Aalto University

# Linearisation of nonlinear state-space models

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# Linearisation of nonlinear state-space models

Many dynamical models used to describe processes in chemical engineering are given as a set of first-order ordinary differential equations, the equations are very often nonlinear

- From the application of material and energy conservation laws

A body of commonly used techniques for system analysis and control uses linear models

- To access such a technology we need to simplify common process models
- This will require approximating the general state-space representation
- The model approximations of our interest are Jacobian linearisation

$$\underbrace{\begin{cases} \dot{x}(t) = f\left(x(t), u(t) | \theta_x\right) \\ y(t) = g\left(x(t), u(t) | \theta_y\right) \end{cases}}_{\text{Nonlinear dynamics and read-out}} \quad \rightsquigarrow \quad \underbrace{\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}}_{\text{Linear dynamics and read-out}}$$

The main idea behind the linearisation of nonlinear process models in state-space form

- $\rightsquigarrow$ Approximate function $f$ (nonlinear) with matrices $A$ and $B$
- $\rightsquigarrow$ Approximate function $g$ (nonlinear) with matrices $C$ and $D$

# Linearisation of nonlinear state-space models

We discuss how to determine a linear approximation of a nonlinear state-space model

- ⇝ We study a number of cases, of increasing complexity
- ⇝ As a result, we will be able to linearise any[1] model

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$$\underbrace{\begin{cases} \dot{x}(t) = f\,(x(t), u(t)|\theta_x) \\ y(t) = g\,(x(t), u(t)|\theta_y) \end{cases}}_{\text{Nonlinear dynamics and read-out}}$$

In general, we have a general state-space model with variables of arbitrary dimension

- ⇝ $x(t) \in \mathcal{R}^{N_x}$
- ⇝ $u(t) \in \mathcal{R}^{N_u}$
- ⇝ $y(t) \in \mathcal{R}^{N_y}$

We start with $N_x = 1$ (one state), $N_u = 0$ (no inputs) and $N_y = 1$ (one measurement)

- Then, we will add complexity (we add more variables)

[1]We only require that model functions $f$ and $g$ are continuous and differentiable.

# A single state variable, no inputs

$$\dot{x}(t) = f\left(x(t)\right), \quad \text{with } x(t) \in \mathcal{R}$$

We suppose that function $f(x(t))$ can be approximated by a Taylor series expansion

- We are interested in an approximation around an fixed point, $x(t) = x_{SS}$
- At the steady-state point $x_{SS}$, time variations are zero $\dot{x}(t) = 0$
- Thus, we also have that $f\left(x_{SS}\right) = 0$, whatever the time $t$

We can perfectly represent function $f(x)$ by using an infinite Taylor series expansion

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}}(x - x_{SS})}_{\text{linear in } x} + \underbrace{\frac{1}{2}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\Big|_{x_{SS}}(x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\mathcal{O}(x^3)}_{\text{H.O. terms}}$$

The expansion is a sum of polynomials of $x$, with the derivatives of $f$ as coefficients

- Given this representation of $f$, we want to use it to approximate $f$

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Note that steady-state, stationary, fixed, equilibrium points are all equivalent terms

# A single state variable, no inputs (cont.)

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}}(x - x_{SS})}_{\text{linear in } x} + \underbrace{\frac{1}{2}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\Big|_{x_{SS}}(x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\mathcal{O}(x^3)}_{\text{H.O. terms}}$$

Suppose that we are interested in an approximation based only on first-order terms

- We accept to neglect (truncate out) second- and higher-order terms

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}}(x - x_{SS})}_{\text{linear in } x} + \underbrace{\mathcal{O}(x^2)}_{\text{quadratic and H.O. terms}}$$

After truncation, we get an approximation of $f$ which is a linear function of $x$

$$f(x) \approx f(x_{SS}) + \frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|_{x_{SS}}(x - x_{SS})$$

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Note the complete expression of $f\,(x(t))$ with explicit dependencies also wrt time $t$

$$f\,(x(t)) \approx f(x_{SS}) + \frac{\mathrm{d}f\,(x(t))}{\mathrm{d}x}\Big|_{x_{SS}}(x(t) - x_{SS})$$

