Aalto University

Linearisation of nonlinear models: Example

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Example II

Linearisation of nonlinear models

Example II

We consider two irreversible chemical reactions in a perfectly mixed chemical reactor

- $A \longrightarrow B \longrightarrow C$
- $2A \longrightarrow D$

The two reactions compete to convert species A, species B is the desired product

The chemical reactor operates in liquid-phase

⇝ Assume constant volume, $V \neq V(t)$

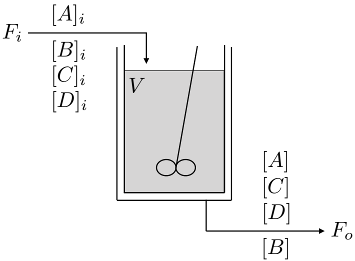
⇝ Assume constant density, $\rho \neq \rho(t)$

Assume constant temperature

⇝ $T(t) \neq T(t)$

Volumetric flow-rates

⇝ $F_i(t)$ and $F_o(t)$



Our interest is in understanding the dynamics of the concentrations inside the reactor

⇝ The concentration of species A, B, C and D, as a function of time

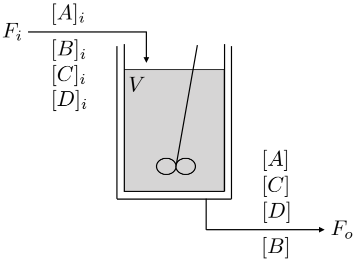
- $[A](t)$, $[B](t)$, $[C](t)$, and $[D](t)$ (molar concentrations, [mol lt^{-1}])

Example II (cont.)

Reaction rate constants (per unit volume)

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D$$



Assume that only component A is fed

- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$

The total material balance, under the assumption of a constant volume in the tank

Total mass balance

$$\frac{\mathrm{d}\,V(t)}{\mathrm{d}t} = F_i(t) - F_o(t) = 0$$

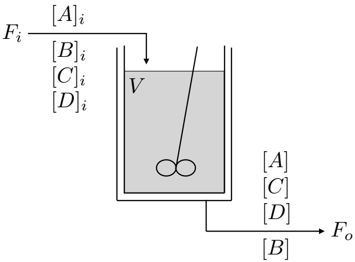
As a result, we simplify notation

- $V(t) = \text{constant} = V$
- $F_i(t) = F_o(t) = F(t)$

Example II (cont.)

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D$$



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$

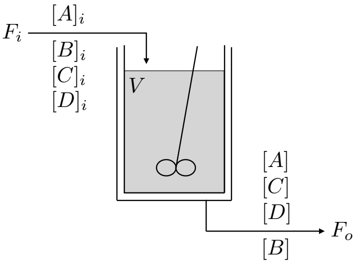
Mass balance for component A

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V[A](t) &= F(t)[A]_i(t) - F(t)[A](t) - Vk_1[A](t) - Vk_3[A]^2(t) \\ &= \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)\end{aligned}$$

Example II (cont.)

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D$$



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$

Mass balance for component B, C, and D

$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = \frac{F(t)}{V}\Big(\underbrace{\cancel{[B]_i(t)}}_{=0} - [B](t)\Big) + k_1[A](t) - k_2[B](t)$$

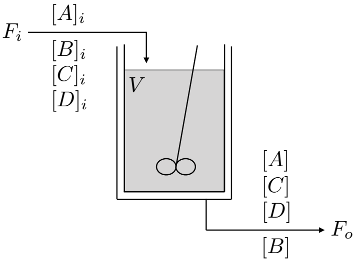
$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = \frac{F(t)}{V}\Big(\underbrace{\cancel{[C]_i(t)}}_{=0} - [C](t)\Big) + k_2[B](t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = \frac{F(t)}{V}\Big(\underbrace{\cancel{[D]_i(t)}}_{=0} - [D](t)\Big) + \frac{1}{2}k_3[A]^2(t)$$

Example II (cont.)

