Aalto University

Linear time-invariant processes: Estimation

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State-estimation

LTI systems - Estimation

State estimation

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$$

From the forced response, we saw that controllability must depend only on A and B

We made some simplifying assumptions that helped us focusing on control

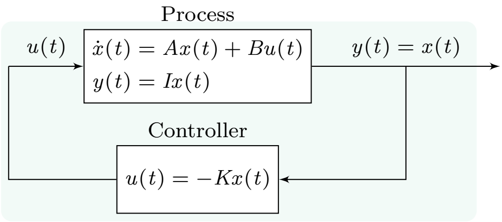
- We stated that the assumptions have no implications

We assumed that we can measure all state variables (and no feedthrough)

- That is, we have $C = I$ and $D = 0$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{N_y}(t) \end{bmatrix}}_{y(t)} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{C=I} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{D=0} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)}$$

State estimation (cont.)



For linear time-invariant system, function $h(x(t)) = -Kx(t)$ is optimal, in some sense

$$\underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{N_u}(t) \end{bmatrix}}_{u(t)} = \underbrace{-\underbrace{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1,N_x} \\ k_{21} & k_{22} & \cdots & k_{2,N_x} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N_u,1} & k_{N_u,2} & \cdots & k_{N_u,N_x} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N_x}(t) \end{bmatrix}}_{x(t)}}_{h(x(t))}$$

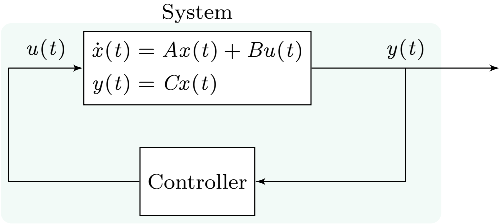
- For a system in state $x(t)$, the optimal control action is $u(t) = -Kx(t)$
- (We will briefly also discuss the underlaying optimality criterion)

State estimation (cont.))

We assumed perfect measurement (observation) variables $y(t)$ (from system's sensors)

- We assumed that $y(t)$ returns all state variables $x(t)$, or $y(t) = Ix(t)$

Not always the case, what if we cannot measure the state of the system, $x(t) \neq y(t)$?



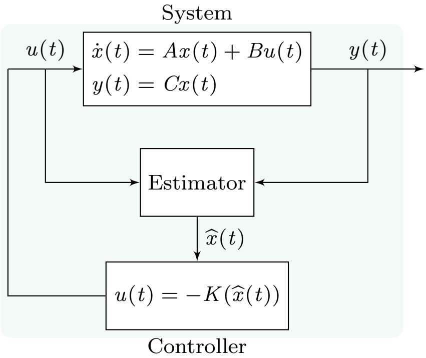
How to design controllers that define an optimal control $u(t)$, when $x(t)$ is unknown?

State estimation (cont.)

We could design another device capable to provide an estimate the state of the system

- Based on the measurable quantities of the system (data)
- That is, measurements $y(t)$ and inputs $u(t)$

A device that approximates the state of the system is a **state observer**, or **estimator**



Were the state estimate $\widehat{x}(t)$ accurate, we could use it with the optimal controller $(-K)$

State estimation (cont.)

The state estimation problem has a solution only for those systems that are observable

Definition

One, formal, definition of observability

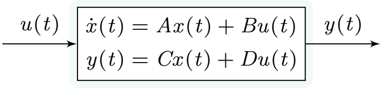
A linear and time-invariant system with state equation $\dot{x}(t) = Ax(t)$ *and measurement equation* $y(t) = Cx(t)$ *is said to be observable if and only if it is possible to determine its state* $x(t)$ *from the force-free response of its measurements over a finite time* ($t_f < \infty$)*, from any arbitrary initial state* $x(t_0)$

Observability

LTI systems - Estimation

Observability

Observability refers to the possibility for estimate the system state from measurements



$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Observability for linear and time-invariant systems depends only on the pair (A, C)

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

We present a formal definition of observability for linear time-invariant systems

- Necessary and sufficient conditions
- (Invariance under similarity)

Observability

We analyse the observablity of a linear time-invariant system by using three criteria

- **Observability gramian**
- **Observability matrix**
- (**Popov-Belevich-Hautus test**)

As with controllability, all these criteria are complementary as for their usefulness

Observability (cont.)

Definition

Observability gramian

Consider the linear and time-invariant system (A, C), with $x(t) \in \mathcal{R}^{N_x}$ and $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

The system's **observability gramian** is a $(N_x \times N_x)$ matrix, real and symmetric

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} \mathrm{d}\tau$$

Observability (cont.)

