Single state va

Single state var and single input

Multiple states and multiple inputs



Linearisation of nonlinear state-space models CHEM-E7190 (was E7140), 2023

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Single state var

Single state var and single input

Multiple states and multiple inputs

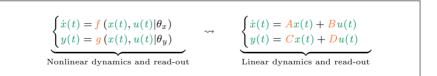
Linearisation of nonlinear state-space models

Many dynamical models used to describe processes in chemical engineering are given as a set of first-order ordinary differential equations, the equations are very often nonlinear

• From the application of material and energy conservation laws

A body of commonly used techniques for system analysis and control uses linear models

- To access such a technology we need to simplify common process models
- This will require approximating the general state-space representation
- The model approximations of our interest are Jacobian linearisation



The main idea behind the linearisation of nonlinear process models in state-space form

- \rightsquigarrow Approximate function f (nonlinear) with matrices A and B
- \rightsquigarrow Approximate function g (nonlinear) with matrices C and D

Single state var

Single state var and single input

Multiple states and multiple inputs

Linearisation of nonlinear state-space models

We discuss how to determine a linear approximation of a nonlinear state-space model \rightsquigarrow We study a number of cases, of increasing complexity

 \rightsquigarrow As a result, we will be able to linearise any¹ model

 $\begin{cases} \dot{x}(t) = f(x(t), u(t)|\theta_x) \\ y(t) = g(x(t), u(t)|\theta_y) \end{cases}$

Nonlinear dynamics and read-out

In general, we have a general state-space model with variables of arbitrary dimension $\rightsquigarrow x(t) \in \mathcal{R}^{N_x}$ $\rightsquigarrow u(t) \in \mathcal{R}^{N_u}$

 $\rightsquigarrow y(t) \in \mathcal{R}^{N_y}$

We start with $N_x = 1$ (one state), $N_u = 0$ (no inputs) and $N_y = 1$ (one measurement)

• Then, we will add complexity (we add more variables)

¹We only require that model functions f and g are continuous and differentiable.

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state variable, no inputs

$$\dot{x}(t) = f(x(t)), \text{ with } x(t) \in \mathcal{R}$$

We suppose that function f(x(t)) can be approximated by a Taylor series expansion

- We are interested in an approximation around an fixed point, $x(t) = x_{SS}$
- At the steady-state point x_{SS} , time variations are zero $\dot{x}(t) = 0$
- Thus, we also have that $f(x_{SS}) = 0$, whatever the time t

We can perfectly represent function f(x) by using an infinite Taylor series expansion

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f(x)}{\mathrm{d}x}}_{\text{linear in } x} \left|_{x_{SS}} (x - x_{SS}) + \underbrace{\frac{1}{2} \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}}_{\text{quadratic in } x} \right|_{x_{SS}} (x - x_{SS})^2 + \underbrace{\mathcal{O}(x^3)}_{\text{H.O. terms}}$$

The expansion is a sum of polynomials of x, with the derivatives of f as coefficients

• Given this representation of f, we want to use it to approximate f

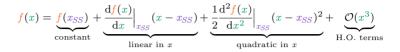
Note that steady-state, stationary, fixed, equilibrium points are all equivalent terms

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state variable, no inputs (cont.)



Suppose that we are interested in an approximation based only on first-order terms

• We accept to neglect (truncate out) second- and higher-order terms



After truncation, we get an approximation of f which is a linear function of x

$$f(x) \approx f(x_{SS}) + \left. \frac{\mathrm{d}f(x)}{\mathrm{d}x} \right|_{x_{SS}} (x - x_{SS})$$

Note the complete expression of f(x(t)) with explicit dependencies also wrt time t

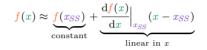
$$f(x(t)) \approx f(x_{SS}) + \frac{\mathrm{d}f(x(t))}{\mathrm{d}x}\Big|_{x_{SS}} (x(t) - x_{SS})$$

Single state var

Single state var and single input

Multiple state and multiple inputs

A single state variable, no inputs (cont.)



