Aalto University

# Dynamic programming

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# Multi-stage optimisation

## Dynamic programming

# Optimising multi-stage functions

Consider the set of decision variables $w$, $y$, and $z$ and the following objective function

$$\underbrace{f(w,x)}_{0}+\underbrace{g(x,y)}_{1}+\underbrace{h(y,z)}_{2}$$

Each stage-cost function in the sum depends only on the adjacent variable pairs

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Consider the case in which $w$ is a known, and we want to solve the optimisation problem

$$\min_{x,y,z|w} \quad f(x|w)+g(x,y)+h(y,z)$$

One possibility would be to optimise for all the three decision variables $(x,y,z)$

- ⇝ This solution is valid, but it does not exploit the problem structure

We can alternatively solve a sequence of single-variable optimisation problems

$$\min_{x|w} \quad \bigg(f(x|w)+\min_{y} \quad \Big(g(x,y)+\min_{z} \quad h(y,z)\Big)\bigg)$$

# Optimising multi-stage functions (cont.)

$$\min_{x|w} \quad \Big( f\,(x|w) + \min_{y} \quad \Big( g\,(x,y) + \min_{z} \quad h\,(y,z) \Big) \Big)$$

Starting from the innermost optimisation problem, we solve with respect to variable $z$

$$\min_{z} \quad h\,(y,z)$$

We obtain the solution for $z$ and the optimal value function in terms of variable $y$,

$$h^*\,(y) = \min_{z} \quad h\,(y,z) \qquad\qquad \text{(optimal value function)}$$

$$z^*\,(y) = \arg\min_{z} \quad h\,(y,z) \qquad\qquad \text{(minimiser)}$$

# Optimising multi-stage functions (cont.)

$$\min_{x|w} \left( f(x|w) + \min_{y} \left( g(x,y) + \underbrace{\min_{z} \quad h(y,z)}_{h^*(y)} \right) \right)$$

Proceeding with the next optimisation problem, we solve it with respect to variable $y$

$$\min_{y} \quad g(x,y) + h^*(y)$$

We obtain the solution for $y$ and the optimal value function in terms of variable $x$,

$$g^*(x) = \min_{y} \quad g(x,y) + h^*(y) \qquad \text{(optimal value function)}$$

$$y^*(x) = \arg\min_{y} \quad g(x,y) + h^*(y) \qquad \text{(minimiser)}$$

# Optimising multi-stage functions (cont.)

$$\min_{x|w} \left( f(x|w) + \underbrace{\min_{y} \left( g(x,y) + \underbrace{\min_{z} \quad h(y,z)}_{h^*(y)} \right)}_{g^*(x)} \right)$$

At the third and final optimisation problem, we solve it with respect to variable $x$

$$\min_{x|w} \quad f(x|w) + g^*(x)$$

We obtain the solution for $x$ and the optimal value function in terms of value $w$

$$f^*(w) = \min_{x} \quad f(x|w) + g^*(x) \qquad \text{(optimal function value)}$$

$$x^*(w) = \arg\min_{x} \quad f(x|w) + g^*(x) \qquad \text{(solution)}$$

# Optimising multi-stage functions (cont.)

$$\underbrace{\min_{x|w}\left(f(x|w)+\underbrace{\min_{y}\left(g(x,y)+\underbrace{\min_{z}\quad h(y,z)}_{h^*(y)\text{ at }z^*(y)}\right)}_{g^*(x)\text{ at }y^*(x)}\right)}_{f^*(w)\text{ at }x^*(w)}$$

Because $w$ is fixed, we know its value, we have that $x^*(w)$ is completely determined

Thus, we also have that $y^*(x^*(w))$ and $z^*(y^*(x^*(w)))$ are completely determined

$$\begin{aligned}\widetilde{y}^*(w) &= y^*(x^*(w))\\ \widetilde{z}^*(w) &= z^*(\widetilde{y}^*(w))\\ &= z^*(y^*(x^*(w)))\end{aligned}$$

Similarly, the optimal value of the obejctive function can be also computed

$$f^*(w)+g^*(x^*(w))+h^*(y^*(x^*(w)))$$

# Optimising multi-stage functions (cont.)

The method to solve (unconstrained) multi-state optimisation problems can be an alternative approach for optimal control problems, **backward dynamic programming**

- The decision variables are solved in reverse order

The solutions expressed as functions of the variables to be optimised at the next stage

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Its application is easiest for discrete-time systems with discrete state and action spaces

- With continuous spaces, applicability is achieved by discretisation
- In continuous-time the problem is formulated as a PDE, the HJB

# Discrete state- and action-spaces

## Dynamic programming

# Discrete state- and action-spaces

We consider the nonlinear dynamic equation of a discrete-time state-space model

$$x_{k+1} = f\left(x_k, u_k\right)$$

Moreover, suppose that the state- and the action-space be discrete and finite

$$\begin{aligned} x_k &\in \mathcal{X}, \quad \text{with } |\mathcal{X}| = N_{\mathcal{X}} \\ u_k &\in \mathcal{U}, \quad \text{with } |\mathcal{U}| = N_{\mathcal{U}} \end{aligned}$$

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Based on the discrete dynamics, we formulate the optimal control problem

$$\begin{aligned} \min_{\substack{x_0, x_1, \ldots, x_{K-1}, x_K \\ u_0, u_1, \ldots, u_{K-1}}} \quad & E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) \\ \text{subject to} \quad & f\left(x_k, u_k\right) - x_{k+1} = 0, \qquad k = 0, 1, \ldots, K-1 \\ & \overline{x}_0 - x_0 = 0 \end{aligned}$$

The initial state $x_0$ is assumed to be know, fixed at value $\overline{x}_0$

# Discrete state- and action-spaces (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_{K-1}, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & f(x_k, u_k) - x_{k+1} = 0, \qquad k = 0, 1, \dots, K-1 \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

The controls $\{u_k\}_{k=0}^{K-1}$ are the true decision variables of the optimisation

The state variables can be eliminated by forward simulation

$$\begin{aligned}
x_1(x_0, u_0) &= f(x_0, u_0) \\
x_2(x_0, u_0, u_1) &= f(x_1, u_1) \\
&= f(f(x_0, u_0), u_1) \\
x_3(x_0, u_0, u_1, u_2) &= f(x_2, u_2) \\
&= f(f(f(x_0, u_0), u_1), u_2) \\
\cdots &= \cdots \\
x_K(x_0, u_0, u_1, \dots, u_{K-2}, u_{K-1}) &= f(x_{K-1}, u_{K-1}) \\
&= f(f(\cdots f(x_0, u_0), u_{K-2}), u_{K-1})
\end{aligned}$$

# Discrete state- and action-spaces (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_{K-1}, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right) \\
\text{subject to} \quad & f\left(x_k, u_k\right) - x_{k+1} = 0, \qquad k = 0, 1, \dots, K-1 \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

This formulation of discrete-time optimal control problem misses path constraints

They can be implicitly included by allowing the stage cost to be equal to infinity

- For infeasible points $(x_k, u_k)$, we have that $L\left(x_k, u_k\right) = \infty$

To be able to include inequality constraints, we thus have

$$L : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{R} \cup \infty$$

# Discrete state- and action-spaces (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \ldots, x_{K-1}, x_K \\ u_0, u_1, \ldots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & f(x_k, u_k) - x_{k+1} = 0, \qquad k = 0, 1, \ldots, K-1 \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

As each $u_k$ can only take on one of $N_{\mathcal{U}}$ values, there are $N_{\mathcal{U}}^K$ possible control sequences

$$\underbrace{N_{\mathcal{U}} \times N_{\mathcal{U}} \times \cdots \times N_{\mathcal{U}}}_{K \text{ times}}$$

Each possible sequence would correspond to a different trajectory $\{\{x_k, u_k\}_{k=0}^{K-1} \cup x_K\}$

- ⇝ Each trajectory is characterised by its specific value of the objective function
- ⇝ The optimal solution corresponds to the sequence of smallest function value

# Discrete state- and action-spaces (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, x_{K-1}, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\
\text{subject to} \quad & f(x_k, u_k) - x_{k+1} = 0, \qquad k = 0, 1, \dots, K-1 \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