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D$$



- $[B]_i(t), [C]_i(t), [D]_i(t) = 0$
- $[A]_i(t) \neq 0$

Putting things together, we get the dynamics of the state-space model of the reactor

$$\frac{\mathrm{d}}{\mathrm{d}t}[A](t) = \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

$$k_1 = 5/6\ [\mathrm{min}^{-1}]$$

$$k_2 = 5/3\ [\mathrm{min}^{-1}]$$

$$k_3 = 1/6\ [\mathrm{lt}(\mathrm{mol}^{-1}\mathrm{min}^{-1})]$$

Example II (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}[A](t) = \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

$\rightsquigarrow$ State variables, $x(t)$

$$x(t) = \begin{bmatrix}[A](t)\\ [B](t)\\ [C](t)\\ [D](t)\end{bmatrix} = \begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix}$$

Example II (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}[A](t) = \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

⇝ Input variables, $u(t)$

$$u(t) = \begin{bmatrix} F_i(t) \\ [A]_i(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Example II (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}[A](t) = \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

$\rightsquigarrow$ Parameters, θ_x

$$\theta_x = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ V \end{bmatrix} = \begin{bmatrix} \theta_{x,1} \\ \theta_{x,2} \\ \theta_{x,3} \\ \theta_{x,4} \end{bmatrix}$$

Example II (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}[A](t) = \frac{F(t)}{V}\Big([A]_i(t) - [A](t)\Big) - k_1[A](t) - k_3[A]^2(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[B](t) = -\frac{F(t)}{V}[B](t) + k_1[A](t) - k_2[B](t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[C](t) = -\frac{F(t)}{V}[C](t) + k_2[B](t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[D](t) = -\frac{F(t)}{V}[D](t) + \frac{1}{2}k_3[A]^2(t)$$

Using the control notation, we get

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{u_1(t)}{\theta_{x,4}}\Big(u_2(t) - x_1(t)\Big) - \theta_{x,1}x_1(t) - k_3x_1^2(t)$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)$$

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t)$$

$$\frac{\mathrm{d}x_4(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)$$

Example II (cont.)

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \underbrace{\frac{u_1(t)}{\theta_{x,4}}\Big(u_2(t) - x_1(t)\Big) - \theta_{x,1}x_1(t) - k_3x_1^2(t)}_{f_1(x,u|\theta_x)}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)}_{f_2(x,u|\theta_x)}$$

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t)}_{f_3(x,u|\theta_x)}$$

$$\frac{\mathrm{d}x_4(t)}{\mathrm{d}t} = \underbrace{-\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)}_{f_4(x,u|\theta_x)}$$

$$\rightsquigarrow \quad \dot{x}(t) = f\left(x(t), u(t)|\theta_x\right)$$

Example II (cont.)

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \frac{u_1(t)}{\theta_{x,4}}\Big(u_2(t) - x_1(t)\Big) - \theta_{x,1}x_1(t) - k_3x_1^2(t)$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)$$

$$\frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_3(t) + \theta_{x,2}x_2(t)$$

$$\frac{\mathrm{d}x_4(t)}{\mathrm{d}t} = -\frac{u_1(t)}{\theta_{x,4}}x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)$$

Suppose that we are capable of measuring the concentration of B, we then also have

$$y(t) = \underbrace{\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{g(x(t),u(t)|\theta_x)}$$

Example II (cont.)

The dynamics are a set of nonlinear equations, the measurement equation is linear

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \\ \dot{x_4}(t) \end{bmatrix} = \begin{bmatrix} \frac{u_1(t)}{\theta_{x,4}}(u_2(t) - x_1(t)) - \theta_{x,1} x_1(t) - k_3 x_1^2(t) \\ -\frac{u_1(t)}{\theta_{x,4}} x_2(t) + \theta_{x,1} x_1(t) - \theta_{x,2} x_2(t) \\ -\frac{u_1(t)}{\theta_{x,4}} x_3(t) + \theta_{x,2} x_2(t) \\ -\frac{u_1(t)}{\theta_{x,4}} x_4(t) + \frac{1}{2}\theta_{x,3} x_1^2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

To be able to proceed with the tools of linear systems theory, we need to linearise