Theorem

Observability test (I)

Consider the linear and time-invariant system (A, C), with $x(t) \in \mathcal{R}^{N_x}$ and $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

Let $W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} \mathrm{d}\tau$ be the observability gramian of the system

- The system is observable iff $W_o(t)$ is non-singular, for all $t > 0$

Observability (cont.)

Proof (Sufficient condition)

From the second Lagrangian equation, we have the force-free evolution of the output

$$y(\tau) = Ce^{A\tau}x(0)$$

We left-multiply the equation by $e^{A^T\tau}$, then we integrate between 0 and some t_f

$$\begin{aligned}\int_0^{t_f} e^{A^T\tau}y(\tau)\mathrm{d}\tau &= \int_0^{t_f} e^{A^T\tau}Ce^{A\tau}x(0)\mathrm{d}\tau \\ &= W_o(t_f)x(0)\end{aligned}$$

Thus, we have

$$x(0) = W_o^{-1}(tf)\int_0^{t_f} e^{A^T\tau}Cy(\tau)\mathrm{d}\tau$$

The initial state is given as a function of the inverse of the observability gramian $W_o(tf)$ and the integral $\int_0^{t_f} e^{A^T\tau}Ce^{A\tau}y(\tau)\mathrm{d}\tau$ which can be computed from measurements $y(\tau)$

- The observability gramian need be non-singular
- Needed for the inverse to exist

■

Observability (cont.)

Example

Consider the linear and time-invariant system (A, B), with $x(t) \in \mathcal{R}^2$ and $u(t) \in \mathcal{R}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) \\ \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

We are interested in verifying the observability of the system and initial state $x(0)$

- We have measured the output, $y(\tau) = 1 + 2\tau$
- Over a finite interval, $\tau \in [0, 1]$

We compute the observability gramian, then we determine whether its invertible

Observability (cont.)

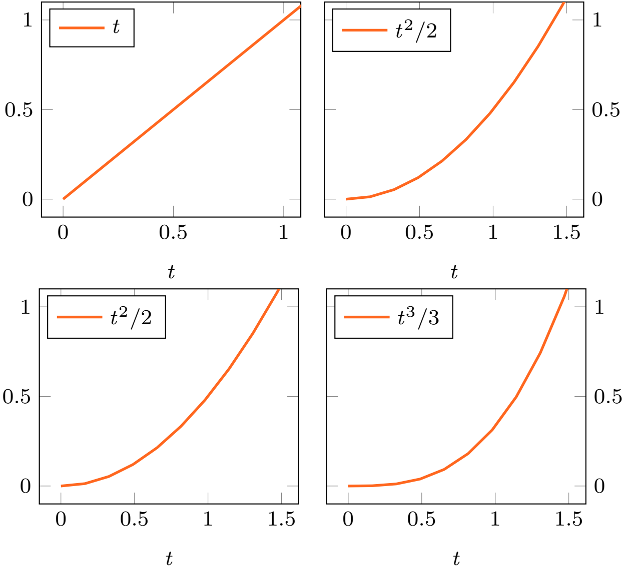
To compute the observability gramian, we already computed the state transition matrix

$$e^{A\tau} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$$

We can compute the observability gramian of the system, by applying the definition

$$\begin{aligned} W_o(t) &= \int_0^t e^{A^T\tau} C^T C e^{A\tau} \mathrm{d}\tau \\ &= \int_0^t \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix} \end{aligned}$$

Observability (cont.)



Observability (cont.)

$$W_o(t) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}$$

To verify the singularity of the observability gramian $W_o(t)$, we check the determinant

- We need to check whether it is zero or it is non-zero
- Whatever the value of t (that is, at any time)

$$\begin{aligned} \det\left(W_o(t)\right) &= t^4/12 \\ &> 0 \quad (\forall t > 0) \end{aligned}$$

Since $\det\left(W_o(t)\right) \neq 0$ for all $t > 0$, we can conclude that the system is observable

Observability (cont.)

$$x(0) = W_o^{-1}(tf) \int_0^{t_f} e^{A^T \tau} C y(\tau) \mathrm{d}\tau$$

For the observability gramian and its inverse at the final time $t_f = 1$, we have

$$W_o(1) = \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

Then,

$$W_o^{-1}(1) = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

We also have,

$$\int_0^1 e^{A^T \tau} C^T y(\tau) \mathrm{d}\tau = \int_0^1 \begin{bmatrix} 1 & 0 \\ \tau & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1 + 2\tau) \mathrm{d}\tau = \begin{bmatrix} 2 \\ 7/6 \end{bmatrix}$$

Observability (cont.)

