Aalto University

Linear time-invariant processes: Dynamics

CHEM-E7190 (was E7140), 2023

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Representation and analysis

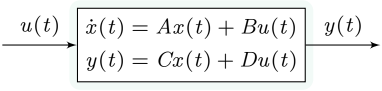
LTI systems - Dynamics

Representation and analysis

Consider a linear and time-invariant system of order N_x, in **state-space** representation

- ⇝ Let N_x be the number of outputs
- ⇝ Let N_u be the number of inputs

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



System

A $(N_x \times N_x)$, B $(N_x \times N_u)$, C $(N_y \times N_x)$ and D $(N_y \times N_u)$ are the system matrices

- ⇝ $x(t)$ is the **state vector**
  - (N_x components)
- ⇝ $\dot{x}(t)$ is the derivative of the state vector
  - (N_x components)
- ⇝ $u(t)$ is the **input vector**
  - (N_u components)
- ⇝ $y(t)$ is the **output vector**
  - (N_y components)

Representation and analysis (cont.)

The analysis problem: Determine the behaviour of state $x(t)$ and output $y(t)$ for $t \geq t_0$

- We are given the input function $u(t)$, for $t \geq t_0$
- We are given the initial state $x(t_0)$

The solution to the analysis, for $t \geq t_0$, an initial state $x(t_0)$ and an input $u(t \geq t_0)$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{Cx(t)} + Du(t)$$

The solution is known as the **Lagrange formula**

- Based on the **state transition matrix**

$\rightsquigarrow$ e^{At}

Force-free and forced evolution

Note that we can write the state solution $x(t)$, for $t \geq t_0$, as the sum of two terms

$$x(t) = \underbrace{e^{A(t-t_0)}x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{x_f(t)}$$

$$= x_u(t) + x_f(t)$$

- The **force-free evolution** of the state, $x_u(t)$
- The **forced evolution** of the state, $x_f(t)$

The **force-free evolution** of the state, from the initial condition $x(t_0)$

- $e^{A(t-t_0)}$ determines the transition from $x(t_0)$ to $x(t)$
- In the absence of contribution from the input

The **forced evolution** of the state, from the contribution of input $u(t)$

- In the absence of an initial condition $x(t_0)$

The state transition matrix

LTI systems - Dynamics

The state transition matrix

Consider a square $(N_x \times N_x)$ matrix A, the exponential e^A is square $(N_x \times N_x)$ matrix

$$\rightsquigarrow \quad e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

The **state transition matrix** is the matrix exponential e^{At} of the matrix At

- ⇝ It is a matrix whose elements are functions of time
- ⇝ We discuss its meaning and how to compute it

The state transition matrix (cont.)

The exponential function

Let z be some scalar, by definition its exponential is a scalar

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

The series always converges

The matrix exponential

Let A be a $(n \times n)$ matrix, by definition its exponential is a $(n \times n)$ matrix

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

The series always converges

The state transition matrix (cont.)

The product of several matrices

The product of matrix A and B is only possible when the matrixes are compatible

- Number of columns of A must equal the number of rows of B

The same applies to the product of several matrixes

$$\underbrace{M}_{(m \times n)} = \underbrace{A_1}_{(m \times m_1)} \underbrace{A_2}_{(m_1 \times m_2)} \cdots \underbrace{A_{k-1}}_{(m_{k-2} \times m_{k-1})} \underbrace{A_k}_{(m_{k-1} \times n)}$$

Powers of a matrix

Let A be an order-n square matrix, we want to define the k-th power of matrix A

The k-th power of matrix A is the n-order matrix A^k

$$A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$$

Some special cases,

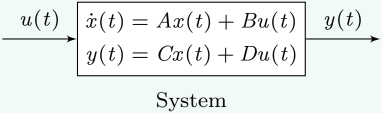
- $\rightsquigarrow$ $A^{k=0} = I$
- $\rightsquigarrow$ $A^{k=1} = A$

The state transition matrix (cont.)

Definition

The state transition matrix

Consider a linear and time-invariant state-space model with $(N_x \times N_x)$ state matrix A



$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

The **state transition matrix** of this system is given by the $(N_x \times N_x)$ matrix e^{At}

$$e^{At} = \underbrace{\frac{A^0 t^0}{0!}}_{I} + \underbrace{\frac{A^1 t^1}{1!}}_{At} + \underbrace{\frac{A^2 t^2}{2!}}_{(A^2 t^2)/2!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

The state transition matrix is well defined for any square matrix A

- (The series always converges)

The state transition matrix (cont.)

It is not convenient to determine the state transition matrix starting from its definition

$\rightsquigarrow$ One exception is when A is (block-)diagonal

The matrix exponential of block-diagonal matrixes

Consider any block-diagonal matrix A, we have

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \quad \rightsquigarrow \quad e^A = \begin{bmatrix} e^{A_1} & 0 & \cdots & 0 \\ 0 & e^{A_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{A_q} \end{bmatrix}$$

The matrix exponential of diagonal matrixes (as special case)

For any diagonal $(n \times n)$ matrix A, we have

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \rightsquigarrow \quad e^A = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

The state transition matrix (cont.)

