CHEM-E7225/2024: Exercise 03

Task 1 (Optimal control for Three-Tank System).
Consider again the benchmark Three-Tank System from Homework 1 (Figure 1): The process consists of three cylindrical tanks (T_i, $i = 1, 2, 3$) connected by two fixed valves (V_i, $i = 1, 2$), with an outflow valve V_0 for the last tank. The liquid levels (h_i, $i = 1, 2, 3$) in each tank are controlled by manipulating the incoming flow-rates to tanks T_1 and T_3 through the pumps P_1 and P_3, respectively.

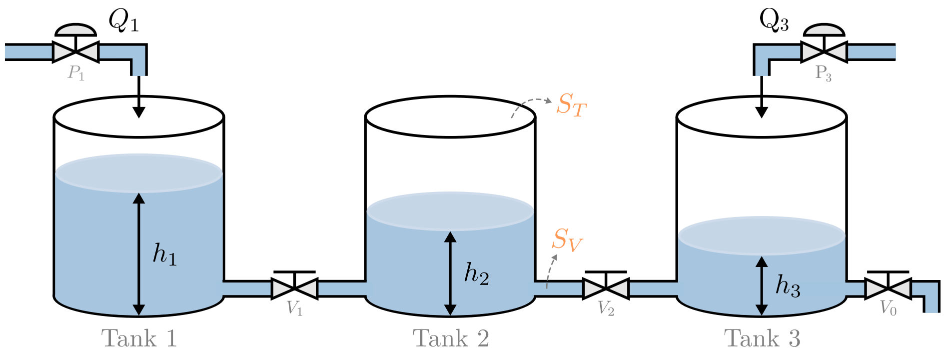

Figure 1: Three-Tank System: Process layout.

Using Torricelli's law to model the flow between tanks, and the `tanh` function to indicate its direction, the mass balances for the individual components lead to the following nonlinear dynamics

$$\frac{dh_1}{dt} = \frac{1}{S_T}\left[Q_1 - \alpha_V S_V \mathtt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|}\right]; \tag{1a}$$

$$\frac{dh_2}{dt} = \frac{1}{S_T}\left[\alpha_V S_V \mathtt{tanh}(h_1 - h_2)\sqrt{2g|h_1 - h_2|} - \alpha_V S_V \mathtt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|}\right]; \tag{1b}$$

$$\frac{dh_3}{dt} = \frac{1}{S_T}\left[Q_3 + \alpha_V S_V \mathtt{tanh}(h_2 - h_3)\sqrt{2g|h_2 - h_3|} - \alpha_0 S_V \sqrt{2gh_3}\right]; \tag{1c}$$

The state vector is $x(t) = (h_1(t), h_2(t), h_3(t))$ and the controls are $u(t) = \big(Q_1(t), Q_3(t)\big)$. The vector of model parameters in nominal operating conditions is $\theta = \big(S_T, S_V, \alpha_V, \alpha_0, g\big) = \big(154,\ 0.5,\ 0.47,\ 0.77,\ 981\big)$. The description of each process variable and the constant parameter values are presented in Table 1.

Table 1: Three-Tank System: Process variables and constant parameters.

Variable	Description	Value	Units
h_1	Water level of tank 1	$\in [0, 60]$	cm
h_2	Water level of tank 2	$\in [0, 60]$	cm
h_3	Water level of tank 3	$\in [0, 60]$	cm
Q_1	Flow-rate to tank 1	$\in [0, 140]$	ml/s
Q_3	Flow-rate to tank 3	$\in [0, 140]$	ml/s
S_T	Cross-section of tanks T_i $(i = 1, 2, 3)$	154	cm^2
S_V	Cross-section of valves V_i $(i = 0, 1, 2)$	0.5	cm^2
α_V	Flow coefficient of valves V_i $(i = 1, 2)$	0.47	–
α_0	Outflow coefficient of valve V_0	0.77	–
g	Gravitational constant	981	cm/s^2

Adapt the starting code in the archive `E7225-E03.zip` to solve this problem.