After substituting, we have the initial state

$$\begin{aligned} x(0) &= W_o^{-1}(1) \int_0^1 e^{A^T \tau} C^T y(\tau) \mathrm{d}\tau \\ &= \begin{bmatrix} 4 & 6 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 7/6 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

The trajectory of the state variable from the initial state $x(0)$,

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau$$

Observability (cont.)

Theorem

Observability matrix and observability test (II)

Consider a linear and time-invariant system (A, C), with $x(t) \in \mathcal{R}^{N_x}$ and $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$$

We define the $(N_x \times (N_y \times N_x))$ **observability matrix**

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N_x - 1} \end{bmatrix}$$

Necessary and sufficient condition for observability,

$$\text{rank}(\mathcal{O}) = N_x$$

Observability (cont.)

Example

Consider the linear and time-invariant system (A, C), with $x(t) \in \mathcal{R}^{N_x}$ and $y(t) \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} x(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t) \end{cases}$$

We are interested in verifying its observability using the observability matrix

The observability matrix has size $((N_y = 2 \times N_x = 3) \times N_x = 3) = (6 \times 3)$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

Observability (cont.)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

We know C, we need to compute CA and CA^2,

$$\begin{aligned} CA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} CA^2 &= (CA)A \\ &= \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0.5 \\ 0 & 4 & 0.5 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 24 & 4 \\ 0 & 0 & 12 \end{bmatrix} \end{aligned}$$

Observability (cont.)

Thus, we have the observability matrix

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 0.5 \\ 0 & 0 & 6 \\ 4 & 24 & 4 \\ 0 & 0 & 12 \end{bmatrix}$$