Because x_{SS} is chosen to be a fixed-point, we have $\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{x_{SS}} = \dot{x}(t)\Big|_{x_{SS}} = f(x_{SS}) = 0$

Thus, we can write

$$f(x) \approx \underbrace{f(x_{SS})}_{=0} + \frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}} (x - x_{SS})$$

$$\rightsquigarrow \dot{x}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state variable, no inputs (cont.)

$$\dot{x}(t) \approx \frac{\mathrm{d}f(x(t))}{\mathrm{d}x}\Big|_{x_{SS}} (x(t) - x_{SS})$$

We can now introduce a perturbation or deviation variable $x'(t) = x(t) - x_{SS}$

- Variable x'(t) encodes how far state variable x(t) is from steady-state x_{SS}
- (And, because variable x(t) varies with time, also x'(t) varies)

Therefore, we can also compute the time-derivative of perturbation variable x'(t)

- It describes the rate of change of variable x'(t) with respect to time
- Or, equivalently, its dynamics

By differenting x'(t) to get dx'(t)dt, we have

$$\frac{\mathrm{d}(x(t) - x_{SS})}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \underbrace{\frac{\mathrm{d}x_{SS}}{\mathrm{d}t}}_{=0}$$
$$= \dot{x}(t)$$
$$\rightsquigarrow f(x(t))$$

Deviation/perturbation variables and state variables have identical dynamics

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state variable, no inputs (cont.)

We can equate the two terms and write

We obtained that
$$\frac{\mathrm{d}(x(t) - x_{SS})}{\mathrm{d}t} = f(x(t)) \text{ and } f(x(t)) \approx \frac{\mathrm{d}f(x(t))}{\mathrm{d}x}\Big|_{x_{SS}}(x(t) - x_{SS})$$

 $\frac{\mathrm{d}(x(t) - x_{SS})}{\mathrm{d}t} = f(x(t)) \approx \underbrace{\frac{\mathrm{d}f(x(t))}{\mathrm{d}x}}_{\text{constant}} (x(t) - x_{SS})$

We have derived the approximated state equation for the deviation variable x'(t)

$$\dot{x'}(t) pprox \frac{\mathrm{d}f(x(t))}{\mathrm{d}x}\Big|_{x_{SS}} (x(t) - x_{SS})$$

This is a linear time-invariant approximation of the (perturbed) state equation

$$\dot{x'}(t) = \alpha x'(t), \quad ext{with constant } \alpha = rac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}} \in \mathcal{R}$$

We also know how solve it for some initial condition x'(0)

$$x'(t) = e^{\alpha t} x'(0)$$

Single state var

Single state var and single input

Multiple state and multiple inputs

A single state and a single input variable

$$\dot{x}(t) = f(x(t), u(t)), \text{ with } x, u \in \mathcal{R}$$

We assume that function f(x, u) can be approximated by a Taylor series expansion

• We are interested in an approximation around some fixed-point point

$$(x = x_{SS}, u = u_{SS})$$

- At fixed points $(x = x_{SS}, u = u_{SS})$, time-variations are zero $\dot{x}(t) = 0$
- \rightsquigarrow (The state remains fixed at x_{SS} as long as the input is fixed, at u_{SS})

Because at steady-state there is no evolution, the right hand-side is zero

$$\rightsquigarrow f(x = x_{SS}, u = u_{SS}) = 0$$

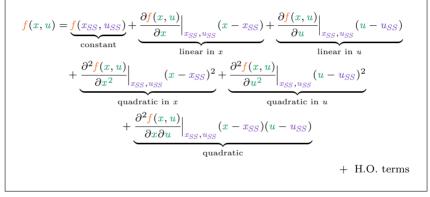
Single state var

Single state var and single input

Multiple states and multiple inputs

A single state and a single input variable

Suppose that we can perfectly represent f(x, u) by using an infinite Taylor expansion



Suppose that we are interested in an approximation based only on first-order terms

• By truncation, we accept to neglect second- and higher-order terms

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state and a single input variable (cont.)

We can again obtain a linear, first-order, approximation of function f by truncation

$$f(x,u) \approx f(x_{SS}, u_{SS}) + \frac{\partial f(x,u)}{\partial x}\Big|_{x_{SS}, u_{SS}}(x - x_{SS}) + \frac{\partial f(x,u)}{\partial u}\Big|_{x_{SS}, u_{SS}}(u - u_{SS})$$

For completeness, again note the explicit dependencies

$$\begin{aligned} f(x(t), u(t)) &\approx f(x_{SS}, u_{SS}) + \frac{\partial f(x(t), u(t))}{\partial x} \Big|_{x_{SS}, u_{SS}} (x(t) - x_{SS}) \\ &+ \frac{\partial f(x(t), u(t))}{\partial u} \Big|_{x_{SS}, u_{SS}} (u(t) - u_{SS}) \end{aligned}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

A single state and a single input variable (cont.)