Task 3.1. Substitute the files `ThreeTank.m` and `rk4fnc.m` with your solutions from Homework 1. (5 points)

Task 3.2. Complete the `E03A_main.m` template to implement the following discrete-time optimal control problem using the simultaneous approach with `CasADi/IPOPT`,

$$\begin{aligned}
&\underset{\substack{x_0,\ldots,x_K \\ u_0,\ldots,u_{K-1}}}{\text{minimize}} && \sum_{k=0}^{K-1}\Big[(x_k - x_k^{ref})^T Q (x_k - x_k^{ref}) + u_k^T R u_k\Big] && && \text{(2a)}\\
&\text{subject to} && x_{k+1} - F(x_k, u_k|\theta_x) = 0, && k = 0,1,\ldots,K-1 && \text{(2b)}\\
& && x_k^{\min} \le x_k \le x_k^{\max}, && k = 0,1,\ldots,K && \text{(2c)}\\
& && u_k^{\min} \le u_k \le u_k^{\max}, && k = 0,1,\ldots,K && \text{(2d)}\\
& && x_0 - \hat{x}_0 = 0, && && \text{(2e)}
\end{aligned}$$

The state-references $x_k^{ref} = (x_{1k}^{ref}, x_{2k}^{ref}, x_{3k}^{ref})$ are obtained from the continuous-time signals

$$x_1^{ref}(t) = 40 + 10\cos(0.03t), \qquad x_2^{ref}(t) = 30, \qquad \text{and} \qquad x_3^{ref}(t) = 20 + 10\sin(0.03t). \tag{3}$$

The weighting matrices $Q \succeq 0$ and $R \succ 0$ are

$$Q = \begin{bmatrix} 25 & & \\ & 10 & \\ & & 25 \end{bmatrix}, \qquad \text{and} \qquad R = \begin{bmatrix} 0.01 & \\ & 0.01 \end{bmatrix}. \tag{4}$$

The function $F(\cdot|\theta_x)$ is an integrator for the dynamics $f(\cdot|\theta_x)$: We consider the RK4 method with a step-size of Δt (which is equal to the control rate). The control bounds are $u_k^{\min} = (0,0)$ and $u_k^{\max} = (140, 140)$, whereas the states bounds are $x_k^{\min} = (0,0,0)$ and $x_k^{\max} = (60,60,60)$, for all $k = 0,\ldots,K$. Assume the initial state $\hat{x}_0 = (10,20,30)$. Finally, consider a horizon of $K = 1200$ time-steps and a control rate of $\Delta t = 0.5$s. (60 points)

Hint: Define x^{ref} using MATLAB's `sin` and `cos` functions over the time-nodes $(t_0,\ldots,t_{K-1})$.

Task 3.3. Consider the linear approximation of dynamics $f(\cdot|\theta_x)$ around some fixed-point $P := (x_p, u_p)$,

$$\dot{x}(t) \approx \underbrace{f(x(t), u(t)|\theta_x)}_{\tilde{c}} + \underbrace{\frac{\partial f}{\partial x}\bigg|_P}_{\widetilde{A}}(x(t) - x_p) + \underbrace{\frac{\partial f}{\partial u}\bigg|_P}_{\widetilde{B}}(u(t) - u_p), \tag{5}$$

which can be *discretised* and converted into a time-invariant affine model $x_{k+1} = c + Ax_k + Bu_k$, as done in Homework 1. In this case, the transition function $F(\cdot|\theta_x)$ corresponds to the right-hand side of this model and Problem (2) becomes a (constrained) *linear-quadratic regulator* (LQR): A class of convex optimization problems with well-understood properties and efficient solution.

In this task, you will solve Problem (2) using this affine model: First, substitute the function file `Linearize.m` with your solution from Homework 1. Then, complete the template `E03B_main.m` exactly as in Task 3.2, but using $F(\cdot|\theta_x) = c + Ax_k + Bu_k$ for the dynamic constraints Eq. (2b). For the linearisation, use the fixed-point $x_p = (40, 30, 20)$ and $u_p = (0,0)$. (35 points)

Execute `E03A_main.m` and `E03B_main.m`, and report the results. Compare the optimisations (*objective value, execution time, etc.*) and interpret the optimal state- and control-trajectories in terms of the physical system (Figure 1), given the controller tuning (i.e., the weights Eq. (4)). Finally, change the linearisation point to $x_p = (0, 30, 60)$ and $u_p = (0,0)$ and re-run `E03B_main.m`: Does the performance of the LQR change? Why?