Example

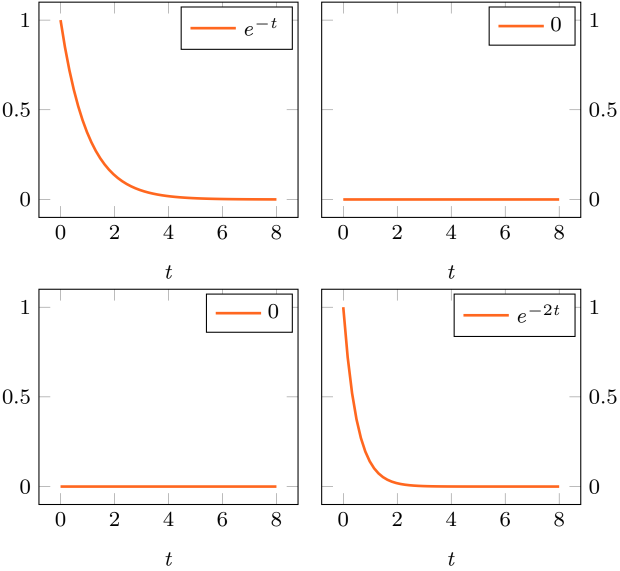
Consider a linear and time-invariant state-space model with (2×2) diagonal matrix A

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

We are interested in the corresponding state transition matrix

We have,

$$e^{At} = \begin{bmatrix} e^{(-1)t} & 0 \\ 0 & e^{(-2)t} \end{bmatrix}$$


e^{-t}
0
0
e^{-2t}
1
0.5
0
0
2
4
6
8
t

Some properties

State transition matrix

Properties

We state without proof some fundamental results about a state transition matrix e^{At}

- ⇝ They are needed to derive Lagrange formula

Proposition

Derivative of the state transition matrix

Consider the state transition matrix e^{At}, we have,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{At} &= Ae^{At} \\ &= e^{At}A\end{aligned}$$

■

By using the derivative property, we have that A commutes with e^{At}

- ⇝ (This result is important)

Properties (cont.)

Proposition

Composition of two state transition matrices

Consider the two state transition matrices e^{At} and $e^{A\tau}$, we have

$$e^{At}e^{A\tau} = e^{A(t+\tau)}$$

Proposition

Inverse of the state transition matrix

Let e^{At} be a state transition matrix, its inverse $\left(e^{At}\right)^{-1}$ is matrix e^{-At}

$$\begin{aligned} e^{At}e^{-At} &= e^{-At}e^{At} \\ &= I \end{aligned}$$

A state transition matrix e^{At} is always invertible (non-singular)

- Even if A were singular

Properties (cont.)

Matrix inverse

Consider a square matrix A of order n

We define the **inverse** of A the square matrix of order n, A^{-1}

$$A^{-1}A = AA^{-1} = I$$

The inverse of matrix A exists if and only if A is non-singular

- When the inverse exists, it is also unique

Sylvester's formula

The state transition matrix

Sylvester's expansion

We determine the analytical expression of the state transition matrix e^{At}

- The procedure is known as **Sylvester expansion**
- (Does not require computing the infinite series)
- There are also other procedures (later)

Proposition

The Sylvester's expansion

Let A be a $(n \times n)$ matrix and let the corresponding state transition matrix be e^{At}

We have,

$$\begin{aligned} e^{At} &= \beta_0(t)I + \beta_1(t)A + \beta_2(t)A^2 + \cdots + \beta_{n-1}(t)A^{n-1} \\ &= \sum_{i=0}^{n-1} \beta_i(t)A^i \end{aligned}$$

The coefficients β_i of the expansion are appropriate functions of time

- ⇝ They can be determined by solving a set of linear equations
- ⇝ There is a finite number (n) of them

Sylvester's expansion (cont.)

We show how to determine the coefficients when A has eigenvalues of multiplicity one

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We will not consider the other cases, because rather involved and tedious to derive

- Matrix A has complex eigenvalues (with multiplicity larger one)
- Matrix A has complex eigenvalues (with multiplicity one)
- Eigenvalues of A have multiplicity larger than one

Sylvester's expansion (cont.)

Eigenvalues with multiplicity one

Let matrix A have distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$
\begin{aligned}
e^{At} &= \sum_{i=0}^{n-1} \beta_i(t) A^i \\
&= \beta_0(t) I + \beta_1(t) A + \beta_2(t) A^2 + \cdots + \beta_{n-1}(t) A^{n-1}
\end{aligned}
$$

The n unknown functions $\beta_i(t)$ are those that solve the system

$$
\rightsquigarrow \quad \begin{cases}
1\beta_0(t) + \lambda_1 \beta_1(t) + \lambda_1^2 \beta_2(t) + \cdots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\
1\beta_0(t) + \lambda_2 \beta_1(t) + \lambda_2^2 \beta_2(t) + \cdots + \lambda_2^{n-1} \beta_{n-1}(t) = e^{\lambda_2 t} \\
\quad \cdots \\
1\beta_0(t) + \lambda_n \beta_1(t) + \lambda_n^2 \beta_2(t) + \cdots + \lambda_n^{n-1} \beta_{n-1}(t) = e^{\lambda_n t}
\end{cases}
$$

Sylvester's expansion (cont.)