- Approximate nonlinearities with first-order Taylor series expansions
- About some convenient steady-state point, (x^{SS}, u^{SS})

$$x^{SS} = \begin{bmatrix} x_1^{SS} & x_2^{SS} & x_2^{SS} & x_2^{SS} \end{bmatrix}^T = \begin{bmatrix} [A]^{SS} & [B]^{SS} & [C]^{SS} & [D]^{SS} \end{bmatrix}^T$$
$$u^{SS} = \begin{bmatrix} u_1^{SS} & u_2^{SS} \end{bmatrix}^T - \begin{bmatrix} F_i^{SS} & [A]_i^{SS} \end{bmatrix}^T$$

Example II (cont.)

How to determine the steady-state point associated to a desirable operating conditions?

- By simulation, integrate the model until stationarity is reached
- By optimisation, solve $f(x,u) = 0$ with respect to x and u

Example II (cont.)

Sometimes, it can also be worked out from the model equations at steady-state (x_{SS}, u_{SS})

At steady-state all derivative are zero, for component $[A]$ we thus have

$$\begin{aligned}\frac{\mathrm{d}[A](t)}{\mathrm{d}t} &= \frac{F_i^{SS}}{V}\left([A]_i^{SS} - [A](t)\right) - k_1[A](t) - k_3[A]^2(t) \\ &= -k_3[A]^2(t) - [A](t)\left(\frac{F_i^{SS}}{V} + k_1\right) + \frac{F_i^{SS}}{V}[A]_i^{SS} \\ &= 0\end{aligned}$$

We get the second-order equation in the variable $[A](t)$,

$$k_3[A]^2(t) + \left(\frac{F_i^{SS}}{V} + k_1\right)[A](t) - \frac{F_i^{SS}}{V}[A]_i = 0$$

Second-order equation: $ax^2 + bx + c = 0$ with solutions $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example II (cont.)

$$\underbrace{k_3[A]^2(t)}_{ax^2} + \underbrace{\left(\frac{F_i^{SS}}{V} + k_1\right)[A](t)}_{bx} - \underbrace{\frac{F_i^{SS}}{V}[A]_i}_{-c} = 0$$

The steady-state values for $[A]$, given F_i^{SS} and $[A]_i^{SS}$

$$[A]_{1,2}^{SS} = \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} \pm \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}}$$

We need to consider only the root where $[A]$ is positive,

$$[A]^{SS} = \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}}$$

Example II (cont.)

Proceeding similarly for component $[B]$, we can write

$$\frac{\mathrm{d}[B](t)}{\mathrm{d}t} = -[B](t)\left(\frac{F_i^{SS}}{V} + k_2\right) + k_1 \underbrace{[A](t)}_{[A]^{SS}}$$
$$= 0$$

We get the first-order equation in $[B](t)$

$$[B](t)\left(\frac{F_i^{SS}}{V} + k_2\right) - k_1 [A]^{SS} = 0$$

The steady-state value for $[B]$,

$$[B]^{SS} = \frac{k_1 [A]^{SS}}{\left(\dfrac{F^{SS}}{V} + k_2\right)}$$

given F_i^{SS}, $[A]_i^{SS}$, and $[A]^{SS}$

Example II (cont.)

Substituting $[A]^{SS}$, we get

$$[B]^{SS} = \frac{k_1 [A]^{SS}}{\left(\frac{F^{SS}}{V} + k_2\right)}$$

$$= k_1 \left(\frac{\frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}}}{\left(\frac{F^{SS}}{V} + k_2\right)} \right)$$

Example II (cont.)

For component $[C]$, we have

$$\frac{\mathrm{d}[C](t)}{\mathrm{d}t} = -[C](t)\left(\frac{F_i^{SS}}{V}\right) + k_2 \underbrace{[B](t)}_{[B]^{SS}}$$
$$= 0$$

We get the equation,

$$\left(\frac{F_i^{SS}}{V}\right)[C](t) - k_2[B]^{SS} = 0$$

The steady-state value for $[C]$,

$$[C]^{SS} = \frac{k2[B]^{SS}}{\left(\frac{F_i^{SS}}{V}\right)}$$

given F_i^{SS}, $[A]_i^{SS}$, $[A]^{SS}$, and $[B]^{SS}$

Example II (cont.)