# A single state variable, no inputs (cont.)

$$f(x) \approx \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\bigg|_{x_{SS}} (x - x_{SS})}_{\text{linear in } x}$$

Because $x_{SS}$ is chosen to be a fixed-point, we have $\frac{\mathrm{d}x(t)}{\mathrm{d}t}\bigg|_{x_{SS}} = \dot{x}(t)\bigg|_{x_{SS}} = f(x_{SS}) = 0$

Thus, we can write

$$f(x) \approx \underbrace{\cancel{f(x_{SS})}}_{=0} + \frac{\mathrm{d}f(x)}{\mathrm{d}x}\bigg|_{x_{SS}} (x - x_{SS})$$

$$\rightsquigarrow \dot{x}$$

# A single state variable, no inputs (cont.)

$$\dot{x}(t) \approx \frac{\mathrm{d}f\,(x(t))}{\mathrm{d}x}\bigg|_{x_{SS}} (x(t) - x_{SS})$$

We can now introduce a **perturbation** or **deviation variable** $x'(t) = x(t) - x_{SS}$

- Variable $x'(t)$ encodes how far state variable $x(t)$ is from steady-state $x_{SS}$
- (And, because variable $x(t)$ varies with time, also $x'(t)$ varies)

Therefore, we can also compute the time-derivative of perturbation variable $x'(t)$

- It describes the rate of change of variable $x'(t)$ with respect to time
- Or, equivalently, its dynamics

By differenting $x'(t)$ to get $\mathrm{d}x'(t)\mathrm{d}t$, we have

$$\begin{aligned}\frac{\mathrm{d}\,(x(t) - x_{SS})}{\mathrm{d}t} &= \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \underbrace{\cancel{\frac{\mathrm{d}x_{SS}}{\mathrm{d}t}}}_{=0} \\ &= \dot{x}(t) \\ &\rightsquigarrow f\,(x(t))\end{aligned}$$

Deviation/perturbation variables and state variables have identical dynamics

# A single state variable, no inputs (cont.)

We obtained that $\frac{\mathrm{d}\left(x(t)-x_{SS}\right)}{\mathrm{d}t} = f\left(x(t)\right)$ and $f\left(x(t)\right) \approx \left.\frac{\mathrm{d}f\left(x(t)\right)}{\mathrm{d}x}\right|_{x_{SS}} \left(x(t)-x_{SS}\right)$

We can equate the two terms and write

$$\frac{\mathrm{d}\left(x(t)-x_{SS}\right)}{\mathrm{d}t} = f\left(x(t)\right) \approx \underbrace{\left.\frac{\mathrm{d}f\left(x(t)\right)}{\mathrm{d}x}\right|_{x_{SS}}}_{\text{constant}} \left(x(t)-x_{SS}\right)$$

We have derived the approximated state equation for the deviation variable $x'(t)$

$$\dot{x}'(t) \approx \left.\frac{\mathrm{d}f\left(x(t)\right)}{\mathrm{d}x}\right|_{x_{SS}} \left(x(t)-x_{SS}\right)$$

This is a linear time-invariant approximation of the (perturbed) state equation

$$\dot{x}'(t) = \alpha x'(t), \quad \text{with constant } \alpha = \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x_{SS}} \in \mathcal{R}$$

We also know how solve it for some initial condition $x'(0)$

$$x'(t) = e^{\alpha t} x'(0)$$

# A single state and a single input variable

$$\dot{x}(t) = f\left(x(t), u(t)\right), \quad \text{with } x, u \in \mathcal{R}$$

We assume that function $f(x, u)$ can be approximated by a Taylor series expansion

- We are interested in an approximation around some fixed-point point

$$(x = x_{SS}, u = u_{SS})$$

- At fixed points $(x = x_{SS}, u = u_{SS})$, time-variations are zero $\dot{x}(t) = 0$

$\rightsquigarrow$ (The state remains fixed at $x_{SS}$ as long as the input is fixed, at $u_{SS}$)