Naive enumeration of all trajectories has a complexity that grows exponentially in $K$

$$\underbrace{N_{\mathcal{U}} \times N_{\mathcal{U}} \times \cdots \times N_{\mathcal{U}}}_{K \text{ times}}$$

The idea behind dynamic programming is to approach the enumeration task differently

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We start by noting that each sub-trajectory of an optimal trajectory must be optimal

- We denote this property as the **principle of optimality**

# Discrete state- and action-spaces (cont.)

We define the **value-function** or **cost-to-go** as the optimal cost that would be attained if, at time $k$ and state $\overline{x}_k$, we would solve the shorter optimal control problem

$$J_k\left(\overline{x}_k\right) = \min_{\substack{x_k, x_{k+1}, \ldots, x_{K-1}, x_K \\ u_k, u_{k+1}, \ldots, u_{K-1}}} \quad E\left(x_K\right) + \sum_{i=k}^{K-1} L\left(x_i, u_i\right)$$

$$\text{subject to} \qquad f\left(x_i, u_i\right) - x_{i+1} = 0, \quad i = k, k+1, \ldots, K-1$$

$$\overline{x}_k - x_k = 0$$

Each function $J_k : \mathcal{X} \rightarrow \mathcal{R} \cup \infty$ summarises the cost-to-go to the end of the horizon

- Starting from the initial state $\overline{x}_k$, under the optimal actions $\{u_i^*\}_{i=k}^{K-1}$

There is a finite number $N_{\mathcal{X}}$ of possible initial states $\overline{x}_k$, at each stage $k$ we have

$$J_k\left(x_k^{(1)}\right)$$

$$\vdots$$

$$J_k\left(x_k^{(N_{\mathcal{X}})}\right)$$

# Discrete state- and action-spaces (cont.)



The **Bellman equation**

The principle of optimality states that for any $k \in \{0, 1, \dots, K-1\}$ the following holds

$$\begin{aligned} J_k(\overline{x}_k) &= \min_u \Big( L(\overline{x}_k, u) + J_{k+1}(f(\overline{x}_k, u)) \Big) \\ &= \min_u \Big( L(\overline{x}_k, u) + J_{k+1}(\overline{x}_{k+1}) \Big) \end{aligned}$$

# Discrete state- and action-spaces (cont.)

The backward recursion is known as the **dynamic programming recursion**

$$u_k^*(x_k) = \arg\min_u L(x_k, u) + J_{k+1}(f(x_k, u))$$

Once all the value-functions $J_k$ are computed, the **optimal feedback control**

$$x_{k+1} = f(x_k, u_k^*(x_k)), \quad k = 0, 1, \dots, K-1$$

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The computationally demanding step is the generation of the $K$ value functions $J_k$

- Each recursion step requires to test $N_{\mathcal{U}}$ controls, for each of the $N_{\mathcal{X}}$ states
- Each recursion requires computing $f(x_k, u)$ and $L(x_k, u)$

The overal complexity is thus $K \times (N_{\mathcal{X}} \times N_{\mathcal{U}})$

# Discrete state- and action-spaces (cont.)



One of the main advantages of the dynamic programming approach to optimal control is the possibility to be extended to continuous state- and action-spaces, by discretisation

- No assumptions on differentiability of the dynamics or convexity of the objective

However, it is important to notice that for a $N_x$ dimensional state-space discretised along each dimension using $M_x$ intervals, the total number of grid points is $N_{\mathcal{X}} = M_x^{N_x}$

- That is, complexity grows exponential with the dimension of the state-space

# An example

## Discrete state and action spaces

# An example

Consider a total stage cost given by the sum of the state cost and control stage cost

$$L_k\left(x_k, u_k\right) = L_x^k\left(x_k\right) + L_u^k\left(x_k, u_k\right)$$

The stage-cost for the states, the positions on a $(4 \times 3)$ board

- The target state is in position $(2, 2)$
- The state-cost per step is zero

$$\underbrace{\begin{array}{cccc} \times & | & \times & \times \\ \times & | & {\color{green}\times} & \times \\ \times & | & \times & \times \\ \times & & \times & \times \end{array}}_{\mathcal{X}} \rightsquigarrow \underbrace{\begin{array}{cccc} 5 & | & 5 & 5 \\ 5 & | & {\color{green}\mathbf{0}} & 5 \\ 5 & | & 5 & 5 \\ 5 & & 5 & 5 \end{array}}_{L_x^k(x_k \in \mathcal{X})}$$

The stage-cost for the controls, the 9 possible 'moves'

- The control-cost per stage is one, or zero

$$\underbrace{\begin{array}{ccc} \nwarrow & \uparrow & \nearrow \\ \leftarrow & \cdot & \rightarrow \\ \swarrow & \downarrow & \searrow \end{array}}_{\mathcal{U}} \rightsquigarrow \underbrace{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}}_{L_u^k(x_k, u_k \in \mathcal{U})}$$

# An example (cont.)

The policy specifies the action that we will perform at time step $k$

- It is a function of the state, at stage $k$

$$\pi(x_k) = u_k(x_k)$$

A random example of policy,

$$\pi(x_k) \quad = \quad \begin{array}{c|cc} \cdot & \nwarrow & \leftarrow \\ \uparrow & \downarrow & \rightarrow \\ \uparrow & \cdot & \swarrow \\ \leftarrow & \nearrow & \cdot \end{array}$$

At $k$, the objective is to find the policy that minimises the cost-to-go

$$\sum_{k}^{K} L_k(x_k, u_k)$$

The value function of the policy at $k$ is the goodness of each policy

$$V_\pi(x_k) = L_k(x_k, u_k) + V_\pi(x_{k+1})$$

# An example (cont.)

## Stage $K$

At the final stage $k = K$, we have the following value function of the policy function

$$\begin{aligned} V_{\pi}(x_K) &= L_K(x_K, \cancel{u_K}) + \cancel{V_{\pi}(x_{K+1})} \\ &= \underbrace{L_x^K(x_K) + \cancel{L_u^K(x_K, u_K)}}_{L_k(u_K, u_K)} + \cancel{V_{\pi}(x_{K+1})} \end{aligned}$$

$$= \begin{array}{c|cc} 5 & 5 & 5 \\ 5 & 0 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{array}$$

As there is no time left to apply any control, we have the optimal policy

$$\pi^*(x_K) = \begin{array}{c|cc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

# An example (cont.)

The value function for the optimal policy corresponds to the terminal cost $E(x_K)$

$$\begin{aligned} V_{\pi^*}(x_K) &= V_{\pi^*}(x_K) \\ &= E(x_K) \end{aligned}$$

We have the optimal policy,

$$\pi^*(x_K) \quad = \quad \begin{array}{c|cc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

The value of the policy,

$$V_{\pi^*}(x_K) = \begin{array}{c|cc} 5 & 5 & 5 \\ 5 & 0 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{array}$$

The value of the optimal policy at stage $K$ gives the total cost that would be incurred if, starting at some state $x_K \in \mathcal{X}$, the best sequence of actions would be performed

- The first optimal action of the sequence (!) was found to be ‘do nothing’

# An example (cont.)

According to the Bellman optimality principle, the optimal policy at stage $K-1$

$$\pi^*(x^{K-1}) = \arg\min_{u} \big(L_{K-1}(x_{K-1}, u_{K-1}) + V_{\pi^*}(x_K)\big)$$

Remaining controls are optimal with respect to the state resulting from the first one

- ⇝ We must compute the stage-cost $L_{K-1}(x_{K-1}, u_{K-1})$ at stage $K-1$
- ⇝ We know the value of the policy $V_{\pi^*}(x_K)$

$$V_{\pi^*}(x_K) = \begin{array}{cccc} 5 & | & 5 & 5 \\ 5 & | & 0 & 5 \\ 5 & | & 5 & 5 \\ 5 & & 5 & 5 \end{array}$$

# An example (cont.)

## Stage $K-1$

$$\underbrace{\begin{array}{cccc} \times & | & \times & \times \\ \times & | & \times & \times \\ \times & | & \times & \times \\ \times & & \times & \times \end{array}}_{\mathcal{X}}$$

For each state $x_{K-1} \in \mathcal{X}$, compute the stage cost $L_{K-1}(x_{K-1}, u_{K-1})$ for all $u_{K-1} \in \mathcal{U}$

