CHEM-E7225/2023: Exercise 03

Task 1 (Optimal control for Three-Tank System).
Consider again the benchmark Three-Tank System from Homework 1 (Figure 1): The process consists of three cylindrical tanks (T_i, $i = 1, 2, 3$) connected by two fixed valves (V_i, $i = 1, 2$), with an outflow valve V_0 for the last tank. The liquid levels (h_i, $i = 1, 2, 3$) in each tank are controlled by manipulating the incoming flow-rates to tanks T_1 and T_3 through the pumps P_1 and P_3, respectively.

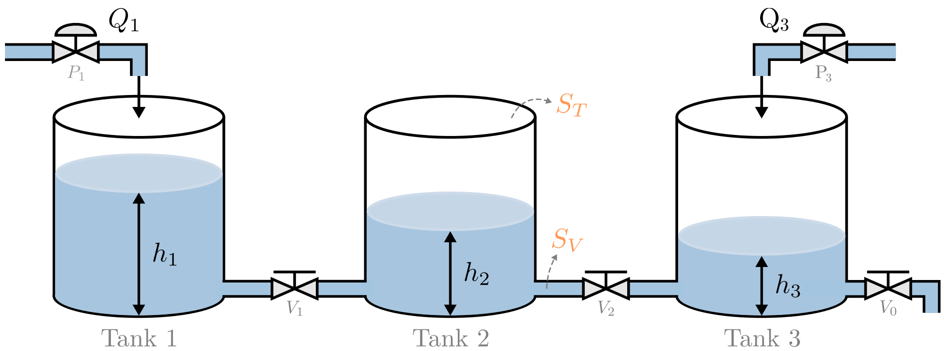

Figure 1: Three-Tank System: Process layout.

Using Torricelli's law to model the flow between tanks, and now using the `tanh` function to indicate its direction, the mass balances for the individual components lead to the following nonlinear dynamics

$$\frac{dh_1}{dt} = \frac{1}{S_T}\left[Q_1 - \alpha_V S_V \texttt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|}\right]; \quad (1a)$$

$$\frac{dh_2}{dt} = \frac{1}{S_T}\left[\alpha_V S_V \texttt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|} - \alpha_V S_V \texttt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|}\right]; \quad (1b)$$

$$\frac{dh_3}{dt} = \frac{1}{S_T}\left[Q_3 + \alpha_V S_V \texttt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|} - \alpha_0 S_V \sqrt{2gh_3}\right]; \quad (1c)$$

The state vector is $x(t) = (h_1(t), h_2(t), h_3(t))$ and the controls are $u(t) = \big(Q_1(t), Q_3(t)\big)$. The vector of model parameters in nominal operating conditions is $\theta = \big(S_T, S_V, \alpha_V, \alpha_0, g\big) = \big(154,\ 0.5,\ 0.56,\ 0.73,\ 981\big)$. The description of each process variable and the constant parameter values are presented in Table 1.

Table 1: Three-Tank System: Process variables and constant parameters.

Variable	Description	Value	Units
h_1	Water level of tank 1	$\in [0, 60]$	cm
h_2	Water level of tank 2	$\in [0, 60]$	cm
h_3	Water level of tank 3	$\in [0, 60]$	cm
Q_1	Flow-rate to tank 1	$\in [0, 140]$	ml/s
Q_3	Flow-rate to tank 3	$\in [0, 140]$	ml/s
S_T	Cross-section of tanks T_i ($i = 1, 2, 3$)	154	cm^2
S_V	Cross-section of valves V_i ($i = 0, 1, 2$)	0.5	cm^2
α_V	Flow coefficient of valves V_i ($i = 1, 2$)	0.56	–
α_0	Outflow coefficient of valve V_0	0.73	–
g	Gravitational constant	981	cm/s^2

Adapt the starting code in the archive `E3_code.zip` to solve this problem.