Because at steady-state there is no evolution, the right hand-side is zero

$$\rightsquigarrow \quad f(x = x_{SS}, u = u_{SS}) = 0$$

# A single state and a single input variable

Suppose that we can perfectly represent $f(x, u)$ by using an infinite Taylor expansion

$$f(x,u) = \underbrace{f(x_{SS}, u_{SS})}_{\text{constant}} + \underbrace{\left.\frac{\partial f(x,u)}{\partial x}\right|_{x_{SS},u_{SS}} (x - x_{SS})}_{\text{linear in } x} + \underbrace{\left.\frac{\partial f(x,u)}{\partial u}\right|_{x_{SS},u_{SS}} (u - u_{SS})}_{\text{linear in } u}$$

$$+ \underbrace{\left.\frac{\partial^2 f(x,u)}{\partial x^2}\right|_{x_{SS},u_{SS}} (x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\left.\frac{\partial^2 f(x,u)}{\partial u^2}\right|_{x_{SS},u_{SS}} (u - u_{SS})^2}_{\text{quadratic in } u}$$

$$+ \underbrace{\left.\frac{\partial^2 f(x,u)}{\partial x \partial u}\right|_{x_{SS},u_{SS}} (x - x_{SS})(u - u_{SS})}_{\text{quadratic}}$$

$$+ \text{ H.O. terms}$$

Suppose that we are interested in an approximation based only on first-order terms

- By truncation, we accept to neglect second- and higher-order terms

# A single state and a single input variable (cont.)

We can again obtain a linear, first-order, approximation of function $f$ by truncation

$$f(x, u) \approx f(x_{SS}, u_{SS}) + \frac{\partial f(x, u)}{\partial x}\Big|_{x_{SS}, u_{SS}} (x - x_{SS}) + \frac{\partial f(x, u)}{\partial u}\Big|_{x_{SS}, u_{SS}} (u - u_{SS})$$

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For completeness, again note the explicit dependencies

$$\begin{aligned} f\left(x(t), u(t)\right) \approx f\left(x_{SS}, u_{SS}\right) &+ \frac{\partial f\left(x(t), u(t)\right)}{\partial x}\Big|_{x_{SS}, u_{SS}} \left(x(t) - x_{SS}\right) \\ &+ \frac{\partial f\left(x(t), u(t)\right)}{\partial u}\Big|_{x_{SS}, u_{SS}} \left(u(t) - u_{SS}\right) \end{aligned}$$

# A single state and a single input variable (cont.)

Since $(x_{SS}, u_{SS})$ is chosen to be a fixed-point, we have $\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{x_{SS},u_{SS}} = f(x_{SS}, u_{SS}) = 0$

Thus, we can write

$$f(x,u) \approx \underbrace{\cancel{f(x_{SS}, u_{SS})}}_{=0} + \left.\frac{\partial f(x,u)}{\partial x}\right|_{x_{SS},u_{SS}} (x - x_{SS}) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{x_{SS},u_{SS}} (u - u_{SS})$$

$$\rightsquigarrow \dot{x}$$

We can define again perturbation/deviation variables $x' = x - x_{SS}$ and $u' = u - u_{SS}$

By computing the time-derivative of the (perturbed) state variables, we get

$$\frac{\mathrm{d}(x(t) - x_{SS})}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} - \underbrace{\cancel{\frac{\mathrm{d}x_{SS}}{\mathrm{d}t}}}_{=0}$$

$$= \dot{x}(t)$$

$$\rightsquigarrow f(x(t), u(t))$$

# A single state and a single input variable (cont.)

As a result, we have

$$\begin{aligned}\dot{x}'(t) &= \left.\frac{\partial f\,(x(t),u(t))}{\partial x}\right|_{x_{SS},u_{SS}}(x(t)-x_{SS}) + \left.\frac{\partial f\,(x(t),u(t))}{\partial u}\right|_{x_{SS},u_{SS}}(u(t)-u_{SS}) \\ &= \underbrace{\left.\frac{\partial f\,(x(t),u(t))}{\partial x}\right|_{x_{SS},u_{SS}}}_{\alpha} x'(t) + \underbrace{\left.\frac{\partial f\,(x(t),u(t))}{\partial u}\right|_{x_{SS},u_{SS}}}_{\beta} u'(t) \\ &= \alpha x'(t) + \beta u'(t)\end{aligned}$$

with constants

$$\rightsquigarrow \ \alpha = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x_{SS},u_{SS}} \in \mathcal{R}$$

$$\rightsquigarrow \ \beta = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x_{SS},u_{SS}} \in \mathcal{R}$$

