



Aalto University

# Linear time-invariant processes: Estimation

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State estimation

Observability

Asymptotic  
observer

# State-estimation

LTI systems - Estimation

Dec 02, 2021  
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## State estimation

## State estimation

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$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

From the forced response, we saw that controllability must depend only on  $A$  and  $B$

We made some simplifying assumptions that helped us focusing on control

- We stated that the assumptions have no implications

We assumed that we can measure all state variables (and no feedthrough)

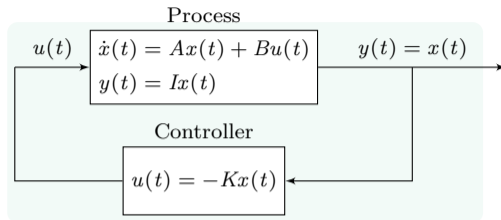
- That is, we have  $C = I$  and  $D = 0$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{C=I} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{D=0} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)}$$

## State estimation (cont.)

## State estimation

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For linear time-invariant system, function  $h(x(t)) = -Kx(t)$  is optimal, in some sense

$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)} = - \underbrace{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,N_x} \\ k_{21} & k_{22} & \cdots & k_{2,N_x} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N_u,1} & k_{N_u,2} & \cdots & k_{N_u,N_x} \end{bmatrix}}_K \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)}$$

$h(x(t))$

- For a system in state  $x(t)$ , the optimal control action is  $u(t) = -Kx(t)$
- (We will briefly also discuss the underlying optimality criterion)

## State estimation (cont.)

## State estimation

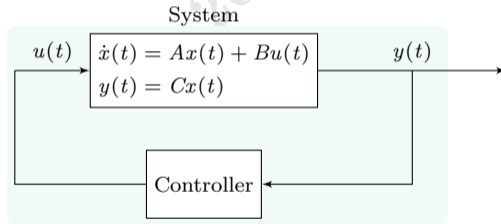
## Observability

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We assumed perfect measurement (observation) variables  $y(t)$  (from system's sensors)

- We assumed that  $y(t)$  returns all state variables  $x(t)$ , or  $y(t) = Ix(t)$

Not always the case, what if we cannot measure the state of the system,  $x(t) \neq y(t)$ ?



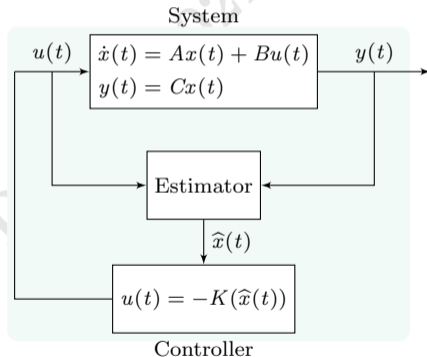
How to design controllers that define an optimal control  $u(t)$ , when  $x(t)$  is unknown?

## State estimation (cont.)

We could design another device capable to provide an estimate the state of the system

- Based on the measurable quantities of the system (data)
- That is, measurements  $y(t)$  and inputs  $u(t)$

A device that approximates the state of the system is a **state observer**, or **estimator**



Were the state estimate  $\hat{x}(t)$  accurate, we could use it with the optimal controller  $(-K)$

## State estimation (cont.)

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The state estimation problem has a solution only for those systems that are observable

### Definition

#### One, formal, definition of observability

*A linear and time-invariant system with state equation  $\dot{x}(t) = Ax(t)$  and measurement equation  $y(t) = Cx(t)$  is said to be observable if and only if it is possible to determine its state  $x(t)$  from the force-free response of its measurements over a finite time ( $t_f < \infty$ ), from any arbitrary initial state  $x(t_0)$*

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# Observability

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# Observability

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Observability refers to the possibility for estimate the system state from measurements

$$\begin{array}{c} u(t) \longrightarrow \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}} \longrightarrow y(t) \end{array} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Observability for linear and time-invariant systems depends only on the pair  $(A, C)$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

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We present a formal definition of observability for linear time-invariant systems

- Necessary and sufficient conditions
- (Invariance under similarity)

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We analyse the observability of a linear time-invariant system by using three criteria

- **Observability gramian**
- **Observability matrix**
- **(Popov-Belevich-Hautus test)**

As with controllability, all these criteria are complementary as for their usefulness

## Definition

## Observability gramian

Consider the linear and time-invariant system  $(A, C)$ , with  $x(t) \in \mathcal{R}^{N_x}$  and  $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