Or, equivalently, in matrix form

$$V\beta = \eta$$

- The vector of unknowns

$$\rightsquigarrow \quad \beta = \begin{bmatrix}\beta_0(t) & \beta_1(t) & \cdots & \beta_{n-1}(t)\end{bmatrix}^T$$

- The coefficients matrix[1]

$$\rightsquigarrow \quad V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

- The known vector

$$\rightsquigarrow \quad \eta = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t}\end{bmatrix}^T$$

[1] A matrix in this form is known to be a Vandermonde matrix.

Sylvester's expansion (cont.)

$$\eta = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \end{bmatrix}^T$$

The components of vector η are special functions of time, $e^{\lambda t}$

- ⇝ Functions $e^{\lambda t}$ are the **modes** of matrix A
- ⇝ Mode $e^{\lambda t}$ associates with eigenvalue λ

Each element of e^{At} is a linear combination of such modes

Sylvester expansion (cont.)

Example

Consider a (2×2) matrix A, we want to determine the state transition matrix e^{At}

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

Matrix A is triangular, the eigenvalues correspond to the diagonal elements

Matrix A has 2 distinct eigenvalues

- $\rightsquigarrow$ $\lambda_1 = -1$
- $\rightsquigarrow$ $\lambda_2 = -2$

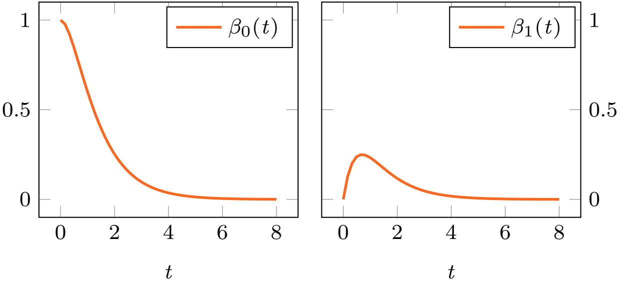
To determine e^{At}, we write the system

$$\begin{cases} 1\beta_0(t) + \lambda_1 \beta_1(t) = e^{\lambda_1 t} \\ 1\beta_0(t) + \lambda_2 \beta_1(t) = e^{\lambda_2 t} \end{cases} \quad \rightsquigarrow \quad \begin{cases} \beta_0(t) + (-1)\beta_1(t) = e^{(-1)t} \\ \beta_0(t) + (-2)\beta_1(t) = e^{(-2)t} \end{cases}$$

Sylvester's expansion (cont.)

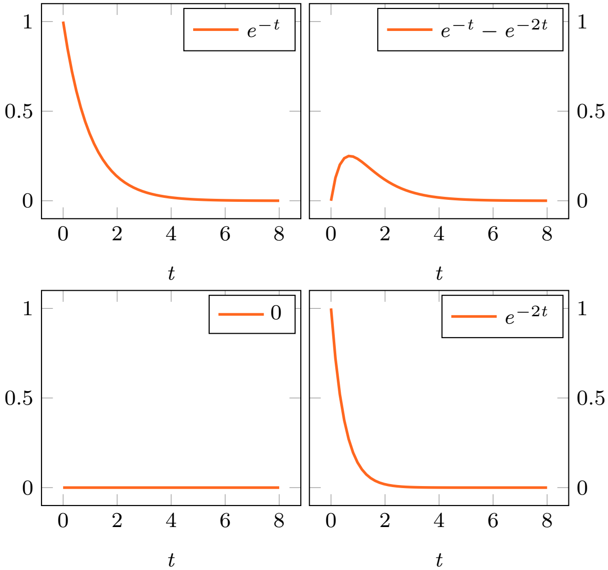
By simple manipulation, we get

$$\rightsquigarrow \quad \begin{cases} \beta_0(t) = 2e^{-t} - e^{-2t} \\ \beta_1(t) = e^{-t} - e^{-2t} \end{cases}$$



Thus,

$$\begin{aligned} e^{At} &= \beta_0(t) I_2 + \beta_1(t) A \\ &= (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$



Each element of e^{At} is a linear combination of the two system modes, e^{-t} and e^{-2t}

Sylvester's expansion (cont.)

Eigenvalues and eigenvectors

Let $\lambda \in \mathcal{R}$ be some scalar and let $v \neq 0$ be some $(n \times 1)$ column vector

Consider a square matrix A of order n, suppose that the identify holds

$$Av = \lambda v$$

The scalar λ is called an **eigenvalue** of A

Vector v is the associated **eigenvector**

Consider a square matrix A of order n whose elements are real numbers

Matrix A has n (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

- They can be real numbers or conjugate-complex pairs
- If $\lambda_i \neq \lambda_j$ for $i \neq j$, A has multiplicity one

Sylvester's expansion (cont.)

Eigenvalues of triangular and diagonal matrices

Let matrix $A = \{a_{i,j}\}$ be a triangular or a diagonal matrix

- The eigenvalues of A are the n diagonal elements $\{a_{i,i}\}$

Sylvester's expansion (cont.)

Characteristic polynomial

The **characteristic polynomial** of a square matrix A of order n

- The n-order polynomial in the variable s

$$P(s) = \det(sI - A)$$

Computing eigenvalues and eigenvectors

The eigenvalues of matrix A of order n solve its characteristic polynomial

$\rightsquigarrow$ The roots of the equation $P(s) = \det(sI - A) = 0$

Let λ be an eigenvalue of matrix A

Each eigenvector v associated to it is a non-trivial solution to the system

$$(\lambda I - A)v = 0$$

Sylvester's expansion (cont.)