We can then add it to the optimal value function at stage $K$ and optimise

$$V_{\pi^*}(x^{K-1}) = \min_{u_{K-1}} \Big( L_{K-1}(x_{K-1}, u_{K-1}) + V_{\pi^*}(x^K) \Big)$$

From a minimisation of the value function, we compute the optimal policy

$$\pi^*(x^{K-1}) = \arg\min_{u} \Big( L_{K-1}(x_{k-1}, u_{k-1}) + V_{\pi^*}(x^K) \Big)$$

$$\underbrace{\begin{array}{ccc} \nwarrow & \uparrow & \nearrow \\ \leftarrow & \cdot & \rightarrow \\ \swarrow & \downarrow & \searrow \end{array}}_{\mathcal{U}}$$

# An example (cont.)

$$\begin{array}{ccccc} \circ & | & \times & & \times \\ \times & | & \times & & \times \\ \times & | & \times & & \times \\ \times & & \times & & \times \end{array}$$

Suppose that the system is at state $\mathcal{X}_{1,1}$ and consider control action $\uparrow$

- As a result the system stays at state $\mathcal{X}_{1,1}$

We have the total stage cost, as sum of state-cost and action-cost

$$\begin{aligned} L_{K-1}(\mathcal{X}_{1,1}, \uparrow) &= L_x^{K-1}(\mathcal{X}_{1,1}) + L_u^{K-1}(\mathcal{X}_{1,1}, \uparrow) \\ &= 5 + 1 \\ &= 6 \end{aligned}$$

The application of action $\downarrow$ leads to state $\mathcal{X}_{1,1}$

$$V_{\pi^*}(\mathcal{X}_{1,1}) = 5$$

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Similarly, for actions $\downarrow$, $\nwarrow$, $\nearrow$, $\swarrow$, $\searrow$, $\leftarrow$, $\cdot$, and $\rightarrow$ applied to state $\mathcal{X}_{1,1}$

# An example (cont.)

$$
\begin{array}{cccc}
\circ & | & \times & \times \\
\times & | & \times & \times \\
\times & | & \times & \times \\
\times & & \times & \times
\end{array}
$$

For action $\downarrow$ applied to state $\mathcal{X}_{1,1}$, we have the total stage-cost

$$
\begin{aligned}
L_{K-1}(\mathcal{X}_{1,1}, \downarrow) &= J_x^{K-1}(\mathcal{X}_{1,1}) + J_u^{K-1}(\mathcal{X}_{1,1}, \downarrow) \\
&= 5 + 1 \\
&= 6
\end{aligned}
$$

The application of action $\downarrow$ leads to state $\mathcal{X}_{2,1}$

$$
V_{\pi^*}(\mathcal{X}_{2,1}) = 5
$$

# An example (cont.)

$$
\begin{array}{cccc}
\circ & | & \times & \times \\
\times & | & \times & \times \\
\times & | & \times & \times \\
\times & & \times & \times
\end{array}
$$

For action $\cdot$ applied to state $\mathcal{X}_{1,1}$, we have the total stage-cost

$$
\begin{aligned}
L_{K-1}(\mathcal{X}_{1,1}, \cdot) &= J_x^{K-1}(\mathcal{X}_{1,1}) + J_u^{K-1}(\mathcal{X}_{1,1}, \cdot) \\
&= 5 + 0 \\
&= 5
\end{aligned}
$$

The application of action $\downarrow$ leads to state $\mathcal{X}_{1,1}$

$$V_{\pi^*}(\mathcal{X}_{1,1}) = 5$$

# An example (cont.)

Summarising, for state $\mathcal{X}_{1,1}$

- At stage $K-1$

$$
\begin{aligned}
L_{K-1}(\mathcal{X}_{1,1}, \uparrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \nwarrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \nearrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \swarrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \searrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \leftarrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \rightarrow) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \downarrow) + V_{\pi^*}(\mathcal{X}_{2,1}) &= 6 + 5 \\
&= 11 \\
L_{K-1}(\mathcal{X}_{1,1}, \cdot) + V_{\pi^*}(\mathcal{X}_{1,1}) &= 5 + 5 \\
&= 10
\end{aligned}
$$

# An example (cont.)

The optimal action that we can do when at state $\mathcal{X}_{1,1}$ at stage $K-1$ is to not move, $\cdot$

$$\pi^*(\mathcal{X}_{1,1}) = \begin{array}{c|cc} \cdot & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{array}$$

The value of the optimal action, at stage $K-1$

$$V_{\pi^*}(x_{K-1}) = \begin{array}{c|cc} 10 & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{array}$$

---

The value function $V_{\pi^*}(\mathcal{X}_{1,1})$ gives the cost that would be incurred if, starting at state $\mathcal{X}_{1,1}$ and from that stage on, we performed the best possible sequence of actions

- The first action would be the one given by the optimal policy $\pi^*(\mathcal{X}_{1,1} \in \mathcal{X})$

# An example (cont.)

Analogously for the other states $x_{K-1} \in \mathcal{X}$ at stage $K-1$, we have the optimal policy

$$\pi^*(x_{K-1} \in \mathcal{X}) = \begin{array}{c|cc} \cdot & \downarrow & \swarrow \\ \cdot & \cdot & \leftarrow \\ \cdot & \uparrow & \nwarrow \\ \cdot & \cdot & \cdot \end{array}$$

The value of the optimal policy, at stage $K-1$

$$V_{\pi^*}(x_{K-1} = \mathcal{X}_{1,1}) = \begin{array}{c|cc} 10 & 6 & 6 \\ 10 & 0 & 6 \\ 10 & 6 & 6 \\ 10 & 10 & 10 \end{array}$$

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The value function $V_{\pi^*}(x_{K-1})$ gives the cost that would be incurred if, starting at any state $x_{K-1}$ and from that stage on, we performed the best possible sequence of actions

- The first action would be the one given by the optimal policy $\pi^*(x_{K-1} \in \mathcal{X})$

# An example (cont.)

## Stage $K-2$

The value of the optimal policy at stage $K-1$ gives the total cost that would be incurred if, starting at state $x_{K-1} \in \mathcal{X}$, the best sequence of actions would be performed

$$V_{\pi^*}(x_{K-1}) = \begin{array}{ccc} 10 & \vert\quad 6 & 6 \\ 10 & \vert\quad 0 & 6 \\ 10 & \vert\quad 6 & 6 \\ 10 & 10 & 10 \end{array}$$

The first optimal action of the sequence

$$\pi^*(x_{K-1} \in \mathcal{X}) = \begin{array}{ccc} \cdot & \vert\quad \downarrow & \swarrow \\ \cdot & \vert\quad \cdot & \leftarrow \\ \cdot & \vert\quad \uparrow & \nwarrow \\ \cdot & \cdot & \cdot \end{array}$$

# An example (cont.)

$$\underbrace{\begin{array}{cccc} \times & | & \times & \times \\ \times & | & \times & \times \\ \times & | & \times & \times \\ \times & & \times & \times \end{array}}_{\mathcal{X}}$$

For each state $x_{K-2} \in \mathcal{X}$, compute the stage cost $L_{K-2}(x_{K-2}, u_{K-2})$ for all $u_{K-2} \in \mathcal{U}$

We can then add it to the optimal value function at stage $K$ and optimise

$$V_{\pi^*}(x_{K-2}) = \min_{u_{K-2}} \left(L_{K-2}(x_{K-2}, u_{K-2}) + V_{\pi^*}(x_{K-1})\right)$$

From a minimisation of the value function, we compute the optimal policy

$$\pi^*(x_{K-2}) = \arg\min_{u} \left(L_{K-2}(x_{K-2}, u_{K-2}) + V_{\pi^*}(x_{K-1})\right)$$

$$\underbrace{\begin{array}{ccc} \nwarrow & \uparrow & \nearrow \\ \leftarrow & \cdot & \rightarrow \\ \swarrow & \downarrow & \searrow \end{array}}_{\mathcal{U}}$$

# An example (cont.)

At stage $K-2$, we have the optimal policy

$$\pi^*(x_{K-2} \in \mathcal{X}) = \begin{array}{cccc} \cdot & | & \downarrow & \swarrow \\ \cdot & | & \cdot & \leftarrow \\ \cdot & | & \uparrow & \nwarrow \\ \nearrow & & \uparrow & \uparrow \end{array}$$