# A single state, a single input, and a single output variable

Now, suppose that there exists also a single measurement variable, $y(t) = g\left(x(t), u(t)\right)$

We treat function $g\left(x, u\right)$ similarly, by using a Taylor expansion around $(x_{SS}, u_{SS})$

- Then, we truncate the expasiom to keep only first-order terms

$$g(x,u) \approx \underbrace{g(x_{SS}, u_{SS})}_{y_{SS}} + \underbrace{\frac{\partial g(x,u)}{\partial x}\Big|_{x_{SS}, u_{SS}}}_{\gamma}(x - x_{SS}) + \underbrace{\frac{\partial g(x,u)}{\partial u}\Big|_{x_{SS}, u_{SS}}}_{\delta}(u - u_{SS})$$

$y_{SS} = g(x_{SS}, u_{SS})$ is the fixed-point $y_{SS}$ of the measurement (not necessarily zero!)

$$\begin{aligned} y(t) \approx\; & g(x_{SS}, u_{SS}) \\ & + \frac{\partial g\left(x(t), u(t)\right)}{\partial x}\Big|_{x_{SS}, u_{SS}}\left(x(t) - x_{SS}\right) + \frac{\partial g\left(x(t), u(t)\right)}{\partial u}\Big|_{x_{SS}, u_{SS}}\left(u(t) - u_{SS}\right) \end{aligned}$$

And, we can also introduce a perturbation variable for the measurements, to get

$$\underbrace{y(t) - y_{SS}}_{y'(t)} = \gamma x'(t) + \delta u'(t)$$

# Multiple state and multiple input variables

We can easily generalise the procedure to process models of arbitrary dimensionality

- That is, with an arbitrary number of state, input and output variables

Consider a system with two state variables $x = (x_1, x_2)'$, one input $u$, one output $y$

$$\begin{cases} \begin{cases} \dot{x}_1(t) = f_1\left(x_1(t), x_2(t), u(t)\right) \\ \dot{x}_2(t) = f_2\left(x_1(t), x_2(t), u(t)\right) \end{cases} \\ y(t) = g\left(x_1(t), x_2(t), u(t)\right) \end{cases} \rightsquigarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1\left(x_1(t), x_2(t), u(t)\right) \\ f_2\left(x_1(t), x_2(t), u(t)\right) \end{bmatrix}$$

We can linearise this system by using truncated Taylor series expansions of $f$ and $g$

- Around a fixed point $(x_{SS}, u_{SS}) = (\underbrace{(x_1^{SS}, x_2^{SS})}_{x_{SS}}, u_{SS})$

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Note that now function $f$ is vector-valued (two values), it is two functions $f = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$

- They need to be treated (linearised) individually

⇝ With respect to each state variable

⇝ With respect to the input variable

# Multiple state and multiple input variables (cont.)

We start with $f_1\left(x_1(t), x_2(t), u(t)\right)$, then after expanding and truncating we obtain

$$f_1\left(x_1(t), x_2(t), u(t)\right) = \underbrace{f_1\left(x_1^{SS}, x_2^{SS}, u^{SS}\right)}_{\text{constant}}$$

$$+ \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial x_1}\right|_{x_{SS}, u_{SS}}\left(x_1(t) - x_1^{SS}\right)}_{\text{linear in } x_1}$$

$$+ \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial x_2}\right|_{x_{SS}, u_{SS}}\left(x_2(t) - x_2^{SS}\right)}_{\text{linear in } x_2}$$

$$+ \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial u}\right|_{x_{SS}, u_{SS}}\left(u(t) - u_{SS}\right)}_{\text{linear in } u}$$

$$+ \text{H.O. terms}$$

At any fixed-point the constant term is equal to zero, thus $f_1(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