Systems of linear equations

Consider a system of n linear equations in n unknowns $Ax = b$

- $\rightsquigarrow$ A is a $(n \times n)$ matrix of **coefficients**
- $\rightsquigarrow$ b is a $(n \times 1)$ vector of **known terms**
- $\rightsquigarrow$ x is a $(n \times 1)$ vector of **unknowns**

If A is non-singular, the system admits one and only one solution

If matrix A is singular, let $M = [A|b]$ be a $[n \times (n+1)]$ matrix

- If $\text{rank}(A) = \text{rank}(M)$, system has infinite solutions
- If $\text{rank}(A) < \text{rank}(M)$, system has no solutions

Sylvester's expansion (cont.)

Matrix rank

The **rank** of a $(m \times n)$ matrix A is equal to the number of columns (or rows) of the matrix that are linearly independent, $\text{rank}(A)$

Matrix kernel or null space

Consider a $(m \times n)$ matrix A, we define its **null space** or **kernel**

$$\text{ker}(A) = \{x \in R^n | Ax = 0\}$$

It is the set of all vectors $x \in \mathcal{R}^n$ that left-multiplied by A produce the null vector

The set is a vector space, its dimension is called the **nullity** of matrix A, $\text{null}(A)$

Lagrange formula

LTI systems - Dynamics

Lagrange formula

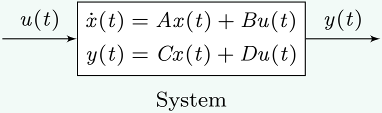
We can now prove the solution to the analysis problem for MIMO systems

- **Lagrange formula**

Theorem

Lagrange formula

Consider the state-space representation of a time-invariant linear system of order n



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The solution for $t \geq t_0$, for an initial state $x(t_0)$ and an input $u(t \geq t_0)$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Lagrange formula (cont.)

Proof

By left-multiplying the state equation $\dot{x}(t) = Ax(t) + Bu(t)$ by e^{-At}, we get

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

The resulting state equation can be rewritten,

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

Then, by using the result on the derivative of the state transition matrix[2],

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-At}x(t)\Big] &= e^{-At}\dot{x}(t) - e^{-At}Ax(t)\\ &= e^{-At}Bu(t)\end{aligned}$$

[2] Derivative of the state transition matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-At}x(t)\Big] = e^{-At}\Big[\frac{\mathrm{d}}{\mathrm{d}t}x(t)\Big] + \Big[\frac{\mathrm{d}}{\mathrm{d}t}e^{At}\Big]x(t) = e^{-At}\dot{x}(t) - e^{-At}Ax(t).$$

Lagrange formula (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

By integrating between t_0 and t, we obtain

$$\left[e^{-A\tau}x(\tau)\right]_{t_0}^{t} = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\mathrm{d}\tau$$

That is,

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(t)\mathrm{d}\tau$$

Thus,

$$e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(t)\mathrm{d}\tau$$

Lagrange formula (cont.)

$$e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(t)$$

The first Lagrange formula is obtained by multiplying both sides by e^{At}

$$\rightsquigarrow \quad x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

The second formula is obtained by substituting $x(t)$ in the output equation

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ \rightsquigarrow \quad & C\underbrace{\left[e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau \right]}_{x(t)} + Du(t) \end{aligned}$$

Force-free and forced evolution

Lagrange formula

Force-free and forced evolution

$$x(t) = \underbrace{e^{A(t-t_0)}x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{x_f(t)}$$

We can write the state solution (for $t \geq t_0$) as the sum of two terms

$$\rightsquigarrow \quad x(t) = x_u(t) + x_f(t)$$

- ⇝ The **force-free evolution** of the state, $x_u(t)$
- ⇝ The **forced evolution** of the state, $x_f(t)$

Force-free and forced evolution (cont.)

$$x(t) = \underbrace{e^{A(t-t_0)}x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau}_{x_f(t)}$$

The **force-free evolution** of the state, from the initial condition $x(t_0)$

$$\rightsquigarrow \quad x_l(t) = e^{A(t-t_0)}x(t_0)$$

- $\rightsquigarrow$ $e^{A(t-t_0)}$ indicates the transition from $x(t_0)$ to $x(t)$
- $\rightsquigarrow$ In the absence of contribution from the input

The **forced evolution** of the state

$$\rightsquigarrow \quad x_f(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

- $\rightsquigarrow$ The contribution of $u(\tau)$ to state $x(t)$
- $\rightsquigarrow$ Through a weighting function, $e^{A(t-\tau)}B$

Force-free and forced evolution (cont.)

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{y_u(t)} + \underbrace{C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)}_{y_f(t)}$$

We can write the output solution (for $t \geq t_0$) as the sum of two terms

$$\rightsquigarrow \quad y(t) = y_l(t) + y_f(t)$$

- ⇝ The **force-free evolution** of the output, $y_u(t)$
- ⇝ The **forced evolution** of the output, $y_f(t)$

Free and forced evolution (cont.)