The system's **observability gramian** is a  $(N_x \times N_x)$  matrix, real and symmetric

$$W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau$$

## Theorem

### Observability test (I)

Consider the linear and time-invariant system  $(A, C)$ , with  $x(t) \in \mathcal{R}^{N_x}$  and  $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

Let  $W_o(t) = \int_0^t e^{A^T \tau} C^T C e^{A \tau} d\tau$  be the observability gramian of the system

- The system is observable iff  $W_o(t)$  is non-singular, for all  $t > 0$

## Observability (cont.)

### Proof (Sufficient condition)

From the second Lagrangian equation, we have the force-free evolution of the output

$$y(\tau) = Ce^{A\tau}x(0)$$

We left-multiply the equation by  $e^{A^T\tau}$ , then we integrate between 0 and some  $t_f$

$$\begin{aligned}\int_0^{t_f} e^{A^T\tau}y(\tau)d\tau &= \int_0^{t_f} e^{A^T\tau}Ce^{A\tau}x(0)d\tau \\ &= W_o(t_f)x(0)\end{aligned}$$

Thus, we have

$$x(0) = W_o^{-1}(t_f) \int_0^{t_f} e^{A^T\tau}Cy(\tau)d\tau$$

The initial state is given as a function of the inverse of the observability gramian  $W_o(t_f)$  and the integral  $\int_0^{t_f} e^{A^T\tau}Ce^{A\tau}y(\tau)d\tau$  which can be computed from measurements  $y(\tau)$

- The observability gramian need be non-singular
- Needed for the inverse to exist

## Example

Consider the linear and time-invariant system  $(A, B)$ , with  $x(t) \in \mathcal{R}^2$  and  $u(t) \in \mathcal{R}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

We are interested in verifying the observability of the system and initial state  $x(0)$

- We have measured the output,  $y(\tau) = 1 + 2\tau$
- Over a finite interval,  $\tau \in [0, 1]$

We compute the observability gramian, then we determine whether its invertible

## Observability (cont.)

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To compute the observability gramian, we already computed the state transition matrix

$$e^{A\tau} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$$

We can compute the observability gramian of the system, by applying the definition

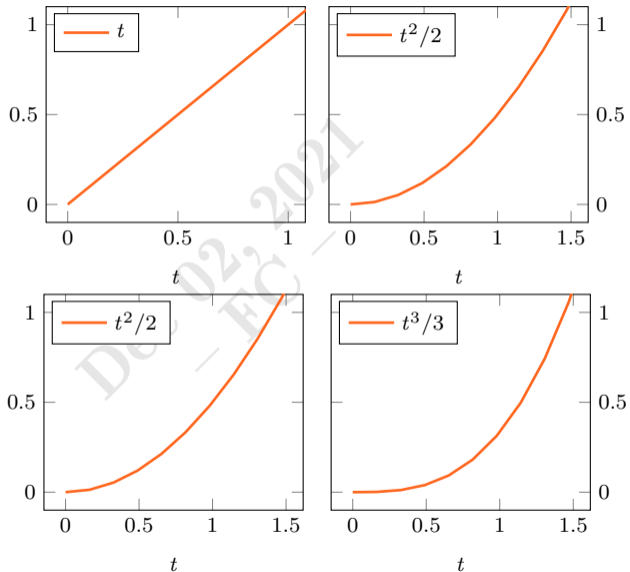
$$\begin{aligned} W_o(t) &= \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau \\ &= \int_0^t \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix} \end{aligned}$$

## Observability (cont.)

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## Observability (cont.)

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$$W_o(t) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}$$

To verify the singularity of the observability gramian  $W_o(t)$ , we check the determinant

- We need to check whether it is zero or it is non-zero
- Whatever the value of  $t$  (that is, at any time)

$$\begin{aligned} \det(W_o(t)) &= t^4/12 \\ &> 0 \quad (\forall t > 0) \end{aligned}$$

Since  $\det(W_o(t)) \neq 0$  for all  $t > 0$ , we can conclude that the system is observable

## Observability (cont.)

State estimation

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$$x(0) = W_o^{-1}(t_f) \int_0^{t_f} e^{A^T \tau} C y(\tau) d\tau$$

For the observability gramian and its inverse at the final time  $t_f = 1$ , we have

$$\begin{aligned} W_o(1) &= \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \end{aligned}$$