The value of the optimal policy, at stage $K-2$

$$V_{\pi^*}(x_{K-2}) = \begin{array}{cccc} 15 & | & 6 & 6 \\ 15 & | & 0 & 6 \\ 15 & | & 6 & 6 \\ 12 & & 12 & 12 \end{array}$$

# An example (cont.)

## Stage $K-3$

At stage $K-3$, we have the optimal policy

$$\pi^*(x_{K-3} \in \mathcal{X}) = \begin{array}{cccc} \cdot & \vert & \downarrow & \swarrow \\ \cdot & \vert & \cdot & \leftarrow \\ \downarrow & \vert & \uparrow & \nwarrow \\ \nearrow & & \uparrow & \uparrow \end{array}$$

The value of the optimal policy, at stage $K-3$

$$V_{\pi^*}(x_{K-3}) = \begin{array}{cccc} 20 & \vert & 6 & 6 \\ 20 & \vert & 0 & 6 \\ 18 & \vert & 6 & 6 \\ 12 & & 12 & 12 \end{array}$$

# An example (cont.)

## Stage $K-4$

At stage $K-4$, we have the optimal policy

$$\pi^*(x_{K-4} \in \mathcal{X}) = \begin{array}{c|ccc} \cdot & \downarrow & & \swarrow \\ \downarrow & \cdot & & \leftarrow \\ \downarrow & \uparrow & & \nwarrow \\ \multicolumn{1}{c}{\nearrow} & \uparrow & & \uparrow \end{array}$$

The value of the optimal policy, at stage $K-4$

$$V_{\pi^*}(x_{K-4}) = \begin{array}{c|ccc} 25 & 6 & & 6 \\ 24 & 0 & & 6 \\ 18 & 6 & & 6 \\ \multicolumn{1}{c}{12} & 12 & & 12 \end{array}$$

# An example (cont.)

## Stage $K-5$

At stage $K-5$, we have the optimal policy

$$\pi^*(x_{K-5} \in \mathcal{X}) = \begin{array}{c|cc} \cdot & \downarrow & \swarrow \\ \downarrow & \cdot & \leftarrow \\ \downarrow & \uparrow & \nwarrow \\ \nearrow & \uparrow & \uparrow \end{array} = \pi^*(x_{K-4} \in \mathcal{X})$$

The value of the optimal policy, at stage $K-4$

$$V_{\pi^*}(x_{K-4}) = \begin{array}{c|cc} 30 & 6 & 6 \\ 24 & 0 & 6 \\ 18 & 6 & 6 \\ 12 & 12 & 12 \end{array}$$

# The linear-quadratic regulator

## Dynamic programming

# The linear-quadratic regulator

An important class of optimal control problems is the linear-quadratic regulator, LQR

- The controller has to take the state of the system to the origin
- The system dynamics are deterministic and linear
- The objective function is quadratic

The problem is unconstrained and the horizon for control can be finite or infinite

- Their solution can be obtained with dynamic programming

# The linear-quadratic regulator (cont.)

Consider first the case in which we are interested in stabilising the system in $K$ steps

We define an objective function to quantify the distance of the pairs $(x_k, u_k)$ from zero

$$V\left(x_0, u_0, x_1, u_1, \ldots, x_{K-1}, u_{K-1}, x_K\right) = E\left(x_K\right) + \sum_{k=0}^{K-1} L\left(x_k, u_k\right)$$

- Terminal-stage cost

$$E\left(x_k\right) = \frac{1}{2} xK^T Q_K x_K^T$$

- Stage-cost

$$L\left(x_k, u_k\right) = \frac{1}{2}\left(x_k^T Q x_k + u_k^T R u_k\right)$$

The objective depends on the control sequence $\{u_k\}_{k=0}^{K_1}$ and the state sequence $\{x_k\}_{k=0}^{K}$

- We assume that the initial state $x_0$ is fixed and known quantity
- Remaining states are determined by the model and $\{u_k\}_{k=0}^{K_1}$

Matrices $Q$ and $Q_K$ are positive semi-definite, $R$ is positive definite

- They are tuning parameters

# The linear-quadratic regulator | Baby LQR

Consider a linear and time-invariant process with single state variable and single input

The system dynamics, in discrete-time

$$x_{k+1} = a x_k + b u_k, \quad \text{with } x_k, u_k \in \mathcal{R}$$

The control problem, in discrete-time

$$\underset{u_0, u_1, \dots, u_{K-1}}{\text{minimise}} \quad \underbrace{\frac{1}{2} x_K^T q_K x_K}_{E(x_K)} + \frac{1}{2} \sum_{k=0}^{K-1} \underbrace{\Big( x_k^T q x_k + u_k^T r u_k \Big)}_{L(x_k, u_k)}$$

Consider a finite-horizon of length one ($K = 1$)

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2} x_1^T q_K x_1 + \frac{1}{2} \sum_{k=0}^{1-1} \Big( x_k^T q x_k + u_k^T r u_k \Big)$$

We have,

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2} \Big( x_1^T q_K x_1 + x_0^T q x_0 + u_0^T r u_0 \Big)$$

# The linear-quadratic regulator | Baby LQR (cont.)

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2}\Big(x_1^T q_K x_1 + x_0^T q x_0 + u_0^T r u_0\Big)$$

In this simple case, we only need to (optimise to) find a single control action, $u_0$

- Under the constraint that $x_1 = a x_0 + b u_0$
- The initial state $x_0$ is fixed and known

We have,

$$\underset{u_0}{\text{minimise}} \ \frac{1}{2}\Bigg(\underbrace{x_1^T}_{a x_0 + b u_0} q_K \underbrace{x_1}_{a x_0 + b u_0} + x_0^T q x_0 + u_0^T r u_0\Bigg)$$

All the terms in the cost function are known, with the exception of $u_0$

- It is the decision variable, it is a scalar

# The linear-quadratic regulator | Baby LQR (cont.)

$$\underset{u_0}{\text{minimise}} \ \frac{1}{2}\Bigg(\underbrace{x_1^T}_{a x_0 + b u_0} \ q_K \ \underbrace{x_1}_{a x_0 + b u_0} + x_0^T q x_0 + u_0^T r u_0\Bigg)$$

Substituting and rearranging, we have a quadratic equation $u_0$

$$\underset{u_0}{\text{minimise}} \ \underbrace{\frac{1}{2}\big(q x_0^2 + r u_0^2 + q_K (a x_0 + b u_0)^2\big)}_{f(u_0)}$$

- We are interested in value $u_0$ that minimises this function

---

After some algebra, we see that the cost function is a parabola

$$\begin{aligned} f(u_0) &= \frac{1}{2}\big(q x_0^2 + r u_0^2 + q_K (a x_0 + b u_0)\big) \\ &= \frac{1}{2}\big((q + a^2 q_K) x_0^2 + 2(b a q_K x_0) u_0 + (b^2 q_K + r) u_0^2\big) \end{aligned}$$

We know how to locate the minimum of parabola, its vertex

# The linear-quadratic regulator | Baby LQR (cont.)

$$f(u_0) = \frac{1}{2}\left((q + a^2 q_K)x_0^2 + 2(b\,a\,q_K\,x_0)u_0 + (b^2 q_K + r)u_0^2\right)$$

$f(u_0)$ is a parabola and it is smallest at the value $u_0$ that makes its derivative zero

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u_0}f(u_0) &= b\,q_K\,a\,x_0 + (b^2 q_K + r)u_0\\ &= 0\end{aligned}$$

We have the solution to the optimisation/control problem

$$\begin{aligned}u_0 &= -\underbrace{\frac{b\,q_K\,a}{b^2 q_K + r}}_{k}x_0\\ &= -k x_0\end{aligned}$$

$\square$

# The linear-quadratic regulator (cont.)

For systems with multiple state variables and multiple inputs, the structure is identical

The system dynamics, in discrete-time

$$x_{k+1} = Ax_k + Bu_k, \quad \text{with } x_k \in \mathcal{R}^{N_x} \text{ and } u_k \in \mathcal{R}^{N_u}$$

The control problem, in discrete-time

$$\underset{u_0, u_1, \dots, u_{K-1}}{\text{minimise}} \quad \underbrace{\frac{1}{2} x_K^T Q_K x_K}_{E(x_K)} + \frac{1}{2} \sum_{k=0}^{K-1} \underbrace{\left( x_k^T Q x_k + u_k^T R u_k \right)}_{L(x_k, u_k)}$$