# Multiple state and multiple input variables (cont.)

By retaining only first-order terms, we get the linear approximation of function $f(x, u)$

$$f_1\left(x_1(t), x_2(t), u(t)\right) \approx \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial x_1}\right|_{x_{SS}, u_{SS}}}_{a_{11}} \left(x_1(t) - x_1^{SS}\right)$$

$$+ \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial x_2}\right|_{x_{SS}, u_{SS}}}_{a_{12}} \left(x_2(t) - x_2^{SS}\right)$$

$$+ \underbrace{\left.\frac{\partial f_1\left(x_1(t), x_2(t), u(t)\right)}{\partial u}\right|_{x_{SS}, u_{SS}}}_{b_1} \left(u(t) - u_{SS}\right)$$

# Multiple state and multiple input variables (cont.)

We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_1(x_1, x_2, u) = a_{11}\underbrace{\left(x_1 - x_1^{SS}\right)}_{x_1'} + a_{12}\underbrace{\left(x_2 - x_2^{SS}\right)}_{x_2'} + b_1\underbrace{\left(u - u_{SS}\right)}_{u'}$$

The constants are the partials of $f_1$ with respect to $x_1$, $x_2$, and $u$, at $(x_{SS}, u_{SS})$

- $\rightsquigarrow$ $a_{11} = \left.\dfrac{\partial f_1}{\partial x_1}\right|_{x_{SS}, u_{SS}}$
- $\rightsquigarrow$ $a_{12} = \left.\dfrac{\partial f_1}{\partial x_2}\right|_{x_{SS}, u_{SS}}$
- $\rightsquigarrow$ $b_1 = \left.\dfrac{\partial f_1}{\partial u}\right|_{x_{SS}, u_{SS}}$

# Multiple state and multiple input variables

Similarly for function $f_2(x_1(t), x_2(t), u(t))$, by Taylor expansion and truncation we get

$$f_2(x_1(t), x_2(t), u(t)) = \underbrace{f_2\left(x_1^{SS}, x_2^{SS}, u^{SS}\right)}_{\text{constant}}$$
$$+ \underbrace{\frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_1}\bigg|_{x_{SS}, u_{SS}}\left(x_1(t) - x_1^{SS}\right)}_{\text{lineair in } x_1}$$
$$+ \underbrace{\frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_2}\bigg|_{x_{SS}, u_{SS}}\left(x_2(t) - x_2^{SS}\right)}_{\text{linear in } x_2}$$
$$+ \frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial u}\bigg|_{x_{SS}, u_{SS}}(u(t) - u_{SS})$$
$$+ \text{H.O. terms}$$

Because at some fixed-point, we have again a zero constant term $f_2(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

# Multiple state and multiple input variables (cont.)

Again, by retaining only the first-order terms we get the second linear approximation

$$f_2\left(x_1(t), x_2(t), u(t)\right) \approx \underbrace{\left.\frac{\partial f_2\left(x_1(t), x_2(t), u(t)\right)}{\partial x_1}\right|_{x_{SS}, u_{SS}}}_{a_{21}} \underbrace{\left(x_1(t) - x_1^{SS}\right)}_{x_1'}$$
$$+ \underbrace{\left.\frac{\partial f_2\left(x_1(t), x_2(t), u(t)\right)}{\partial x_2}\right|_{x_{SS}, u_{SS}}}_{a_{22}} \underbrace{\left(x_2(t) - x_2^{SS}\right)}_{x_2'}$$
$$+ \underbrace{\left.\frac{\partial f_2\left(x_1(t), x_2(t), u(t)\right)}{\partial u}\right|_{x_{SS}, u_{SS}}}_{b_2} \underbrace{\left(u(t) - u_{SS}\right)}_{u'}$$

# Multiple state and multiple input variables (cont.)

We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_2(x_1, x_2, u) = a_{21}\underbrace{\left(x_1 - x_1^{SS}\right)}_{x_1'} + a_{22}\underbrace{\left(x_2 - x_2^{SS}\right)}_{x_2'} + b_2\underbrace{\left(u - u_{SS}\right)}_{u'}$$