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{y_u(t)} + \underbrace{C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)}_{y_f(t)}$$

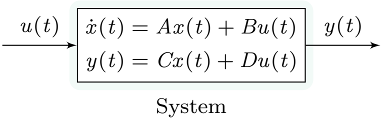
The **force-free evolution** of the output, from initial condition $y(t_0) = Cx(t_0)$

$$\begin{aligned} \rightsquigarrow \quad y_u(t) &= Ce^{A(t-t_0)}x(t_0) \\ &= Cx_u(t) \end{aligned}$$

The **forced-evolution** of the output

$$\begin{aligned} \rightsquigarrow \quad y_f(t) &= C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t) \\ &= Cx_f(t) + Du(t) \end{aligned}$$

Free and forced evolution (cont.)



Note that for $t_0 = 0$, we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Free and forced evolution (cont.)

Example

Consider a linear time-invariant system with the state-space representation,

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

We want to determine the state and the output evolution for $t \geq 0$

- We consider the input signal $u(t) = 2\delta_{-1}(t)$, a 2-step
- We consider the initial state $x(0) = (3, 4)^T$

The state transition matrix for this state-space representation,

$$e^{At} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

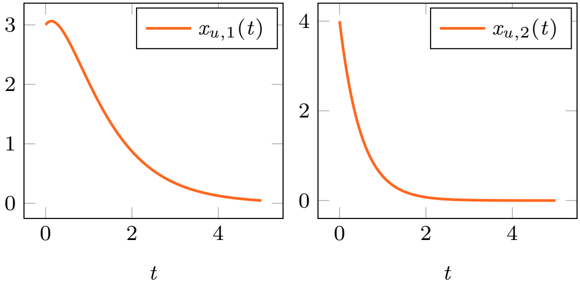
We computed it earlier

Free and forced evolution (cont.)

The force-free evolution of the state, for $t \geq 0$

$$\rightsquigarrow \quad x_u(t) = e^{At}x(0) = \underbrace{\begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}}_{e^{At}} \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{x(0)} = \begin{bmatrix} (7e^{-t} - 4e^{-2t}) \\ 4e^{-2t} \end{bmatrix}$$

That is,

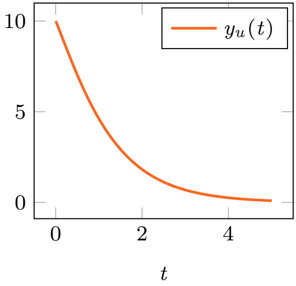


Free and forced evolution (cont.)

The force-free evolution of the output, for $t \geq 0$

$$\rightsquigarrow \quad y_u(t) = Cx_u(t) = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} (7e^{-t} - 4e^{-2t}) \\ 4e^{-2t} \end{bmatrix}}_{x_u(t)} = 14e^{-t} - 4e^{-2t}$$

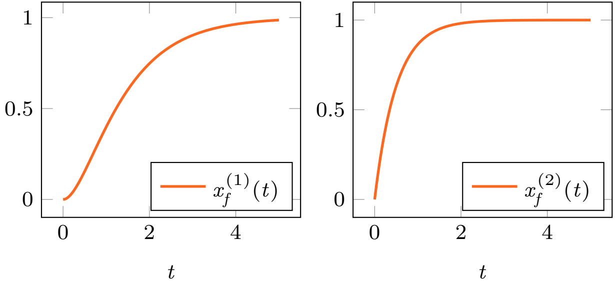
That is,



Free and forced evolution (cont.)

The forced evolution of the state, for $t \geq 0$

$$
\begin{aligned}
\rightsquigarrow \quad x_f(t) &= \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \\
&= \int_0^t \underbrace{\begin{bmatrix} e^{t-\tau} & (e^{t-\tau} - e^{-2(t-\tau)}) \\ 0 & e^{-2(t-\tau)} \end{bmatrix}}_{e^{A(t-\tau)}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{2}_{u(\tau)} \mathrm{d}\tau \\
&= 2 \int_0^t \begin{bmatrix} (e^{t-\tau} - e^{-2(t-\tau)}) \\ e^{-2(t-\tau)} \end{bmatrix} \mathrm{d}\tau = 2 \begin{bmatrix} \int_0^t (e^{t-\tau} - e^{-2(t-\tau)}) \mathrm{d}\tau \\ \int_0^t e^{-2(t-\tau)} \mathrm{d}\tau \end{bmatrix} \\
&= 2 \begin{bmatrix} (1 - e^{-t}) - 1/2(1 - e^{-2t}) \\ 1/2(1 - e^{-2t}) \end{bmatrix} = \begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix}
\end{aligned}
$$

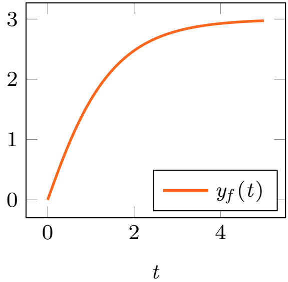


Free and forced evolution (cont.)