We check observability from its rank,

$$\begin{aligned} \text{rank}(\mathcal{C}) &= 3 \\ &= N_x \end{aligned}$$

Observability (cont.)

```
>> help ctrb                                % OBSV computes the observability matrix
                                            % of pair (A,C)
                                            % Read about it and how to use it

>> A = [?];                                 % Define state matrix A
>> C = [?];                                 % Define control matrix C

>> [Ny,Nx] = size(C);                       % Ny and Nx

>> Omat = obsv(A,C)                         % Observability matrix

>> rnkOmat = rank(Omat)                     % Rank of the observability matrix

>> rnkOmat == Nx                            % Return 0/1 for observability
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Observability (cont.)

Example

Consider the linear and time-invariant system (A, B, C), $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$, $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 3 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix A, $\mathcal{C}$, and $\mathcal{O}$

What changes when we have $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$?

Observability (cont.)

Example

Consider the linear and time-invariant system (A, B, C), $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$, $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix A, $\mathcal{C}$, and $\mathcal{O}$

Observability (cont.)

Example

Consider the linear and time-invariant system (A, B, C), $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$, $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix A, $\mathcal{C}$, and $\mathcal{O}$

Observability (cont.)

Example

Consider the linear and time-invariant system (A, B, C), $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$, $y \in \mathcal{R}^{N_y}$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{cases}$$

Determine the stability, controllability, and observability using matrix A, $\mathcal{C}$, and $\mathcal{O}$

Asymptotic state observers

LTI systems - Estimation

Asymptotic state observers

We show that for linear and time-invariant systems which are observable it is possible to exactly estimate the state vector at some infinite $t \to \infty$ (that is, asymptotically)

$$\lim_{t \to \infty} \| x(t) - \widehat{x}(t) \| = 0$$

The device that estimates the state of s system is itself a dynamic system

Asymptotic state observers

Definition

Luenberger observer

Consider a linear and time-invariant dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases},$$

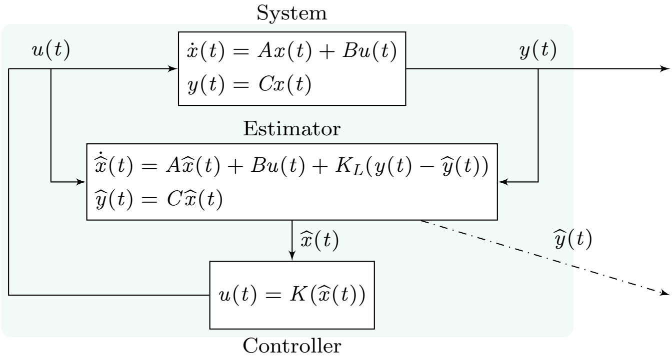
with $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

The linear and time-invariant dynamical system

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) + K_L\left(y(t) - \widehat{y}(t)\right) \\ \widehat{y}(t) = C\widehat{x}(t) \end{cases},$$

with $\widehat{x} \in \mathcal{R}^{N_x}$, $\widehat{y}(t) \in \mathcal{R}^{N_y}$ is a Luenberger observer of the system iff $K_L \in \mathcal{R}^{N_x \times N_y}$ is any matrix such that the eigenvalues of matrix $A - K_L C$ all have a negative real part

Asymptotic state observers (cont.)



$$
\begin{aligned}
\dot{\widehat{x}}(t) &= A\widehat{x}(t) + Bu(t) + K_L y(t) - K_L \widehat{y}(t) \\
&= A\widehat{x}(t) + Bu(t) + K_L y(t) - K_L C\widehat{x}(t) \\
&= (A - K_L C)\widehat{x}(t) + Bu(t) + K_L y(t) \\
&= \underbrace{(A - K_L C)}_{A_L}\widehat{x}(t) + \underbrace{\begin{bmatrix} B & K_L \end{bmatrix}}_{B_L} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}
\end{aligned}
$$

Asymptotic state observers (cont.)

Theorem

Consider a linear and time-invariant dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases},$$

with $x(t) \in \mathcal{R}^{N_x}$, $u(t) \in \mathcal{R}^{N_u}$, and $y(t) \in \mathcal{R}^{N_y}$

The Luenberger observer is an asymptotic state observer

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) + K_L\left(y(t) - \widehat{y}(t)\right) \\ \widehat{y}(t) = C\widehat{x}(t) \end{cases},$$

with $\widehat{x} \in \mathcal{R}^{N_x}$, $\widehat{y}(t) \in \mathcal{R}^{N_y}$

Asymptotic state observers (cont.)

Proof

Let us define the estimation error $e(t)$, the difference between true state and estimate

$$e(t) = x(t) - \widehat{x}(t)$$

We want to show that the error has linear and time-invariant homogenous dynamics

- ⇝ The state matrix of this system is given by $A - K_L C$
- ⇝ Because $A - K_L C$ is stable, the error vanishes
- ⇝ (Asymptotically)

Asymptotic state observers (cont.)

We differentiate with respect to time the definition of estimation error and rearrange

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\widehat{x}}(t) \\
&= \underbrace{(Ax(t) + Bu(t))}_{\dot{x}(t)} - \underbrace{(A\widehat{x}(t) + Bu(t) + K_L(y(t) - \widehat{y}(t)))}_{\dot{\widehat{x}}(t)} \\
&= (Ax(t) + \cancel{Bu(t)}) - (A\widehat{x}(t) + \cancel{Bu(t)} + K_L(y(t) - \widehat{y}(t))) \\
&= A\left(\underbrace{x(t) - \widehat{x}(t)}_{e(t)}\right) - K_L y(t) + K_L \widehat{y}(t) \\
&= A\left(\underbrace{x(t) - \widehat{x}(t)}_{e(t)}\right) - K_L \underbrace{(Cx(t))}_{y(t)} - K_L \underbrace{(C\widehat{x}(t))}_{\widehat{y}(t)} \\
&= A\left(\underbrace{x(t) - \widehat{x}(t)}_{e(t)}\right) - K_L C\left(\underbrace{x(t) - \widehat{x}(t)}_{e(t)}\right) \\
&= (A - K_L C)(x(t) - \widehat{x}(t)) \\
&= (A - K_L C)\, e(t)
\end{aligned}
$$

Asymptotic state observers (cont.)

The dynamics of the state estimation error $e(0) = x(0) - \widehat{x}(0)$,

$$\dot{e}(t) = \underbrace{(A - K_L C)}_{A_L} e(t),$$

We have $\lim_{t\to\infty} \|x(t) - \widehat{x}(t)\| = 0$, for all inputs $u(t)$ and initial states $x(0)$ and $\widehat{x}(0)$

Asymptotic state observers (cont.)

$$\begin{cases} \dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t) + K_L\left(y(t) - \widehat{y}(t)\right) \\ \widehat{y}(t) = \widehat{x}(t) \end{cases},$$

A Luenberger observer is a linear and time-invariant dynamical system

- The observer has the same order N_x of the original system
- The observer state variable is the state estimate $\widehat{x}(t)$
- The observer input variables are $u(t)$ and $y(t)$

The system and observer share the same measurement equation

While the estimation error tends to zero as time tends to infinity, the rate at which the error becomes practically negligible depends on the (estimated) initial state $\widehat{x}(0)$

- The closer to the actual state $x(0)$, the better