CHEM-E7225/2024: Exercise 01

Task 1 (Explicit integration methods).
Consider the Three-Tank System, a benchmark process representative of several typical industrial control applications (e.g., liquid level control in petrochemical plants). The process consists of three cylindrical tanks (T_i, $i = 1, 2, 3$) connected by two fixed valves (V_i, $i = 1, 2$), with an outflow valve V_0 for the last tank. We are interested in controlling the liquid levels (h_i, $i = 1, 2, 3$) in each tank by manipulating the incoming flow-rates to tanks T_1 and T_3 through the pumps P_1 and P_3, respectively. The process is shown in Figure 1.

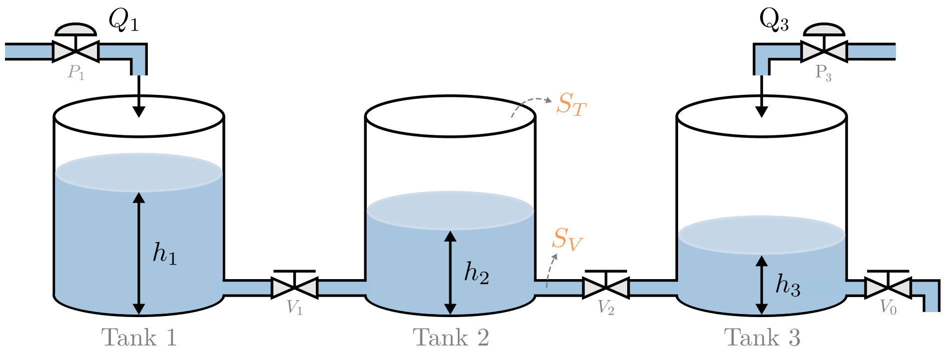

Figure 1: Three-Tank System: Process layout.

Using Torricelli's law to model the flow between tanks, and the `tanh` function to indicate its direction[1], the mass balances for the individual components lead to the following nonlinear dynamics

$$\frac{dh_1}{dt} = \frac{1}{S_T}\left[Q_1 - \alpha_V S_V \texttt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|}\right]; \tag{1a}$$

$$\frac{dh_2}{dt} = \frac{1}{S_T}\left[\alpha_V S_V \texttt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|} - \alpha_V S_V \texttt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|}\right]; \tag{1b}$$

$$\frac{dh_3}{dt} = \frac{1}{S_T}\left[Q_3 + \alpha_V S_V \texttt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|} - \alpha_0 S_V \sqrt{2gh_3}\right]; \tag{1c}$$

The state vector is $x(t) = (h_1(t), h_2(t), h_3(t))$ and the controls are $u(t) = \big(Q_1(t), Q_3(t)\big)$. The vector of model parameters in nominal operating conditions is $\theta = \big(S_T, S_V, \alpha_V, \alpha_0, g\big) = \big(154,\ 0.5,\ 0.47,\ 0.77,\ 981\big)$. The description of each process variable and the constant parameter values are presented in Table 1.

Table 1: Three-Tank System: Process variables and constant parameters.

Variable	Description	Value	Units
h_1	Water level of tank 1	$\in [0, 60]$	cm
h_2	Water level of tank 2	$\in [0, 60]$	cm
h_3	Water level of tank 3	$\in [0, 60]$	cm
Q_1	Flow-rate to tank 1	$\in [0, 140]$	ml/s
Q_3	Flow-rate to tank 3	$\in [0, 140]$	ml/s
S_T	Cross-section of tanks T_i $(i = 1, 2, 3)$	154	cm^2
S_V	Cross-section of valves V_i $(i = 0, 1, 2)$	0.5	cm^2
α_V	Flow coefficient of valves V_i $(i = 1, 2)$	0.47	–
α_0	Outflow coefficient of valve V_0	0.77	–
g	Gravitational constant	981	cm/s^2

[1] The hyperbolic tangent is $\texttt{tanh}(x) = (e^x - e^{-x})/(e^x + e^{-x})$. This function satisfies $\texttt{tanh}(h_i - h_j) \approx 1$ if $h_i - h_j \geq 0$, indicating the flow direction from tank i to tank j, and $\texttt{tanh}(h_i - h_j) \approx -1$ if $h_i - h_j < 0$, indicating the opposite direction. We use it as a smooth alternative to the discontinuous `sign` function, as our tasks require differentiating the dynamics $f(\cdot)$.

In this assignment, you will implement different integration schemes to simulate the dynamics of this system. Re-write the dynamic model in Eq. (1) using the control notation to get it in the form $\dot{x}(t) = f(x(t), u(t)|\theta_x)$, then adapt the starting code in the archive `E01_code.zip` for the following tasks:

Task 1.1. Complete the function file `threeTank.m` to implement the state-equation $\dot{x}(t) = f(x(t), u(t)|\theta_x)$ for the Three-Tank System according to Eq. (1). (10 points)

Task 1.2. Complete the function files `eefunc.m` (Explicit Euler, EE) and `rk4func.m` (4^{th}-Order Range-Kutta, RK4) which implement explicit integration methods. (10 points)

Task 1.3. Complete the function file `Linearize.m` which linearizes the dynamics $\dot{x}(t) = f(x(t), u(t)|\theta_x)$ around a fixed-point $P \coloneqq (x_p, u_p)$ through Jacobians $(\partial f/\partial x)|_{(x_p,u_p)}$ and $(\partial f/\partial u)|_{(x_p,u_p)}$, and converts the resulting model to *discrete-time*. Then, complete the function file `fsimfnc.m` which implements the analytical forward simulation map of affine time-invariant systems. (30 points)

Task 1.4. Complete the first section of the script `E01_main.m` to define the simulation and model parameters, the initial state, and an input signal. We suggest implementing the signal $u(t) = (u_1(t), u_2(t))$,

$$u_1(t) = \begin{cases} u_Z & \text{for } t \leq t_Z \\ 0 & \text{otherwise} \end{cases}, \qquad u_2(t) = \begin{cases} u_Y & \text{for } t \geq t_Y \\ 0 & \text{otherwise} \end{cases},$$

for different values of (u_Z, t_Z, u_Y, t_Y) satisfying $0 \leq t_Z, t_Y \leq T_f$ and $0 \leq u_Z, u_Y \leq 140$. Remember to define these signals in discrete-time form, $u_1 = (u_{1,1}, \ldots, u_{1,K})$ and $u_2 = (u_{2,1}, \ldots, u_{2,K})$.

Hint: Use concatenations of MATLAB's ***`zeros`*** *and* ***`ones`*** *functions with appropriate sizes, or, more elegantly, use the* ***`heaviside`*** *step-function applied to time-nodes* $(t_1, \ldots, t_K)$. (20 points)

Task 1.5. Complete the remaining sections of script `E01_main.m` to simulate the Three-Tank system using the explicit integration methods (EE and RK4), and using CasADi's built-in integrator. Finally, execute the entire script to generate and plot the system responses. (30 points)

Consider a simulation time T_f of 240 seconds (with $K = 300$ time-nodes k, or more), and a valid initial state $x(0)$ satisfying $x_1(0) \neq x_2(0) \neq x_3(0)$. Report the simulation plots and comment on the results **in terms of the process variables in Table 1 and layout depicted in Figure 1**, i.e., describe what is happening at the "physical plant" as the controls are applied. Moreover, compare the simulation results against those obtained using the affine model: Is the linear approximation valid somewhere during the simulation?