Consider a finite-horizon of length one ($K = 1$)

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2} x_1^T Q_K x_1 + \frac{1}{2} \sum_{k=0}^{1-1} \left( x_k^T Q x_k + u_k^T R u_k \right)$$

# The linear-quadratic regulator (cont.)

After substituting the dynamics, we get

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2}\left(\underbrace{x_1}_{Ax_0+Bu_0}{}^T Q_K \underbrace{x_1}_{Ax_0+Bu_0} + x_0^T Q x_0 + u_0^T R u_0\right)$$

After some algebra and rearranging, we have

$$\underset{u_0}{\text{minimise}} \quad \frac{1}{2}\left(x_0^T \left(Q + A^T P A\right) x_0 + 2u_0^T B^T Q_K A x_0 + u_0^T \left(B^T Q_K B + R\right) u_0\right)$$

Taking the derivative and setting it to zero, we get

$$\begin{aligned}\frac{\mathrm{d}f(u_0)}{\mathrm{d}u_0} &= B^T Q_K A x_0 + \left(B^T Q_K B + R\right) u_0 \\ &= 0\end{aligned}$$

Solving this linear system of equations for the unknown $u_0$, we get

$$u_0 = -\underbrace{\left(B^T Q_f B + R\right)^{-1} B^T Q_K A}_{K} x_0$$

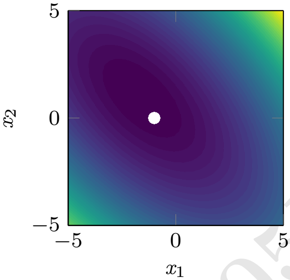
To be able to solve for longer control-horizons, we use backward dynamic programming

# Intermezzo

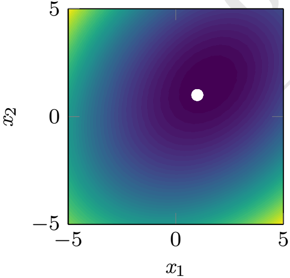
## Sum of quadratic functions

# The LQR | Sum of quadratic functions

Consider two quadratic functions



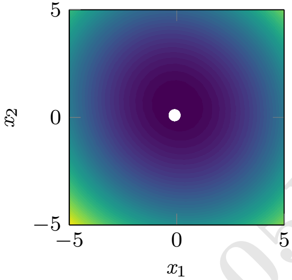

$$
\begin{aligned}
&V_1(x)\\
&= \frac{1}{2}(x-a)^T A\,(x-a)\\
&= \frac{1}{2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)^T \underbrace{\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}}_{\succ 0} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)
\end{aligned}
$$




$$
\begin{aligned}
&V_2(x)\\
&= \frac{1}{2}(x-b)^T B\,(x-b)\\
&= \frac{1}{2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T \underbrace{\begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}}_{\succ 0} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)
\end{aligned}
$$

# The LQR | Sum of quadratic functions (cont.)

We compute function $V(x) = V_1(x) + V_2(x)$ and show that it is a quadratic function




$$V(x) = \frac{1}{2}\Big((x - v)^T H (x - v) + d\Big)$$

$$H = A + B$$
$$v = H^{-1}(Aa - Bb)$$
$$d = -(Aa + Bb)^T H^{-1}(Aa + Bb) + a^T Aa + b^T Bb$$

Matrix $H$ is a positive definite matrix, because both $A$ and $B$ are positive definite

$$V(x) = \frac{1}{2}\Big((x - v)^T H (x - v) + d\Big)$$
$$= \frac{1}{2}\left(\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}\right)^T \underbrace{\begin{bmatrix} 2.75 & 0.25 \\ 0.25 & 2.75 \end{bmatrix}}_{\succ 0} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}\right) + 3.2\right)$$

# The LQR | Sum of quadratic functions (cont.)

Consider two quadratic functions, one of which with a linear combination of variable $x$

$$V_1(x) = \frac{1}{2}(x-a)^T A (x-a)$$
$$V_2(x) = \frac{1}{2}(Cx-b)^T B (Cx-b)$$

We can compute function $V(x) = V_1(x) + V_2$,

$$V(x) = \frac{1}{2}\Big((x-v)^T H (x-v) + d\Big)$$

$$H = A + C^T BC$$
$$v = H^{-1}(Aa - CBb)$$
$$d = -(Aa + CBb)^T H^{-1}(Aa + CBb) + a^T Aa + b^T Bb$$

$\square$

# The linear quadratic regulator (cont.)

## Dynamic programming

# The linear-quadratic regulator (cont.)

We have the optimal control problem, with quadratic cost terms and linear dynamics

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_{K-1}, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & Ax_k + Bu_k - x_{k+1} = 0, \qquad k = 0, 1, \dots, K-1 \\ & \overline{x}_0 - x_0 = 0 \end{aligned}$$

The optimisation problem can be re-written in the equivalent form

$$\min_{\substack{\overline{x}_0 \\ x_1, \dots, x_{K-1}, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad \underbrace{L(\overline{x}_0, u_0) + L(x_1, u_1) + \cdots L(x_{K-1}, u_{K-1}) + E(x_K)}_{V(u_0, x_1, u_1, \dots, u_{K-1} | x_0)}$$

After isolating the last two stages, we get

$$\begin{aligned} \min_{\substack{\overline{x}_0 \\ x_1, \dots, x_{K-2} \\ u_0, u_1, \dots, u_{K-2}}} \quad & L(\overline{x}_0, u_0) + L(x_1, u_1) + \cdots + L(x_{K-2}, u_{K-2}) + \\ \min_{u_{K-1}, x_K} \quad & L(x_{K-1}, u_{K-1}) + E(x_K) \end{aligned}$$

# The linear-quadratic regulator (cont.)

$$\min_{\substack{\overline{x}_0 \\ x_1,\ldots,x_{K-2} \\ u_0,u_1,\ldots,u_{K-2}}} \quad L\left(\overline{x}_0, u_0\right) + L\left(x_1, u_1\right) + \cdots + L\left(x_{K-2}, u_{K-2}\right) +$$

$$\min_{u_{K-1},x_K} \quad L\left(x_{K-1}, u_{K-1}\right) + E\left(x_K\right)$$

At the last stage, we have the optimisation problem

$$\begin{aligned} \min_{u_{K-1},x_K} \quad & L\left(x_{K-1}, u_{K-1}\right) + E\left(x_K\right) \\ \text{subject to} \quad & Ax_{K-1} + Bu_{K-1} - x_K = 0 \end{aligned}$$

The state $x_{K-1}$ appears as parameter

---

We define optimal cost (the minimum) and optimal decision variables (the minimiser)

- The optimal decision variables $u^*_{K-1}\left(x_{K-1}\right)$ and $x^*_K\left(x_{K-1}\right)$
- The optimal cost $V^*\left(x_{K-1}\right)$

# The linear-quadratic regulator (cont.)

$$
\begin{aligned}
\min_{u_{K-1}, x_K} \quad & L\left(x_{K-1}, u_{K-1}\right) + E\left(x_K\right) \\
\text{subject to} \quad & A x_{K-1} + B u_{K-1} - x_K = 0
\end{aligned}
$$

To solve this optimisation problem, we first substitute the dynamics

$$
\begin{aligned}
E\left(x_K\right) + L\left(x_{K-1}, u_{K-1}\right) &= \underbrace{\frac{1}{2}\left(A x_{K-1} + B u_{K-1}\right)^T Q_K \left(A x_{K-1} + B u_{K-1}\right)}_{E(E_K)} \\
&+ \underbrace{\frac{1}{2}\Big(x_{K-1}^T Q x_{K-1} + u_{N-1}^T R u_{N-1}\Big)}_{L(x_{K-1}, u_{K-1})} \\
&= \frac{1}{2}\Big(x_{K-1}^T Q x_{K-1} + \left(u_{K-1} - v\right)^T H \left(u_{K-1} - v\right) + d\Big)
\end{aligned}
$$

We used,

$$
\begin{aligned}
H &= R + B^T Q_K B \\
v &= -\underbrace{\Big(B^T Q_K B + R\Big)^{-1} B^T Q_K A}\, x_{K-1} \\
d &= x_{K-1}^T \left(A^T Q_K A - A^T Q_K B \Big(B^T Q_K B + R\Big)^{-1} B^T Q_K A\right) x_{K-1}
\end{aligned}
$$