The constants are the partials of $f_2$ with respect to $x_1$, $x_2$, and $u$, at $(x_{SS}, u_{SS})$

- $\rightsquigarrow$ $a_{21} = \left.\dfrac{\partial f_2}{\partial x_1}\right|_{x_{SS}, u_{SS}}$
- $\rightsquigarrow$ $a_{22} = \left.\dfrac{\partial f_2}{\partial x_2}\right|_{x_{SS}, u_{SS}}$
- $\rightsquigarrow$ $b_2 = \left.\dfrac{\partial f_2}{\partial u}\right|_{x_{SS}, u_{SS}}$

# Multiple state and multiple input variables (cont.)

By collecting the linear approximation results for function $f_1$ and $f_2$, we have

$$f_1(x_1, x_2, u) = a_{11} \underbrace{\left(x_1 - x_1^{SS}\right)}_{x_1'(t)} + a_{12} \underbrace{\left(x_2 - x_2^{SS}\right)}_{x_2'(t)} + b_1 \underbrace{\left(u - u_{SS}\right)}_{u'(t)}$$

$$= \frac{\mathrm{d}x_1(t)}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}\left(x_1(t) - x_1^{SS}\right)}{\mathrm{d}t} = \dot{x}_1'(t)$$

$$\rightsquigarrow \quad \dot{x}_1'(t) = a_{11}x_1'(t) + a_{12}x_2'(t) + b_1 u'(t)$$

$$\rightsquigarrow \quad \dot{x}_2'(t) = a_{21}x_1'(t) + a_{22}x_2'(t) + b_2 u'(t)$$

$$= \frac{\mathrm{d}x_2(t)}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}\left(x_2(t) - x_2^{SS}\right)}{\mathrm{d}t} = \dot{x}_2'(t)$$

$$f_2(x_1, x_2, u) = a_{21} \underbrace{\left(x_1 - x_1^{SS}\right)}_{x_1'(t)} + a_{22} \underbrace{\left(x_2 - x_2^{SS}\right)}_{x_2(t)} + b_2 \underbrace{\left(u - u_{SS}\right)}_{u'(t)}$$

# Multiple state and multiple input variables (cont.)

We can combine the equations, to get the linearised state-space model

$$\underbrace{\begin{bmatrix} \dot{x}_1'(t) \\ \dot{x}_2'(t) \end{bmatrix}}_{\dot{x}'(t)} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{x'(t)} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b} u'(t)$$

In matrix form, we get the compact formulation

$$\dot{x}'(t) = Ax'(t) + bu'(t)$$

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Remember that matrix $A$ and $b$ contain the partials of $f$ with respect to $x$ and $u$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_1(x_1, x_2, u)}{\partial x_2} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_2(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial u} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial u} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

# Multiple state and multiple input variables (cont.)

We proceed similarly to linearise the read-out function $g$ around the point $(x_{SS}, u_{SS})$

In the case of a single output measurement $y \in \mathcal{R}$, we have

$$y(t) = g\left(x_1(t), x_2(t), u(t)\right)$$

The first-order approximation of $g(x, u)$,

$$\begin{aligned}
g\left(x_1(t), x_2(t), u(t)\right) \approx & \underbrace{g\left(x_1^{SS}, x_2^{SS}, u_{SS}\right)}_{y_{SS}} \\
& + \underbrace{\left.\frac{\partial g\left(x_1(t), x_2(t), u(t)\right)}{\partial x_1}\right|_{x_{SS}, u_{SS}}}_{c_1} \left(x_1(t) - x_1^{SS}\right) \\
& + \underbrace{\left.\frac{\partial g\left(x_1(t), x_2(t), u(t)\right)}{\partial x_2}\right|_{x_{SS}, u_{SS}}}_{c_2} \left(x_2(t) - x_2^{SS}\right) \\
& + \underbrace{\left.\frac{\partial g\left(x_1(t), x_2(t), u(t)\right)}{\partial u}\right|_{x_{SS}, u_{SS}}}_{d} \left(u(t) - u_{SS}\right)
\end{aligned}$$