Since (x_{SS}, u_{SS}) is chosen to be a fixed-point, we have $\frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big|_{x_{SS}, u_{SS}} = f(x_{SS}, u_{SS}) = 0$

Thus, we can write

$$f(x,u) \approx \underbrace{f(x_{SS}, u_{SS})}_{=0} + \frac{\partial f(x,u)}{\partial x}\Big|_{x_{SS}, u_{SS}}(x - x_{SS}) + \frac{\partial f(x,u)}{\partial u}\Big|_{x_{SS}, u_{SS}}(u - u_{SS})$$

$$\rightsquigarrow \dot{x}$$

We can define again perturbation/deviation variables $x' = x - x_{SS}$ and $u' = u - u_{SS}$ By computing the time-derivative of the (perturbed) state variables, we get

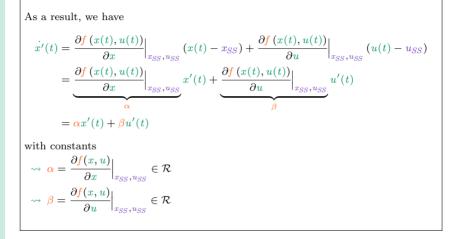
$$\frac{\mathrm{d}(x(t) - x_{SS})}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \underbrace{\frac{\mathrm{d}x_{SS}}{\mathrm{d}t}}_{=0}$$
$$= \dot{x}(t)$$
$$\rightsquigarrow f(x(t), u(t))$$

Single state var

Single state var and single input

Multiple state and multiple inputs

A single state and a single input variable (cont.)



Single state var

Single state var and single input

Multiple states and multiple inputs

A single state, a single input, and a single output variable

Now, suppose that there exists also a single measurement variable, y(t) = g(x(t), u(t))

We treat function g (x, u) similarly, by using a Taylor expansion around (x_{SS}, u_{SS})
Then, we truncate the expasion to keep only first-order terms

$$g(x,u) \approx \underbrace{g(x_{SS}, u_{SS})}_{y_{SS}} + \underbrace{\frac{\partial g(x,u)}{\partial x}}_{\gamma} |_{x_{SS}, u_{SS}} (x - x_{SS}) + \underbrace{\frac{\partial g(x,u)}{\partial u}}_{\delta} |_{x_{SS}, u_{SS}} (u - u_{SS})$$

 $y_{SS} = g(x_{SS}, u_{SS})$ is the fixed-point y_{SS} of the measurement (not necessarily zero!)

$$y(t) \approx g(x_{SS}, u_{SS}) + \frac{\partial g(x(t), u(t))}{\partial x}\Big|_{x_{SS}, u_{SS}} (x(t) - x_{SS}) + \frac{\partial g(x(t), u(t))}{\partial u}\Big|_{x_{SS}, u_{SS}} (u(t) - u_{SS})$$

And, we can also introduce a perturbation variable for the measurements, to get

$$\underbrace{y(t) - y_{SS}}_{y'(t)} = \gamma x'(t) + \delta u'(t)$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables

We can easily generalise the procedure to process models of arbitrary dimensionality

• That is, with an arbitrary number of state, input and output variables

Consider a system with two state variables $x = (x_1, x_2)'$, one input u, one output y

 $\begin{cases} \begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t), u(t)) \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t), u(t)) \\ y(t) = g(x_1(t), x_2(t), u(t)) \end{cases} \rightsquigarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1(t), x_2(t), u(t)) \\ f_2(x_1(t), x_2(t), u(t)) \end{bmatrix}$

We can linearise this system by using truncated Taylor series expansions of f and g

• Around a fixed point $(x_{SS}, u_{SS}) = ((x_1^{SS}, x_2^{SS}), u_{SS})$

 x_{SS}

Note that now function f is vector-valued (two values), it is two functions $f = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$

- They need to be treated (linearised) individually
- $\rightsquigarrow~$ With respect to each state variable
- \leadsto With respect to the input variable

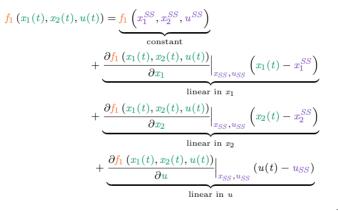
Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

We start with $f_1(x_1(t), x_2(t), u(t))$, then after expanding and truncating we obtain



+ H.O. terms

At any fixed-point the constant term is equal to zero, thus $f_1(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