# The linear-quadratic regulator (cont.)

The optimal control action $u_{K-1}^* = v$ is a linear function of the state $x_{K-1}$

$$u_{K-1}^* = \underbrace{Y - \left(B^T Q_K B + R\right)^{-1} B^T Q_K A}_{K_{K-1}} x_{K-1}$$

By using the dynamics, we compute the terminal state $x_K^*$ from the optimal action

$$\begin{aligned} x_K^* &= A x_{K-1} + B u_{K-1}^* \\ &= A x_{K-1} + B \left(B^T Q_K B + R\right)^{-1} B^T Q_K A x_{K-1} \\ &= \left(A + B \underbrace{\left(B^T Q_K B + R\right)^{-1} B^T Q_K A}_{-K_{K-1}}\right) x_{K-1} \end{aligned}$$

The cost associated to the optimal control action is quadratic in $x_{K-1}$

$$V_K^* = \frac{1}{2}\left( x_{K-1}^T Q x_{K-1} + \underbrace{\left(u_{K-1}^* - \underbrace{v}_{u_{K-1}^*}\right)^T H \left(u_{K-1}^* - \underbrace{v}_{u_{K-1}^*}\right)}_{=0} + d \right)$$

# The linear-quadratic regulator (cont.)

$$
\begin{aligned}
V_K^* &= \frac{1}{2}\left( x_{K-1}^T Q x_{K-1} + \underbrace{\left( u_{K-1}^* - \underbrace{v}_{u_{K-1}^*} \right)^T H \left( u_{K-1}^* - \underbrace{v}_{u_{K-1}^*} \right)}_{=0} + d \right) \\
&= \frac{1}{2}\left( x_{K-1}^T Q x_{K-1} + \underbrace{x_{K-1}^T \left( A^T Q_K A - A^T Q_K B \left( B^T Q_K B + R \right)^{-1} B^T Q_K A \right) x_{K-1}}_{d} \right) \\
&= \frac{1}{2} x_{K-1}^T \underbrace{\left( Q + A^T Q_K A - A^T Q_K B \left( B^T Q_K B + R \right)^{-1} B^T Q_K A \right)}_{\Pi_{K-1}} x_{K-1}
\end{aligned}
$$

# The linear-quadratic regulator (cont.)

$$K_{K-1} = \left(B^T Q_K B + R\right)^{-1} B^T Q_K A$$

Summarising, we have

$$\begin{aligned} u_{K-1}^* \left(x_{K-1}\right) &= K_{K-1} x_{K-1} \\ x_K^* \left(x_{K-1}\right) &= \left(A + BK_{K-1}\right) x_{K-1} \\ V_K^* \left(x_{K-1}\right) &= \frac{1}{2} x_{K-1}^T \Pi_{K-1} x_{K-1} \end{aligned}$$

---

Function $V_K^*$ defines the optimal cost-to-go from $x_{K-1}$, under optimal control $u_{K-1}^*$

- As it depends only on $x_{K-1}$ it allows to move to stage $K-2$

$$\min_{\substack{\overline{x}_0 \\ x_1, \dots, x_{K-2} \\ u_0, u_1, \dots, u_{K-2}}} \quad L\left(\overline{x}_0, u_0\right) + L\left(x_1, u_1\right) + \cdots + L\left(x_{K-2}, u_{K-2}\right) + V^*\left(x_{K-1}\right)$$

# The linear-quadratic regulator (cont.)

$$\min_{\substack{\overline{x}_0 \\ x_1,\dots,x_{K-2} \\ u_0,u_1,\dots,u_{K-2}}} \underbrace{L(\overline{x}_0, u_0) + L(x_1, u_1) + \cdots + L(x_{K-2}, u_{K-2}) + V^*(x_{K-1})}_{V(u_0,x_1,u_1,\dots,u_{K-2}|x_0)}$$

After isolating the last two stages, we get

$$\min_{\substack{\overline{x}_0 \\ x_1,\dots,x_{K-3} \\ u_0,u_1,\dots,u_{K-3}}} L(\overline{x}_0, u_0) + L(x_1, u_1) + \cdots + L(x_{K-3}, u_{K-3}) +$$
$$\min_{u_{K-2},x_{K-1}} L(x_{K-2}, u_{K-2}) + V^*(x_{K-1})$$

At the last stage, we have the optimisation problem

$$\begin{aligned} \min_{u_{K-1},x_K} \quad & V^*(x_{K-1}) + L(x_{K-2}, u_{K-2}) \\ \text{subject to} \quad & Ax_{K-2} + Bu_{K-2} - x_{K-1} = 0 \end{aligned}$$

The state $x_{K-2}$ appears as parameter

# The linear-quadratic regulator (cont.)

$$\begin{aligned}\min_{u_{K-1}, x_K} \quad & V^*\left(x_{K-1}\right) + L\left(x_{K-2}, u_{K-2}\right) \\ \text{subject to} \quad & A x_{K-2} + B u_{K-2} - x_{K-1} = 0\end{aligned}$$

We define optimal cost (the minimum) and optimal decision variables (the minimiser)

- The optimal decision variables $u^*_{K-2}\left(x_{K-2}\right)$ and $x^*_{K-2}\left(x_{K-2}\right)$

$$\begin{aligned}u^*_{K-2}\left(x_{K-2}\right) &= K_{K-2} x_{K-2} \\ x^*_{K-1}\left(x_{K-2}\right) &= \left(A + B K_{K-2}\right) x_{K-2}\end{aligned}$$

- The optimal cost $V^*\left(x_{K-2}\right)$ from stage $K-2$ to $K$

$$V^*_{K-1}\left(x_{K-2}\right) = \frac{1}{2} x^T_{K-2} \Pi_{K-2} x_{K-2}$$

We used,

$$\begin{aligned}K_{K-2} &= -\left(B^T \Pi_{K-1} B + R\right)^{-1} B^T \Pi_{K-1} A \\ \Pi_{K-2} &= Q + A^T \Pi_{K-1} A - A^T \Pi_{K-1} B \left(B^T \Pi_{K-1} B + R\right)^{-1} B^T \Pi_{K-1} A\end{aligned}$$

# The linear-quadratic regulator (cont.)

The recursion from $\Pi_{K-1}$ to $\Pi_{K-2}$ is known as the **backward Riccati iteration**

In the general form, the recursion from $\Pi_K = Q_K$

$$\Pi_{k-1} = Q + A^T \Pi_k A - A^T \Pi_k B \left( B^T \Pi_k B + R \right)^{-1} B^T \Pi_k A$$

$$(k = K, K-1, \cdots, 1)$$

We can also define the general form of the optimal cost and optimal decision variables

- The optimal decision variables $u_k^* (x_k)$ and $x_k^* (x_k)$

$$\begin{aligned} u_k^* (x_k) &= -K_k x_k \\ x_k^* (x_k) &= (A + BK_k) x_k \end{aligned}$$

- The optimal cost to go $V^* (x_k)$ from stage $k$ to $K$

$$V_k^* (x_k) = \frac{1}{2} x_k^T \Pi_{k+1} x_k$$

# An example

## The linear quadratic regulator

# The linear-quadratic regulator (cont.)

## Example

Consider the linear and time-invariant dynamical system with measurement process

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Consider the following system matrices and associate IO representation

$$\begin{aligned} A &= -b \\ B &= -(a+b) \\ C &= k \\ D &= k \end{aligned}$$

$$y(s) = g(s)u(s)$$

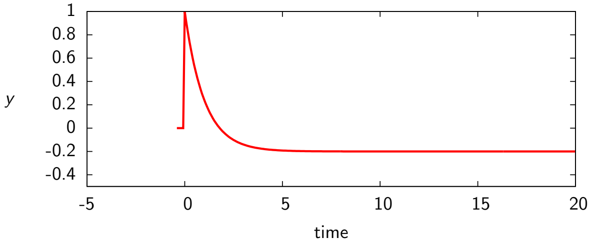
$$g(s) = k\frac{s-a}{s+b}$$

For $(a, b) = (0.2, 1) > 0$ and $k = 1$, system has inverse response (right-half-plane zero)