Then,

$$W_o^{-1}(1) = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

We also have,

$$\begin{aligned} \int_0^1 e^{A^T \tau} C^T y(\tau) d\tau &= \int_0^1 \begin{bmatrix} 1 & 0 \\ \tau & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 + 2\tau) d\tau \\ &= \begin{bmatrix} 2 \\ 7/6 \end{bmatrix} \end{aligned}$$

After substituting, we have the initial state

$$\begin{aligned}x(0) &= W_o^{-1}(1) \int_0^1 e^{A^T \tau} C^T y(\tau) d\tau \\ &= \begin{bmatrix} 4 & 6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 7/6 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}\end{aligned}$$

The trajectory of the state variable from the initial state  $x(0)$ ,

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$



## Observability (cont.)

### Theorem

#### Observability matrix and observability test (II)

Consider a linear and time-invariant system  $(A, C)$ , with  $x(t) \in \mathcal{R}^{N_x}$  and  $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

We define the  $(N_x \times (N_y \times N_x))$  **observability matrix**

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N_x-1} \end{bmatrix}$$

Necessary and sufficient condition for observability,

$$\text{rank}(\mathcal{O}) = N_x$$

## Example

Consider the linear and time-invariant system  $(A, C)$ , with  $x(t) \in \mathcal{R}^{N_x}$  and  $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t) \end{cases}$$

We are interested in verifying its observability using the observability matrix

The observability matrix has size  $((N_y = 2 \times N_x = 3) \times N_x = 3) = (6 \times 3)$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

## Observability (cont.)

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$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

We know  $C$ , we need to compute  $CA$  and  $CA^2$ ,

$$\begin{aligned} CA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} CA^2 &= (CA)A \\ &= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 24 & 4 \\ 0 & 0 & 12 \end{bmatrix} \end{aligned}$$

## Observability (cont.)

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Thus, we have the observability matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 0.5 \\ 0 & 0 & 6 \\ 4 & 24 & 4 \\ 0 & 0 & 12 \end{bmatrix}$$