By retaining only first-order terms, we get the linear approximation of function f(x, u)

$$f_{1}(x_{1}(t), x_{2}(t), u(t)) \approx \underbrace{\frac{\partial f_{1}(x_{1}(t), x_{2}(t), u(t))}{\partial x_{1}}}_{a_{11}} \Big|_{x_{SS}, u_{SS}} \left(x_{1}(t) - x_{1}^{SS}\right) \\ + \underbrace{\frac{\partial f_{1}(x_{1}(t), x_{2}(t), u(t))}{\partial x_{2}}}_{a_{12}} \Big|_{x_{SS}, u_{SS}} \left(x_{2}(t) - x_{2}^{SS}\right) \\ + \underbrace{\frac{\partial f_{1}(x_{1}(t), x_{2}(t), u(t))}{\partial u}}_{b_{1}} \Big|_{x_{SS}, u_{SS}} (u(t) - u_{SS})$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_1(x_1, x_2, u) = a_{11} \underbrace{(x_1 - x_1^{SS})}_{x_1'} + a_{12} \underbrace{(x_2 - x_2^{SS})}_{x_2'} + b_1 \underbrace{(u - u_{SS})}_{u'}$$

The constants are the partials of f_1 with respect to x_1 , x_2 , and u, at (x_{SS}, u_{SS})

$$\Rightarrow a_{11} = \frac{\partial f_1}{\partial x_1} \Big|_{x_{SS}, u_{SS}}$$

$$\Rightarrow a_{12} = \frac{\partial f_1}{\partial x_2} \Big|_{x_{SS}, u_{SS}}$$

$$\Rightarrow b_1 = \frac{\partial f_1}{\partial u} \Big|_{x_{SS}, u_{SS}}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables

Similarly for function $f_2(x_1(t), x_2(t), u(t))$, by Taylor expansion and truncation we get

$$\begin{aligned} f_{2}\left(x_{1}(t), x_{2}(t), u(t)\right) &= \underbrace{f_{2}\left(x_{1}^{SS}, x_{2}^{SS}, u^{SS}\right)}_{\text{constant}} \\ &+ \underbrace{\frac{\partial f_{2}\left(x_{1}(t), x_{2}(t), u(t)\right)}{\partial x_{1}}\Big|_{x_{SS}, u_{SS}}\left(x_{1}(t) - x_{1}^{SS}\right)}_{\text{lineair in } x_{1}} \\ &+ \underbrace{\frac{\partial f_{2}\left(x_{1}(t), x_{2}(t), u(t)\right)}{\partial x_{2}}\Big|_{x_{SS}, u_{SS}}\left(x_{2}(t) - x_{2}^{SS}\right)}_{\text{linear in } x_{2}} \\ &+ \underbrace{\frac{\partial f_{2}\left(x_{1}(t), x_{2}(t), u(t)\right)}{\partial u}\Big|_{x_{SS}, u_{SS}}\left(u(t) - u_{SS}\right)}_{x_{SS}, u_{SS}} + \end{aligned}$$

Because at some fixed-point, we have again a zero constant term $f_2(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

H.O. terms

Single state var

Single state var and single input

Multiple states and multiple inputs

 f_2

Multiple state and multiple input variables (cont.)

Again, by retaining only the first-order terms we get the second linear approximation

$$(x_{1}(t), x_{2}(t), u(t)) \approx \underbrace{\frac{\partial f_{2}(x_{1}(t), x_{2}(t), u(t))}{\partial x_{1}}}_{a_{21}} \Big|_{x_{SS}, u_{SS}} \underbrace{(x_{1}(t) - x_{1}^{SS})}_{x_{1}'} \\ + \underbrace{\frac{\partial f_{2}(x_{1}(t), x_{2}(t), u(t))}{\partial x_{2}}}_{a_{22}} \Big|_{x_{SS}, u_{SS}} \underbrace{(x_{2}(t) - x_{2}^{SS})}_{x_{2}'} \\ + \underbrace{\frac{\partial f_{2}(x_{1}(t), x_{2}(t), u(t))}{\partial u}}_{b_{2}} \Big|_{x_{SS}, u_{SS}} \underbrace{(u(t) - u_{SS})}_{u'}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_{2}(x_{1}, x_{2}, u) = a_{21} \underbrace{(x_{1} - x_{1}^{SS})}_{x_{1}'} + a_{22} \underbrace{(x_{2} - x_{2}^{SS})}_{x_{2}'} + b_{2} \underbrace{(u - u_{SS})}_{u'}$$