# The linear-quadratic regulator (cont.)

Step response, by solving the ODE with $u(t) = 1$ and initial condition $x(0) = 0$

- We observe what happens from the measurements $y(t)$
- The response to a unit step of the control $u(t)$




Suppose that we request a unit step of the output $y(t)$, as a set-point change

- We ask what is the optimal control action
- The best action capable to deliver it

# The linear-quadratic regulator (cont.)

$$y(s) = \underbrace{k\frac{s-a}{s+b}}_{g(s)} u(s)$$

In the Laplace domain, we have the requested output

$$\overline{y}(s) = \frac{1}{s}$$

By solving for $\overline{u}(s)$, we get

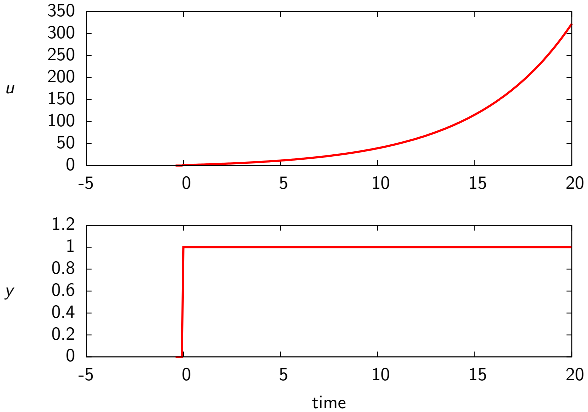
$$\begin{aligned}\overline{u}(s) &= \frac{\overline{y}}{g(s)} \\ &= \frac{s+b}{ks(s-a)}\end{aligned}$$

Back to the time-domain,

$$u(t) = \frac{1}{ka}\left(-b + (a+b)\underbrace{e^{at}}_{a>0\ (!)}\right)$$

# The linear-quadratic regulator (cont.)

Output response, with an exponentially growing input and $y(t)$ is perfectly on target



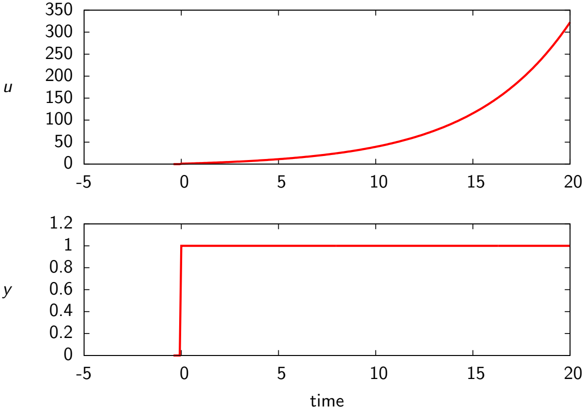

We are capable of achieving perfect tracking in $y(t)$ by using applying an optimal $u(t)$

# The linear-quadratic regulator (cont.)

$$g(s) = k\frac{s-a}{s+b}, \text{ with } \overline{u}(s) = \frac{1}{s-a}\frac{s+b}{ks}$$

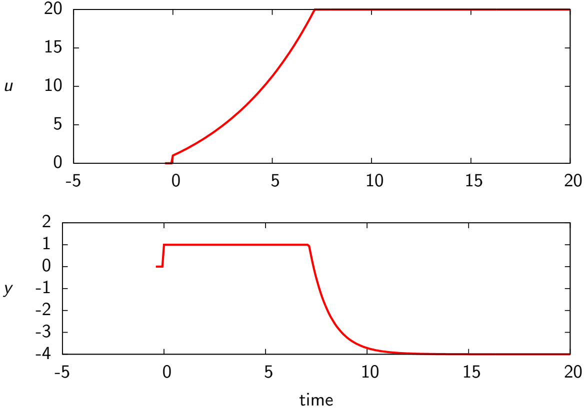
The zeros at $s = a$ in $g(s)$ and $\overline{u}(s)$ cancel out, tracking of output $y(t)$ looks perfect

- The input-blocking property of the zero in the transfer function



# The linear-quadratic regulator (cont.)

The inputs in reality cannot grow unboundedly, at some point they will hit constraints




The saturation of the input at the constraint destroys the perfect output response $y(t)$

# Linear-quadratic optimal control | LTV-QR

We can also consider the more general formulation of a linear-quadratic optimal control

$$\begin{aligned}\min_{x,u} \quad & \underbrace{x_K^T Q_K x_K}_{E(x_K)} + \sum_{k=0}^{K-1} \underbrace{\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}}_{L_k(x_k, u_k)} \\ \text{subject to} \quad & x_{k+1} - A_k x_k - B_k u_k = 0, \quad k = 0, 1, \dots, K-1 \\ & x_0 - \overline{x}_0 = 0 \end{aligned}$$

At each recursion step, we must compute the (now varying) stage-cost $L_k(x_k, u_k)$,

$$L_k(x_k, u_k) = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

Matrices $Q_k$ and $R_k$ are time-varying and positive semi definite and positive definite

- Matrix $Q_K$ is positive definite

Moreover, we allow for further flexibility in tuning by including the mixing matrix $S_k$

# Linear-quadratic optimal control | LTV-QR (cont.)

$$\begin{aligned}
\min_{x,u} \quad & \underbrace{x_K^T Q_K x_K}_{E(x_K)} + \sum_{k=0}^{K-1} \underbrace{\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} Q_k & S_k^T \\ S_k & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}}_{L_k(x_k, u_k)} \\
\text{subject to} \quad & x_{k+1} - A_k x_k - B_k u_k = 0, \qquad k = 0, 1, \dots, K-1 \\
& x_0 - \overline{x}_0 = 0
\end{aligned}$$

Furthermore, we allow the system dynamics to be time-varying,

$$f_k\left(x_k, u_k\right) = A_k x_k + B_k u_k$$

Under these conditions, the optimal cost $V_k^*\left(x_k\right)$ from stage $k$ to $k+1$ is still quadratic

$$V_k^*\left(x_k\right) = \frac{1}{2} x_k^T \Pi_{k+1} x_k$$

The backward Riccati recursion is used to compute $\Pi_{k+1}$

# Linear-quadratic optimal control | LTV-QR (cont.)

Using the terminal condition $\Pi_K = Q_K$, we have

$$\Pi_k = Q_k + A_k^T \Pi_{k+1} A_k - \left(S_k^T + A_k^T \Pi_{k+1} B_k\right)\left(R_k + B_k^T \Pi_{k+1} B_k\right)^{-1}\left(S_k + B_k^T \Pi_{k+1} A_k\right)$$

The optimal decision variables are obtained from the feedback law,

$$u_k^*\left(x_k\right) = -\left(R_k + B_k^T \Pi_{k+1} B_k\right)^{-1}\left(S_k + B_k^T \Pi_{k+1} A_k\right) x_k$$

The forward simulation from $\overline{x}_0$ determines the state variables

$$x_{k+1} = A_k x_k + B_k u_k^*$$

# Linear-quadratic optimal control | AQR

Consider the even more general formulation of an affine-quadratic optimal control

$$\min_{x,u} \quad \underbrace{\begin{bmatrix} 1 \\ x_K \end{bmatrix}^T \begin{bmatrix} * & q_K^T \\ q_K & Q_K \end{bmatrix} \begin{bmatrix} 1 \\ x_K \end{bmatrix}}_{E(x_K)} + \sum_{k=0}^{K-1} \underbrace{\begin{bmatrix} 1 \\ x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} * & q_k^T & s_k^T \\ q_k & Q_k & S_k^T \\ s_k & S_k & R_k \end{bmatrix} \begin{bmatrix} 1 \\ x_k \\ u_k \end{bmatrix}}_{L_k(x_k, u_k)}$$

$$\begin{aligned} \text{subject to} \quad & x_{k+1} - A_k x_k - B_k u_k - c_k = 0, \quad k = 0, 1, \dots, K-1 \\ & x_0 - \overline{x}_0 = 0 \end{aligned}$$

These optimisations often result from trajectory linearisation of nonlinear dynamics

The general dynamic programming solution is retained by augmenting the state

$$\widetilde{x}_k = \begin{bmatrix} 1 \\ x_k \end{bmatrix}$$

The augmented dynamics,

$$\widetilde{x}_{k+1} = \begin{bmatrix} 1 & 0 \\ c_k & A_k \end{bmatrix} \widetilde{x}_k + \begin{bmatrix} 0 \\ B_k \end{bmatrix} u_k$$