Substituting $[B]^{SS}$, we get

$$
[C]^{SS} = \frac{k2[B]^{SS}}{\left(\frac{F_i^{SS}}{V}\right)}
$$

$$
= k_2 \frac{k_1 \left(\frac{ \frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{ \frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3} } }{\left(\frac{F^{SS}}{V} + k_2\right)} \right)}{\left(\frac{F_i^{SS}}{V}\right)}
$$

Example II (cont.)

Finally, for component $[D]$ we have

$$\frac{\mathrm{d}[D](t)}{\mathrm{d}t} = -[D](t)\left(\frac{F_i^{SS}}{V}\right) + \frac{1}{2}k_3 \underbrace{[A]^2(t)}_{([A]^{SS})^2}$$
$$= 0$$

We get the equation,

$$\left(\frac{F_i^{SS}}{V}\right)[D](t) - \frac{1}{2}k_3([A]^{SS})^2 = 0$$

The steady-state value for $[D]$,

$$[D]^{SS} = \frac{\frac{1}{2}k_3([A]^{SS})^2}{\left(\frac{F_i^{SS}}{V}\right)}$$

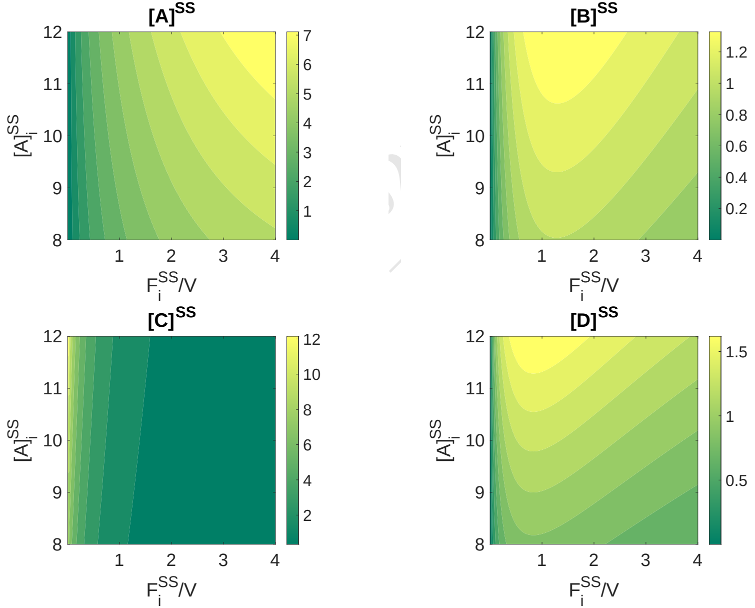
given F_i^{SS}, $[A]_i^{SS}$, $[A]^{SS}$, and $[B]^{SS}$

Example II (cont.)

Substituting $[A]^{SS}$, we get

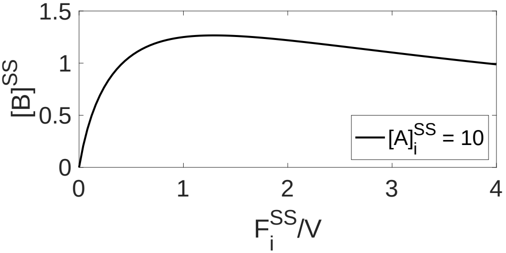
$$[D]^{SS} = \frac{1}{2}k_3 \frac{\left(\frac{-\left(k_1 + \frac{F_i^{SS}}{V}\right)}{2k_3} + \sqrt{\frac{\left(k_1 + \frac{F_i^{SS}}{V}\right)^2 + 4k_3 \frac{F_i^{SS}}{V}[A]_i^{SS}}{2k_3}} \right)^2}{\left(\frac{F_i^{SS}}{V}\right)}$$

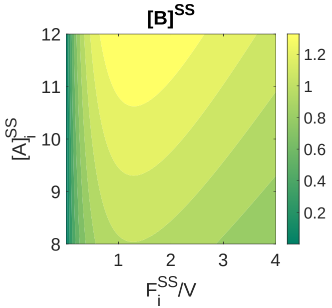
Example II (cont.)



