
&= P^{-1}\left[B | AB | \cdots | A^{n-1}B\right] \\
&= P^{-1}\mathcal{C}
\end{aligned}
$$

Matrix P is non-singular, the controllability matrices have the same rank

■

Controllability (cont.)

Example

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^2$ and $u(t) \in \mathcal{R}$

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} -4 \\ 7 \end{bmatrix} u(t)$$

Consider the following similarity transformation matrix and its inverse

$$P = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

As $A' = P^{-1}AP$ and $B' = P^{-1}B$, we can write the realisation

$$\begin{aligned} \dot{z}(t) &= A'z(t) + B'u(t) \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u(t) \end{aligned}$$

We are interested in the controllability of the system

Controllability (cont.)

We can compute the controllability matrix $\mathcal{C}$ and $\mathcal{C}'$ associated to the two realisations

$$\begin{aligned}\mathcal{C} &= \left[B|AB\right] \\ &= \begin{bmatrix} -4 & 10 \\ 7 & -16 \end{bmatrix} \\ \mathcal{C}^{-1} &= \left[B'|A'B'\right] \\ &= \begin{bmatrix} 2 & -2 \\ 3 & -6 \end{bmatrix}\end{aligned}$$

We have that $\mathcal{C} = P^{-1}\mathcal{C}$, with both matrices that are square and full-rank

$$\begin{aligned}\text{rank}(\mathcal{C}) &= \text{rank}(\mathcal{C}') \\ &= 2 \quad (N_x)\end{aligned}$$

Controllability (cont.)

Example

Consider a linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^3$ and $u(t) \in \mathcal{R}^2$

$$\dot{x}(t) = \begin{bmatrix} 2 & -3 & -2 \\ 0 & 1 & 0 \\ 0 & 3 & 4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} u(t)$$

Consider the following similarity transformation matrix and its inverse

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We are interested in the controllability of the system

Controllability (cont.)

As matrix A has distinct eigenvalues, we write a realisation with a diagonal matrix A'

$$\dot{z}(t) = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A'=P^{-1}AP} z(t) + \underbrace{\begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 4 & 2 \end{bmatrix}}_{B'=P^{-1}B} u(t)$$

Since the input matrix B has no null rows, we conclude that the system is controllable

- Controllability could be checked also from the controllability matrix