Since $D = 0$, the forced evolution of the output for $t \geq 0$

$$\rightsquigarrow \quad y_f(t) = Cx_f(t) = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix}}_{x_f(t)} = 3 - 4e^{-t} + e^{-2t}$$

That is,



Similarity transformation

LTI systems - Dynamics

Similarity transformation

The form of the state space representation depends on the choice of state variables

- The choice is not unique, even when we are coming from a physical model

There is an infinite number of different representations of the same system

- The representations are related by a **similarity transformation**
- These transformations allow flexibility in the analysis
- (We can change to easier system representations)

The state matrix can be transformed to have a **canonical form**

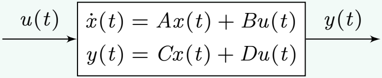
$\rightsquigarrow$ **Diagonal form**, **Jordan form**, . . .

Similarity transformation (cont.)

Definition

Similarity transformation

Consider the state-space representation of a linear time-invariant system of order N_x



System

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

- $x(t)$ and $\dot{x}(t)$, state vector and its derivative (N_x components)
- $u(t)$, input vector (N_u components)
- $y(t)$, output vector (N_y components)

Let vector $z(t)$ be related to $x(t)$ by some linear transformation P, $x(t) = Pz(t)$

P can be any $(N_x \times N_x)$ non-singular matrix (its inverse always exists)

- Because of non-signularity, we have $z(t) = P^{-1}x(t)$

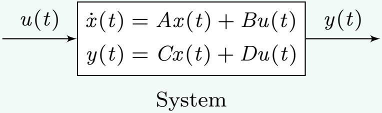
The transformation/matrix P is called a **similarity transformation**/**matrix**

Similarity transformation (cont.)

Proposition

Similar representations

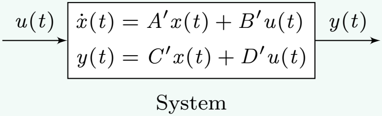
Consider the state-space representation of a linear time-invariant system of order N_x



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Let P be some similarity transformation matrix such that $x(t) = Pz(t)$

Vector $z(t) = P^{-1}x(t)$ satisfies the new state-space representation



$$\begin{cases} \dot{z}(t) = A'z(t) + B'u(t) \\ y(t) = C'z(t) + D'u(t) \end{cases}$$

$\rightsquigarrow$ $A' = P^{-1}AP$

$\rightsquigarrow$ $B' = P^{-1}B$

$\rightsquigarrow$ $C' = CP$

$\rightsquigarrow$ $D' = D$

Similarity transformation (cont.)

Proof

By taking the time-derivative of the state vector $x(t) = Pz(t)$, we have

$$\rightsquigarrow \quad \dot{x}(t) = P\dot{z}(t)$$

By substituting $x(t)$ and $\dot{x}(t)$ into the state-space representation,

$$\rightsquigarrow \quad \begin{cases} \underbrace{P\dot{z}(t)}_{\dot{x}(t)} = A\,\underbrace{Pz(t)}_{x(t)} + Bu(t) \\ y(t) = C\,\underbrace{Pz(t)}_{x(t)} + Du(t) \end{cases}$$

Pre-multiply the state equation by P^{-1}, to complete the proof

$$P^{-1}\,\underbrace{P\dot{z}(t)}_{\dot{x}(t)} = P^{-1}A\,\underbrace{Pz(t)}_{x(t)} + P^{-1}Bu(t)$$

$$P^{-1}y(t) = P^{-1}C\,\underbrace{Pz(t)}_{x(t)} + P^{-1}Du(t)$$

Similarity transformation (cont.)

$$P^{-1}P\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$
$$P^{-1}y(t) = P^{-1}CPz(t) + P^{-1}Du(t)$$

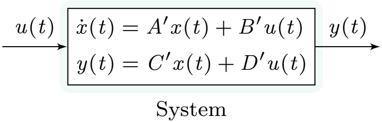
For the state equation, we have

$$\underbrace{P^{-1}P}_{I}\dot{z}(t) = \underbrace{P^{-1}AP}_{A'}z(t) + \underbrace{P^{-1}B}_{B'}u(t)$$
$$\dot{z}(t) = A'z(t) + B'u(t)$$

For the measurements, we have

$$\underbrace{PP^{-1}}_{I}y(t) = \underbrace{PP^{-1}}_{I}CPz(t) + \underbrace{PP^{-1}}_{I}Du(t)$$
$$\underbrace{CP}_{C'}z(t) + \underbrace{D}_{D'}u(t)$$
$$y(t) = C'z(t) + D'u(t)$$

Similarity transformation (cont.)



$$\begin{cases} \dot{z}(t) = A'z(t) + B'u(t) \\ y(t) = C'z(t) + D'u(t) \end{cases}$$

We obtained a different state-space representation of the same dynamical system

- Input $u(t)$ and output $y(t)$ are left unchanged (problem data)
- We defined some new (transformed) state variables, $z(t)$

There is an infinite number of non-singular matrixes P that could be used

- ⇝ Thus, there is also an infinite number of equivalent representations
- ⇝ $A' = P^{-1}AP$
- ⇝ $B' = P^{-1}B$
- ⇝ $C' = CP$
- ⇝ $D' = D$

Similarity transformation (cont.)

Example

Consider a linear time-invariante system with state-space representation $\{A, B, C, D\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}^{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1.5 \\ 0 \end{bmatrix}}_{D} u(t) \end{cases}$$

Consider the similarity transformation of the state using some matrix P

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

What is the $\{A', B', C', D'\}$ state-space representation for state $z(t)$

Similarity transformation (cont.)