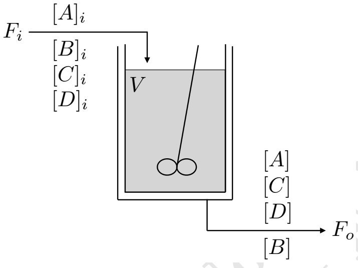
Example II (cont.)

Where would you operate the reactor if told that the feed composition is $[A]_i^{SS} = 10$?





Example II (cont.)



We could define desirable operating conditions

$$u^{SS} = \begin{bmatrix} \dfrac{F_i^{SS}}{V} = \dfrac{4}{7} \text{ min}^{-1} \\ \\ [A]_i^{SS} = 10 \text{ mol l}^{-1} \end{bmatrix} = \begin{bmatrix} F_i^{SS} \\ [A]_i^{SS} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix}$$

Then, determine the corresponding fixed point

$$x^{SS} = \begin{bmatrix} 3.0000 \text{ mol lt}^{-1} \\ 1.1170 \text{ mol lt}^{-1} \\ 3.2580 \text{ mol lt}^{-1} \\ 1.3125 \text{ mol lt}^{-1} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} x_1^{SS} & x_2^{SS} & x_2^{SS} & x_2^{SS} \end{bmatrix}^T$$

Note that we replaced the first input variable (the feed flow-rate, $F_i(t)$)

- We will use the space-velocity $F_i(t)/V$, instead
- No difference, as the volume V is constant

Example II (cont.)

Given a steady-state point $((x_1^{SS}, x_2^{SS}, x_3^{SS}, x_4^{SS}), (u_1^{SS}, u_2^{SS}))$, we linearise the model

We start by defining the deviation variables, for both state- and input variables

- For the state variables, we have

$$x'(t) = \begin{bmatrix} x_1(t) - x_1^{SS} \\ x_2(t) - x_2^{SS} \\ x_3(t) - x_3^{SS} \\ x_4(t) - x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A](t) - [A]^{SS} \\ [B](t) - [B]^{SS} \\ [C](t) - [C]^{SS} \\ [D](t) - [D]^{SS} \end{bmatrix}$$

- For the input variables, we have

$$u'(t) = \begin{bmatrix} u_1(t) - u_1^{SS} \\ u_2(t) - u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i(t)/V - F_i^{SS}/V \\ [A]_i(t) - [A]_i^{SS} \end{bmatrix}$$

Then proceed by computing the Jacobians of dynamics at steady-state (x_{SS}, u_{SS})

- ⇝ State matrix A and input matrix B
- ⇝ $\dot{x}'(t) = Ax'(t) + Bu'(t)$

Example II (cont.)

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \\ \dot{x_4}(t) \end{bmatrix} = \begin{bmatrix} \underbrace{u_1(t)\left(u_2(t) - x_1(t)\right) - \theta_{x,1}x_1(t) - \theta_{x,3}x_1^2(t)}_{f_1} \\ \underbrace{-u_1(t)x_2(t) + \theta_{x,1}x_1(t) - \theta_{x,2}x_2(t)}_{f_2} \\ \underbrace{-u_1(t)x_3(t) + \theta_{x,2}x_2(t)}_{f_3} \\ \underbrace{-u_1(t)x_4(t) + \frac{1}{2}\theta_{x,3}x_1^2(t)}_{f_4} \end{bmatrix}$$

$$\rightsquigarrow \quad A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}_{SS} = \begin{bmatrix} -u_1 - \theta_{x,1} - 2\theta_{x,3}x_1 & 0 & 0 & 0 \\ \theta_{x,1} & -u_1 - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1 & 0 \\ \theta_{x,3}x_1 & 0 & 0 & -u_1 \end{bmatrix}_{SS}$$

Example II (cont.)