We check observability from its rank,

$$\begin{aligned} \text{rank}(C) &= 3 \\ &= N_x \end{aligned}$$



## Observability (cont.)

State estimation

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```
1 >> help ctrb           % OBSV computes the observability matrix
2                       % of pair (A,C)
3                       % Read about it and how to use it
4
5 >> A = [?];           % Define state matrix A
6 >> C = [?];           % Define control matrix C
7
8 >> [Ny,Nx] = size(C); % Ny and Nx
9
10 >> Omat = obsv(A,C)   % Observability matrix
11
12 >> rnkOmat = rank(Omat) % Rank of the observability matrix
13
14 >> rnkOmat == Nx      % Return 0/1 for observability
```



## Example

Consider the linear and time-invariant system  $(A, B, C)$ ,  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$ ,  $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix  $A$ ,  $C$ , and  $\mathcal{O}$

What changes when we have  $C = [1 \quad 1]$ ?



## Example

Consider the linear and time-invariant system  $(A, B, C)$ ,  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$ ,  $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix  $A$ ,  $C$ , and  $\mathcal{O}$



## Example

Consider the linear and time-invariant system  $(A, B, C)$ ,  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$ ,  $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix  $A$ ,  $C$ , and  $\mathcal{O}$



## Example

Consider the linear and time-invariant system  $(A, B, C)$ ,  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$ ,  $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix  $A$ ,  $\mathcal{C}$ , and  $\mathcal{O}$



State estimation

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# Asymptotic state observers

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# Asymptotic state observers

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We show that for linear and time-invariant systems which are observable it is possible to exactly estimate the state vector at some infinite  $t \rightarrow \infty$  (that is, asymptotically)

$$\lim_{t \rightarrow \infty} \|x(t) - \hat{x}(t)\| = 0$$

The device that estimates the state of a system is itself a dynamic system

## Definition

## Luenberger observer

Consider a linear and time-invariant dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases},$$

with  $x(t) \in \mathcal{R}^{N_x}$ ,  $u(t) \in \mathcal{R}^{N_u}$ , and  $y(t) \in \mathcal{R}^{N_y}$

The linear and time-invariant dynamical system

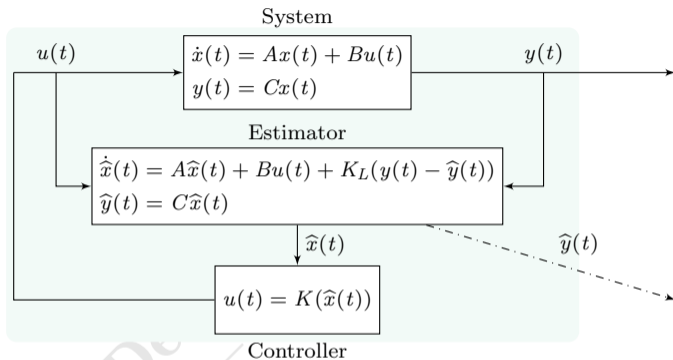
$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases},$$

with  $\hat{x} \in \mathcal{R}^{N_x}$ ,  $\hat{y}(t) \in \mathcal{R}^{N_y}$  is a Luenberger observer of the system iff  $K_L \in \mathcal{R}^{N_x \times N_y}$  is any matrix such that the eigenvalues of matrix  $A - K_L C$  all have a negative real part

## Asymptotic state observers (cont.)

State estimation

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$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K_L y(t) - K_L \hat{y}(t) \\ &= A\hat{x}(t) + Bu(t) + K_L y(t) - C\hat{x}(t) \\ &= (A - K_L C)\hat{x}(t) + Bu(t) + K_L y(t) \\ &= \underbrace{(A - K_L C)}_{A_L} \hat{x}(t) + \underbrace{[B \quad K_L]}_{B_L} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \end{aligned}$$



## Theorem

Consider a linear and time-invariant dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases},$$

with  $x(t) \in \mathcal{R}^{N_x}$ ,  $u(t) \in \mathcal{R}^{N_u}$ , and  $y(t) \in \mathcal{R}^{N_y}$

The Luenberger observer is an asymptotic state observer

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases},$$

with  $\hat{x} \in \mathcal{R}^{N_x}$ ,  $\hat{y}(t) \in \mathcal{R}^{N_y}$

## Asymptotic state observers (cont.)

State estimation

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### Proof

Let us define the estimation error  $e(t)$ , the difference between true state and estimate

$$e(t) = x(t) - \hat{x}(t)$$

We want to show that the error has linear and time-invariant homogenous dynamics

- ↪ The state matrix of this system is given by  $A - K_L C$
- ↪ Because  $A - K_L C$  is stable, the error vanishes
- ↪ (Asymptotically)

## Asymptotic state observers (cont.)

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We differentiate with respect to time the definition of estimation error and rearrange

$$\begin{aligned}
 \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
 &= \underbrace{(Ax(t) + Bu(t))}_{\dot{x}(t)} - \underbrace{(A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t)))}_{\dot{\hat{x}}(t)} \\
 &= (Ax(t) + \cancel{Bu(t)}) - (A\hat{x}(t) + \cancel{Bu(t)} + K_L(y(t) - \hat{y}(t))) \\
 &= A \left( \underbrace{x(t) - \hat{x}(t)}_{e(t)} \right) - K_L y(t) + K_L \hat{y}(t) \\
 &= A(x(t) - \hat{x}(t)) - K_L \underbrace{(Cx(t))}_{y(t)} + K_L \underbrace{(C\hat{x}(t))}_{\hat{y}(t)} \\
 &= A(x(t) - \hat{x}(t)) - K_L C(x(t) - \hat{x}(t)) \\
 &= (A - K_L C)(x(t) - \hat{x}(t)) \\
 &= (A - K_L C)e(t)
 \end{aligned}$$

## Asymptotic state observers (cont.)

State estimation


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The dynamics of the state estimation error  $e(0) = x(0) - \hat{x}(0)$ ,

$$\dot{e}(t) = \underbrace{(A - K_L C)}_{A_L} e(t),$$

We have  $\lim_{t \rightarrow \infty} \|x(t) - \hat{x}(t)\| = 0$ , for all inputs  $u(t)$  and initial states  $x(0)$  and  $\hat{x}(0)$



## Asymptotic state observers (cont.)

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$$\begin{cases} \hat{\dot{x}}(t) = A\hat{x}(t) + Bu(t) + K_L(y(t) - \hat{y}(t)) \\ \hat{y}(t) = \hat{x}(t) \end{cases},$$

A Luenberger observer is a linear and time-invariant dynamical system

- The observer has the same order  $N_x$  of the original system
- The observer state variable is the state estimate  $\hat{x}(t)$
- The observer input variables are  $u(t)$  and  $y(t)$

The system and observer share the same measurement equation

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While the estimation error tends to zero as time tends to infinity, the rate at which the error becomes practically negligible depends on the (estimated) initial state  $\hat{x}(0)$

- The closer to the actual state  $x(0)$ , the better