1. Substitute the function files `ThreeTank.m` and `rk4fnc.m` with your solutions from Homework 1, and then change any `sign()` function in the model with the smooth `tanh()` function;

2. Complete the `E03A_main.m` template to implement the following discrete-time optimal control problem using the simultaneous approach with `CasADi/IPOPT`,

$$\min_{\substack{x_0,\ldots,x_K \\ u_0,\ldots,u_{K-1}}} \quad \sum_{k=0}^{K-1} \Big[(x_k - x_k^{ref})^T Q (x_k - x_k^{ref}) + u_k^T R u_k\Big] \tag{2a}$$

$$\text{subject to} \quad x_{k+1} - F(x_k, u_k|\theta_x) = 0, \qquad k = 0, 1, \ldots, K{-}1 \tag{2b}$$

$$x_k^{\min} \le x_k \le x_k^{\max}, \qquad k = 0, 1, \ldots, K \tag{2c}$$

$$u_k^{\min} \le u_k \le u_k^{\max}, \qquad k = 0, 1, \ldots, K \tag{2d}$$

$$x_0 - \hat{x}_0 = 0. \tag{2e}$$

   The state-references $x_k^{ref} = (x_{1k}^{ref}, x_{2k}^{ref}, x_{3k}^{ref})$ are defined by

$$x_{1k}^{ref} = 40 + 10\cos(0.03k\Delta t), \qquad x_{2k}^{ref} = 30, \qquad \text{and} \qquad x_{3k}^{ref} = 20 + 10\sin(0.03k\Delta t) \tag{3}$$

   for $k = 0, \ldots, K$, given the discretisation interval Δt. The weighting matrices $Q \succeq 0$ and $R \succ 0$ are

$$Q = \begin{bmatrix} 25 & & \\ & 10 & \\ & & 25 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} 0.01 & \\ & 0.01 \end{bmatrix}. \tag{4}$$

   The control bounds are $u_k^{\min} = (0, 0)$ and $u_k^{\max} = (140, 140)$, whereas the states are bounded by $x_k^{\min} = (0, 0, 0)$ and $x_k^{\max} = (60, 60, 60)$, for all $k = 0, \ldots, K$. Assume an initial state $\hat{x}_0 = (10, 20, 30)$. Consider a horizon of $K = 1500$ time-steps with a control discretisation period $\Delta t = 0.5$s.

   Hint: Define reference x^{ref} using MATLAB's `sin` and `cos` functions over the time-nodes $(t_0, \ldots, t_{K-1})$.

3. Consider the linear approximation of dynamics $f(x, u|\theta_x)$ around a fixed-point $P = (x^{(P)}, u^{(P)})$,

$$f(x, u|\theta_x) \approx z + A(x - x^{(P)}) + B(u - u^{(P)}), \tag{5}$$

   with affine term $z = f(x^{(P)}, u^{(P)}|\theta_x)$, and Jacobian matrices $A = (\partial f/\partial x)|_P$ and $B = (\partial f/\partial u)|_P$ evaluated at P. Using this linearisation, the integrator $F(\cdot|\theta_x)$ in Eq. (2b) is also a linear function and thus Problem (2) becomes a *convex quadratic program* (QP): A common class of optimization problems that can be solved efficiently with proven convergence properties.

   Complete the `E03B_main.m` template to solve Problem (2) using the linearized model. First, implement the linear dynamics Eq. (5) by using CasADi's `jacobian` function and fixed-point $P = (x^{(P)}, u^{(P)})$ with $x^{(P)} = (40, 30, 20)$ and $u^{(P)} = (40, 34)$. Then, include your solution from *Task 3.1* (`E03A_main.m`) to implement and solve the modified optimisation problem.

4. Run the scripts `E03A_main.m` and `E03B_main.m`, and report the results generated by the plotting functions provided in the scripts. Compare both results (*objective function, number of iterations, execution time, etc.*) and interpret the optimal state- and control-trajectories (e.g., given weights Eq. (4)).