We get,

$$
\begin{aligned}
A &= \begin{bmatrix} -u_1 - \theta_{x,1} - 2\theta_{x,3}x_1 & 0 & 0 & 0 \\ \theta_{x,1} & -u_1 - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1 & 0 \\ \theta_{x,3}x_1 & 0 & 0 & -u_1 \end{bmatrix}_{SS} \\
&= \begin{bmatrix} -u_1^{SS} - \theta_{x,1} - 2\theta_{x,3}x_1^{SS} & 0 & 0 & 0 \\ \theta_{x,1} & -u_1^{SS} - \theta_{x,2} & 0 & 0 \\ 0 & \theta_{x,2} & -u_1^{SS} & 0 \\ \theta_{x,3}x_1^{SS} & 0 & 0 & -u_1^{SS} \end{bmatrix} \\
&= \begin{bmatrix} -(4/7) - (5/6) - 2 \times (1/6) \times 3 & 0 & 0 & 0 \\ (5/6) & -(4/7) - (5/3) & 0 & 0 \\ 0 & (5/3) & (-4/7) & 0 \\ (1/6) \times 3 & 0 & 0 & -(4/7) \end{bmatrix}
\end{aligned}
$$

We used $\theta_x = \begin{bmatrix} \theta_{x,1} & \theta_{x,2} & \theta_{x,3} \end{bmatrix}^T = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T = \begin{bmatrix} (5/6) & (5/3) & (1/6) \end{bmatrix}^T$ and

$$
x^{SS} = \begin{bmatrix} x_1^{SS} \\ x_2^{SS} \\ x_3^{SS} \\ x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 1.1170 \\ 3.2580 \\ 1.3125 \end{bmatrix}
$$

$$
u^{SS} = \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i^{SS}/V \\ [A]_i^{SS} \end{bmatrix} = \begin{bmatrix} 4/7 \\ 10 \end{bmatrix}
$$

Example II (cont.)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \underbrace{u_1(t)\left(u_2(t) - x_1(t)\right) - \theta_{x,1} x_1(t) - \theta_{x,3} x_1^2(t)}_{f_1} \\ \underbrace{-u_1(t) x_2(t) + \theta_{x,1} x_1(t) - \theta_{x,2} x_2(t)}_{f_2} \\ \underbrace{-u_1(t) x_3(t) + \theta_{x,2} x_2(t)}_{f_3} \\ \underbrace{-u_1(t) x_4(t) + \frac{1}{2}\theta_{x,3} x_1^2(t)}_{f_4} \end{bmatrix}$$

$$\rightsquigarrow \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix}_{SS} = \begin{bmatrix} u_2 - x_1 & u_1 \\ -x_2 & 0 \\ -x_3 & 0 \\ -x_4 & 0 \end{bmatrix}_{SS}$$

Example II (cont.)

We get,

$$B = \begin{bmatrix} u_2 - x_1 & u_1 \\ -x_2 & 0 \\ -x_3 & 0 \\ -x_4 & 0 \end{bmatrix}_{SS}$$
$$= \begin{bmatrix} u_2^{SS} - x_1^{SS} & u_1^{SS} \\ -x_2^{SS} & 0 \\ -x_3^{SS} & 0 \\ -x_4^{SS} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 10 - 3 & (4/7) \\ -1.1170 & 0 \\ -3.2580 & 0 \\ -1.3125 & 0 \end{bmatrix}$$

We used,

$$x^{SS} = \begin{bmatrix} x_1^{SS} \\ x_2^{SS} \\ x_3^{SS} \\ x_4^{SS} \end{bmatrix} = \begin{bmatrix} [A]^{SS} \\ [B]^{SS} \\ [C]^{SS} \\ [D]^{SS} \end{bmatrix} = \begin{bmatrix} 3.0000 \\ 1.1170 \\ 3.2580 \\ 1.3125 \end{bmatrix}$$
$$u^{SS} = \begin{bmatrix} u_1^{SS} \\ u_2^{SS} \end{bmatrix} = \begin{bmatrix} F_i^{SS}/V \\ [A]_i^{SS} \end{bmatrix} = \begin{bmatrix} 4/7 \\ 10 \end{bmatrix}$$