# Multiple state and multiple input variables (cont.)

$$\underbrace{y(t) - y_{SS}}_{y'(t)} = c_1 \underbrace{\left(x_1(t) - x_1^{SS}\right)}_{x_1'(t)} + c_2 \underbrace{\left(x_2(t) - x_2^{SS}\right)}_{x_2'(t)} + d \underbrace{\left(u(t) - u_{SS}\right)}_{u'(t)}$$

We can again get a compact formulation,

$$y'(t) = \underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{x'(t)} + d\, u'(t)$$

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Remember that matrix $C$ and $d$ contain the partials of $g$ with respect to $x$ and $u$

$$C = \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial g(x_1, x_2, u)}{\partial x_1} & \dfrac{\partial g(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

$$d = \begin{bmatrix} d \end{bmatrix} = \begin{bmatrix} \dfrac{\partial g(x_1, x_2, u)}{\partial u} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

# Multiple state and multiple input variables (cont.)

For the general case where $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$ and $y \in \mathcal{R}^{N_y}$, we have

⇝ State-space equation,

$$\underbrace{\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix}}_{\dot{x}'(t)} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{N_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial x_1} & \cdots & \frac{\partial f_{N_x}}{\partial x_{N_x}} \end{bmatrix}_{SS}}_{A} \underbrace{\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix}}_{x'(t)} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_{N_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial u_1} & \cdots & \frac{\partial f_{N_x}}{\partial u_{N_u}} \end{bmatrix}_{SS}}_{B} \underbrace{\begin{bmatrix} u_1'(t) \\ \vdots \\ u_{N_u}'(t) \end{bmatrix}}_{u'(t)}$$

$$= \underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix}}_{(N_x \times N_x)} \underbrace{\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix}}_{(N_x \times 1)} + \underbrace{\begin{bmatrix} b_{1,1} & \cdots & b_{1,N_u} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_u} \end{bmatrix}}_{(N_x \times N_u)} \underbrace{\begin{bmatrix} u_1'(t) \\ \vdots \\ u_{N_u}'(t) \end{bmatrix}}_{(N_u \times 1)}$$

# Multiple state and multiple input variables (cont.)

For general case where $x \in \mathcal{R}^{N_x}$, $u \in \mathcal{R}^{N_u}$ and $y \in \mathcal{R}^{N_y}$, we have

⇝ Read-out map,

$$\underbrace{\begin{bmatrix} y_1'(t) \\ \vdots \\ y_{N_y}'(t) \end{bmatrix}}_{y'(t)} = \underbrace{\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_{N_x}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_{N_y}}{\partial x_1} & \cdots & \dfrac{\partial g_{N_y}}{\partial x_{N_x}} \end{bmatrix}_{SS}}_{C} \underbrace{\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix}}_{x'(t)} + \underbrace{\begin{bmatrix} \dfrac{\partial g_1}{\partial u_1} & \cdots & \dfrac{\partial g_1}{\partial u_{N_u}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_{N_y}}{\partial u_1} & \cdots & \dfrac{\partial g_{N_y}}{\partial u_{N_u}} \end{bmatrix}_{SS}}_{D} \underbrace{\begin{bmatrix} u_1'(t) \\ \vdots \\ u_{N_u}'(t) \end{bmatrix}}_{u'(t)}$$

$$= \underbrace{\begin{bmatrix} c_{1,1} & \cdots & c_{1,N_x} \\ \vdots & \ddots & \vdots \\ c_{N_y,1} & \cdots & c_{N_y,N_x} \end{bmatrix}}_{(N_y \times N_x)} \underbrace{\begin{bmatrix} x_1'(t) \\ \vdots \\ x_{N_x}'(t) \end{bmatrix}}_{(N_x \times 1)} + \underbrace{\begin{bmatrix} d_{1,1} & \cdots & d_{1,N_u} \\ \vdots & \ddots & \vdots \\ d_{N_y,1} & \cdots & d_{N_x,N_u} \end{bmatrix}}_{(N_y \times N_u)} \underbrace{\begin{bmatrix} u_1'(t) \\ \vdots \\ u_{N_u}'(t) \end{bmatrix}}_{(N_u \times 1)}$$