The fixed initial value is $\overline{\widetilde{x}}_0 = \begin{bmatrix} 1 & \overline{x}_0 \end{bmatrix}^T$

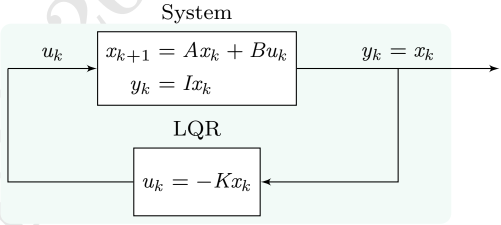
# The linear-quadratic regulator | Infinite-horizon

We discussed the linear quadratic regulator over a finite horizon of some length $K$

Linear quadratic regulators can destabilise a stable system over finite horizons

- Setting $Q, R \succ 0$ is not sufficient to guarantee closed-loop stability

$$\begin{cases} x_{k+1} = Ax_k + B\left(\underbrace{-Kx_k}_{u_k}\right) \\ y(t) = x(t) \end{cases}$$




The stability of the closed-loop is determined by the eigenvalues of matrix $A_{\text{CL}}$

The closed-loop dynamics,

$$\begin{aligned} x_{k+1} &= Ax_k - BKx_k \\ &= \underbrace{(A - BK)}_{A_{CL}} x_k \end{aligned}$$

# An example

## The linear quadratic regulator

# The linear-quadratic regulator | Infinite-horizon (cont.)

## Example

Consider a discrete-time linear time-invariant dynamical system with LQR ($K = 5$)

$$x_{k+1} = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$$

The discrete-time transfer function has a zero ($z = 3/2$), non-minimum phase system

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_4, x_5 \\ u_0, u_1, \dots, u_4}} \quad & x_5^T Q_5 x_5 + \sum_{k=0}^{4} x_k^T Q x_k + u_k^T R u_k \\ \text{subject to} \quad & A x_k + B u_k - x_{k+1} = 0, \quad k = 0, 1, \dots, 4 \\ & \overline{x}_0 - x_0 = 0 \end{aligned}$$

We use $Q = Q_5 = C^T C + 0.001 I$ and $R = 0.001$ that barely penalises controls

# The linear-quadratic regulator | Infinite-horizon (cont.)

Based on the Riccati equation, we iterate four times from $\Pi_K = Q_K = Q$

$$K_4^{(5)}, K_3^{(5)}, K_2^{(5)}, K_1^{(5)}, K_0^{(5)}$$

Assuming that we use the first feedback gain $K_0^{(5)}$, we have

$$u_k = K_0^{(5)} x_k$$
$$x_k = \left(A + BK_0^{(5)}\right)^k x_0$$

The eigenvalues of $\left(A + BK_0^{(5)}\right)$

$$\lambda\left(A_{\text{CL}}^{(5)}\right) = (\underbrace{1.307}_{>1}, 0.001)$$

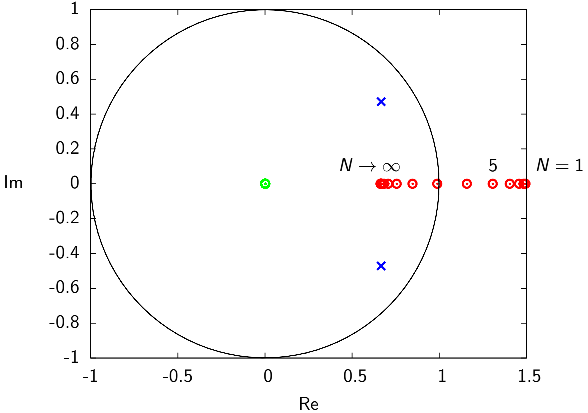
As one of the eigenvalues is outside the unit circle

- The closed-loop system is unstable
- The state grows exponentially
- $x_k \to \infty$ as $k \to \infty$

# The linear-quadratic regulator | Infinite-horizon (cont.)

The closed-loop eigenvalues of $\big(A + BK_0^K\big)$ for control horizons of different lengths, $\circ$

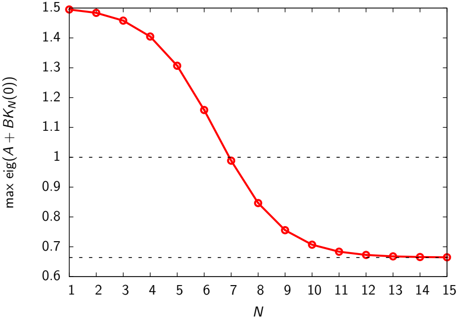
- For reference, the open-loop eigenvalues of $A$, $\times$, are both stable




When we start with a finite horizon LQR, we move both the open-loop eigenvalues

- From $K = 1$, until we enter the unit disc at $K = 7$
- The stability margin grows with $K$

# The linear-quadratic regulator | Infinite-horizon (cont.)




Stability margin as function of the control horizon

- Finite-horizon may return unstable controllers
- More robustness is gained as the horizon grows

$$\lambda\left(A_{\mathrm{CL}}^{(\infty)}\right) = (\underbrace{0.664}_{<1}, 0.001)$$

A feedback gain $K_0^{(\infty)}$ corresponds to an infinite horizon linear quadratic regulator

$$\begin{aligned}
\min_{\substack{x_0, x_1, \dots, \\ u_0, u_1, \dots}} \quad & \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \\
\text{subject to} \quad & A x_k + B u_k - x_{k+1} = 0, \quad k = 0, 1, \dots \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

# The linear-quadratic regulator | Infinite-horizon (cont.)

$$\begin{aligned}
\min_{\substack{x_0, x_1, \ldots, \\ u_0, u_1, \ldots}} \quad & \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \\
\text{subject to} \quad & A x_k + B u_k - x_{k+1} = 0, \quad k = 0, 1, \ldots \\
& \overline{x}_0 - x_0 = 0
\end{aligned}$$

If we are interested in controlling a continuous process, without a final time, then the natural formulation of the optimal control problem is with an infinite horizon cost

- In this case, the Riccati recursion has a stationary solution $\Pi_k = \Pi_{k+1}$,

$$\Pi = Q + A^T \Pi A - A^T \Pi B \left( B^T \Pi B + R \right)^{-1} B^T \Pi A$$

Given $\Pi$, we have the classic optimal control feedback

$$u^* = - \underbrace{\left( R + B^T \Pi B \right)^{-1} B^T \Pi A}_{K} x_k$$

---

Closed-loop stability is not relevant for batch processes, finite-horizon LQRs are fine

# The linear-quadratic regulator | Infinite-horizon (cont.)



$$
\begin{aligned}
\min_{\substack{x_0, x_1, \dots, \\ u_0, u_1, \dots}} \quad & \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k \\
\text{subject to} \quad & A x_k + B u_k - x_{k+1} = 0, \quad k = 0, 1, \dots \\
& \overline{x}_0 - x_0 = 0
\end{aligned}
$$

Infinite-horizon solutions exist as long as the cost function is bounded

- In this case, the cost function is an infinite sum
- The result must not be infinitely big

This is possible when the linear-time invariant systems is controllable

- $\rightsquigarrow$ We can transfer its state from anywhere to anywhere
- $\rightsquigarrow$ And, there exists a control sequence to do that
- $\rightsquigarrow$ And, it can be done in finite time

# The linear-quadratic regulator | Infinite-horizon (cont.)

If the pair $(A, B)$ is controllable, the there exists a finite horizon of length $K$ and a sequence of inputs that can transfer the state of the system from any $x$ to any $x'$

That is, by forward simulation

$$x' = A^K x + \begin{bmatrix} B & AB & \cdots & A^{K-1}B \end{bmatrix} \begin{bmatrix} u_{K_1} \\ u_{K-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Similarly,

$$\underbrace{\begin{bmatrix} B & AB & \cdots & A^{K-1}B \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} u_{K_1} \\ u_{K-1} \\ \vdots \\ u_0 \end{bmatrix} = x' - A^K x +$$

Controllability matrix $\mathcal{C}$ must be full rank for the equation to have a solution $\{u_k\}_{k=0}^{K-1}$

- If cannot reach $x'$ in $K$ moves, then we cannot reach it in any number of moves