The constants are the partials of f_2 with respect to x_1 , x_2 , and u, at (x_{SS}, u_{SS})

$$\Rightarrow a_{21} = \frac{\partial f_2}{\partial x_1} \Big|_{x_{SS}, u_{SS}}$$

$$\Rightarrow a_{22} = \frac{\partial f_2}{\partial x_2} \Big|_{x_{SS}, u_{SS}}$$

$$\Rightarrow b_2 = \frac{\partial f_2}{\partial u} \Big|_{x_{SS}, u_{SS}}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

By collecting the linear approximation results for function f_1 and f_2 , we have

$$f_{1}(x_{1}, x_{2}, u) = a_{11} \underbrace{(x_{1} - x_{1}^{SS})}_{x_{1}'(t)} + a_{12} \underbrace{(x_{2} - x_{2}^{SS})}_{x_{2}'(t)} + b_{1} \underbrace{(u - u_{SS})}_{u'(t)}$$
$$= \frac{dx_{1}(t)}{dt}$$
$$= \frac{dx_{1}(t)}{dt}$$
$$= \frac{d(x_{1}(t) - x_{1}^{SS})}{dt} = \dot{x}_{1}'(t)$$

$$\begin{array}{l} \rightsquigarrow \quad \dot{x_1'}(t) = a_{11}x_1'(t) + a_{12}x_2'(t) + b_1u'(t) \\ \\ \rightsquigarrow \quad \dot{x_2'}(t) = a_{21}x_1'(t) + a_{22}x_2'(t) + b_2u'(t) \end{array}$$

$$= \frac{\mathrm{d}x_{2}(t)}{\mathrm{d}t}$$

= $\frac{\mathrm{d}(x_{2}(t) - x_{2}^{SS})}{\mathrm{d}t} = \dot{x}_{2}'(t)$
 $f_{2}(x_{1}, x_{2}, u) = a_{21}\underbrace{(x_{1} - x_{1}^{SS})}_{x_{1}'(t)} + a_{22}\underbrace{(x_{2} - x_{2}^{SS})}_{x_{2}(t)} + b_{2}\underbrace{(u - u_{SS})}_{u'(t)}$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

We can combine the equations, to get the linearised state-space model



In matrix form, we get the compact formulation

 $\dot{x'}(t) = \mathbf{A}x'(t) + \mathbf{b}u'(t)$

Remember that matrix A and b contain the partials of f with respect to x and u

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_1(x_1, x_2, u)}{\partial x_2} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_2(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$
$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial u} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial u} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

Single state var

Single state var and single input

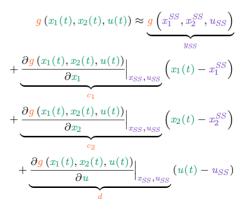
Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

We proceed similarly to linearise the read-out function g around the point (x_{SS}, u_{SS}) In the case of a single output measurement $y \in \mathcal{R}$, we have

 $y(t) = g(x_1(t), x_2(t), u(t))$

The first-order approximation of g(x, u),

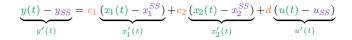


Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)



We can again get a compact formulation,

$$y'(t) = \underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix}}_{x'(t)} + \frac{du'(t)}{du'(t)}$$

Remember that matrix C and d contain the partials of g with respect to x and u

$$egin{aligned} C &= egin{bmatrix} c_1 & c_2 \end{bmatrix} &= egin{bmatrix} rac{\partial g(x_1, x_2, u)}{\partial x_1} & rac{\partial g(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}} \ d &= egin{bmatrix} d \end{bmatrix} = egin{bmatrix} rac{\partial g(x_1, x_2, u)}{\partial u} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}} \end{aligned}$$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

For the general case where $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$ and $y \in \mathcal{R}^{N_y}$, we have

 \rightsquigarrow State-space equation, $\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{N_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial x_1} & \cdots & \frac{\partial f_{N_x}}{\partial x_{N_x}} \end{bmatrix} \begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_{N_x}}{\partial u_{N_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial x_1} & \cdots & \frac{\partial f_{N_x}}{\partial u_{N_u}} \end{bmatrix}$ $\dot{x'}(t)$ $= \begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix} \begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,N_u} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_u} \end{bmatrix} \begin{bmatrix} u_1'(t) \\ \vdots \\ u_{N_u}'(t) \end{bmatrix}$ $(N_x \times N_x)$ $(N_x \times N_u)$ $(N_T \times 1)$ $(N_u \times 1)$

Single state var

Single state var and single input

Multiple states and multiple inputs

Multiple state and multiple input variables (cont.)

For general case where $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$ and $y \in \mathcal{R}^{N_y}$, we have