We are given the similarity transformation P,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We compute its inverse,

$$P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since $z(t) = P^{-1}x(t)$, we have

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_1(t) - x_2(t) \end{bmatrix}$$

The second component of z is the difference between first and second component of x

$\rightsquigarrow$ The first component of z simply equals the second component of x

Similarity transformation (cont.)

We conclude by calculating the resulting state-space representation

$$\begin{aligned} A' &= P^{-1}AP \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} \\ B' &= P^{-1}B \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ C' &= CP \\ &= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \\ D' &= D \\ &= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} \end{aligned}$$

Similarity transformation (cont.)

Proposition

Similarity and state transition matrix

Consider the state matrix $A' = P^{-1}AP$ from some similarity transformation P

The corresponding state transition matrix,

$$e^{A't} = P^{-1}e^{At}P$$

Proof

Note that

$$(A')^k = \underbrace{(P^{-1}AP)\cdot(P^{-1}AP)\cdots(P^{-1}AP)}_{k \text{ times}} = P^{-1}\underbrace{AA\cdots A}_{k \text{ times}}P = P^{-1}A^kP$$

Thus, by definition

$$e^{A't} = \sum_{k=0}^{\infty}\frac{(A')^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{(P^{-1}A^kP)t^k}{k!} = P^{-1}\Big(\sum_{k=0}^{\infty}\frac{A^k t^k}{k!}\Big)P = P^{-1}e^{At}P$$

■

Similarity transformation (cont.)

We show how two similar state-space representations describe the same IO relation

Proposition

Invariance of the IO relationship under similarity

Consider two similar state-space representations of a linear time-invariant system

- $\rightsquigarrow$ $\{A, B, C, D\}$ and $\{A', B', C', D'\}$
- $\rightsquigarrow$ P is the transformation matrix

Suppose that the system be subjected to some known input

- $\rightsquigarrow$ $u(t)$

The two representations produce the same forced response

- $\rightsquigarrow$ $y_f(t)$

Similarity transformation (cont.)

Proof

Consider the Lagrange formula

The forced response of the second representation due to input $u(t)$

$$y_f(t) = C' \underbrace{\int_{t_0}^{t} e^{A'(t-\tau)} B' u(\tau) \mathrm{d}\tau}_{z_f(t)} + Du(t)$$

$$= \underbrace{CP}_{C'} \int_{t_0}^{t} \underbrace{P^{-1} e^{A(t-\tau)} P}_{e^{A'(t-\tau)}} \underbrace{P^{-1} B}_{B'} u(\tau) \mathrm{d}\tau + Du(t)$$

$$= C \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) \mathrm{d}\tau + Du(t)$$

This response corresponds to the one of the original representation

$$y_f(t) = C \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) \mathrm{d}\tau + Du(t)$$

■

Similarity transformation (cont.)

Proposition

Invariance of the eigenvalues under similarity transformations

Matrix A and $P^{-1}AP$ have the same characteristic polynomial

Proof

The characteristic polynomial of matrix A'

$$\begin{aligned}\det(\lambda I - A') &= \det(\lambda I - \underbrace{P^{-1}AP}_{A'}) \\ &= \det(\lambda \underbrace{P^{-1}P}_{I} - P^{-1}AP) \\ &= \det[P^{-1}(\lambda I - A)P] \\ &= \det(P^{-1})\det(\lambda I - A)\det(P) \\ &= \det(\lambda I - A)\end{aligned}$$

The last equality is obtained from $\det(P^{-1})\det(P) = 1$

A and A' share the same characteristic polynomial

$\rightsquigarrow$ Thus, also the eigenvalues are the same

■

Similarity transformation (cont.)

Two similar representations have the same modes, the modes characterise the dynamics

The modes are therefore independent of the representation

- $\rightsquigarrow$ This is important

Similarity transformation (cont.)

Example

Consider two similar state-space representations of a linear time-invariant system

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$A' = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

The similarity transformation matrix

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We are interested in the eigenvalues and modes of the system

Matrix A and A have two eigenvectors

- $\lambda_1 = -1$
- $\lambda_2 = -2$

The system modes are e^{-t} and e^{-2t}

Diagonalisation

LTI systems - Dynamics

Diagonalisation

We consider a special similarity transformation P, we seek for a diagonal matrix A'

- $\rightsquigarrow$ A state-space representation with a diagonal state matrix
- $\rightsquigarrow$ **Diagonal canonical form**
- $\rightsquigarrow$ $\Lambda = A' = P^{-1}AP$

Consider the linear time-invariant system with a single input (and, say, single output)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

The evolution of the i-th component of the state vector

$$\rightsquigarrow \quad \dot{x}_i(t) = \lambda_i x_i(t) + b_i u(t)$$

State derivatives are not related to other components

Diagonalisation (cont.)

We can understand a system with diagonal matrix A as a collection of sub-systems

- ⇝ Each sub-system is described by a single state component
- ⇝ Each state component evolves independently
- ⇝ The representation is **decoupled**
- ⇝ N_x first-order subsystems

The characteristic polynomial of the system for the i-th component

$$\rightsquigarrow \quad P_i(s) = (s - \lambda_i)$$

This subsystem has mode $e^{-\lambda_i t}$

We show how to determine a similarity transformation that leads to a diagonal form

- This can be understood as a somehow special similarity transformation

Modal matrix

Diagonalisation

Diagonalisation (cont.)

Definition

Modal matrix

Consider a system in state-space representation with $(N_x \times N_x)$ matrix A

- Let $v_1, v_2, \ldots, v_n$ be a set of all the eigenvectors of matrix A
- Suppose that they correspond to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Suppose that eigenvectors in this set are also linearly independent

We define the **modal matrix** of A as the $(N_x \times N_x)$ matrix V

$$V = \left[v_1 | v_2 | \cdots | v_n \right]$$

Diagonalisation (cont.)

If a matrix A has N_x distinct eigenvalues λ, then the modal matrix A always exists

- Its N_x eigenvectors v are linearly independent

Distinct eigenvalues

Let A be a n-order matrix whose n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct

Then, there is a set of n linearly independent eigenvectors

- Vectors $v_1, v_2, \dots, v_n$ form a basis for $\mathcal{R}^n$

Diagonalisation (cont.)

Example

Consider a state-space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

We are interested in the modal matrix V of A

The eigenvalues and eigenvectors of A

- $\lambda_1 = 1$ and $v_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$
- $\lambda_2 = 5$ and $v_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$

The modal matrix V,

$$V = \begin{bmatrix} v_1 | v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Diagonalisation (cont.)

$$V = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

It is important to note that the eigenvectors are determined up to a scaling constant

- (Plus, the ordering of the eigenvalues is arbitrary)
- There can be more than one modal matrix

These two modal matrices of matrix A are equivalent

$$V' = [2v_1 | 3v_2] = \begin{bmatrix} 2 & 3 \\ -2 & 9 \end{bmatrix}$$

$$V'' = [v_2 | v_1] = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

Diagonalisation (cont.)

Consider a matrix A with some eigenvalues λ that have multiplicity ν larger than one

- The modal matrix V exists if and only if to each eigenvalue λ_i with multiplicity ν_i is possible to associate ν_i linearly independent eigenvectors $\{v_{i,1}, v_{i,2}, \dots, v_{i,\nu_i}\}$

This is not always possible

But, ...

If a matrix admits a modal matrix, then it can be diagonalised

- (This is what matters to us)

Diagonalisation (cont.)

Example

Consider a state space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Its only eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $\big[\lambda I - A\big] v = 0$

$$\big[2I - A\big] v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$$

We can choose any two linearly independent eigenvectors for λ

- As the equation is satisfied for any value of a and b

A modal matrix with the eigenvectors from the canonical basis

$$\rightsquigarrow \quad V = \big[v_1 | v_2\big] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonalisation (cont.)

Example

Consider a state space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Its only eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $[\lambda I - A]\, v = 0$

$$[2I - A]v = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} -b = 0 \\ 0 = 0 \end{cases}$$

As $b = 0$, we can choose only one linearly independent eigenvector for λ

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix A does not admit a modal matrix

Diagonalisation (cont.)

Proposition

Diagonalisation

Consider a state-space representation of a linear time-invariant system with matrix A

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues and $V = \big[v_1 | v_2 | \cdots | v_n\big]$ one of its modal matrices

Let Λ be the state matrix transformed according to $\Lambda = V^{-1}AV$

$\rightsquigarrow$ Λ is diagonal

Diagonalisation (cont.)

Example

Consider a linear time-invariant system with matrixes $\{A, B, C, D\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

We are interested in a diagonal representation by similarity

We can compute the eigenvalues and eigenvectors of A

- $\lambda_1 = -1$ and $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $\lambda_2 = -2$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Diagonalisation (cont.)

Then, we can determine a modal matrix and its inverse

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

From the similarity transformation expressions, we get

$$A' = V^{-1}AV = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \Lambda$$
$$B' = V^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$C' = CV = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$$
$$D' = D = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

State transition matrix by diagonalisation

Diagonalisation

State transition matrix by diagonalisation

We show a procedure alternative to Sylvester's formula for the state transition matrix

- We assume a linear time-invariant state-space system representation
- We assume that the state matrix A can be diagonalised

Transition matrix by diagonalisation (cont.)

Proposition

State transition matrix by diagonalisation

Consider a $(n \times n)$ state matrix A and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues

Suppose that A admits the modal matrix V

We have for the state transition matrix

$$e^{At} = V e^{\Lambda t} V^{-1} = V \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

Because we have a diagonal state matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

State transition matrix by diagonalisation (cont.)

Proof

We have shown that the identity holds (see similarity and state transition matrices[3])

$$e^{\Lambda t} = V^{-1} e^{At} V$$

To complete the proof, multiply both sides by V on the left and by V^{-1} on the right

■

[3] Given $A' = P^{-1}AP$, we have $e^{A't} = P^{-1}e^{At}P$.

State transition matrix by diagonalisation (cont.)

Example

Consider a linear time-invariant system with matrixes $\{A, B, C, D\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

We are interested in computing the state transition matrix e^{At}

State transition matrix by diagonalisation (cont.)

We have already computed the modal matrix of A and its inverse, V and V^{-1}

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Thus, we have

$$e^{At} = V \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

This is the same result we determined by using the Sylvester expansion