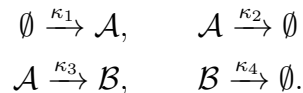


## Classroom problems

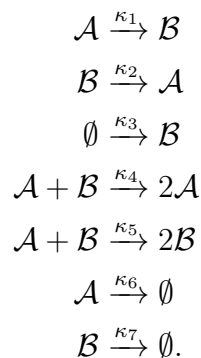
**Problem 1.** Consider a chemical system consisting of two chemical species  $\mathcal{A}$  and  $\mathcal{B}$  in the volume  $V$  subject to the following chemical reactions:



Denote by  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  the probability that there are  $n_{\mathcal{A}}$  molecules of  $\mathcal{A}$  and  $n_{\mathcal{B}}$  molecules of  $\mathcal{B}$  at time  $t$ . Suppose that there are initially 0 molecules of both  $\mathcal{A}$  and  $\mathcal{B}$ .

- prepare an algorithm to solve  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  numerically, given the initial condition and the values of reaction rates.
- solve the chemical master equation numerically for the first 20 seconds when  $\kappa_1 = 2, \kappa_2 = 0.2, \kappa_3 = 0.2, \kappa_4 = 0.1$  and  $V = 1$ . Plot the solution for different times. You may consider  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  for a subset of the state space, that is,  $n_{\mathcal{A}}, n_{\mathcal{B}} \in \{0, 1, \dots, \bar{n}\}$ .
- compare the result from (b) with the empirical distribution for the number of molecules of  $\mathcal{A}$  and  $\mathcal{B}$ . For this, use  $1 \times 10^4$  realisations of the system with the Gillespie algorithm. It is enough to use a single time  $t$  for the comparison, namely,  $t = 20$ .

**Problem 2.** Consider now a different chemical system consisting of two chemical species  $\mathcal{A}$  and  $\mathcal{B}$  in the volume  $V$  subject to the following chemical reactions:

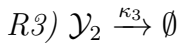
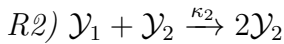
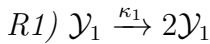


Denote by  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  the probability that there are  $n_{\mathcal{A}}$  molecules of  $\mathcal{A}$  and  $n_{\mathcal{B}}$  molecules of  $\mathcal{B}$  at time  $t$ . Suppose that there are initially 5 molecules of  $\mathcal{A}$  and no molecules of  $\mathcal{B}$ .

- prepare an algorithm to solve  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  numerically, given the initial condition and the values of reaction rates.
- solve the chemical master equation numerically for the first 20 seconds. Let  $\kappa$  be the collection of reaction rates. Consider  $\kappa = (0.001, 0.05, 1, 0.5, 0.5, 0.1, 0.1)$  and  $V = 1$ . Plot the solution for different times. You may again consider  $p(t, n_{\mathcal{A}}, n_{\mathcal{B}})$  for a subset of the state space, that is,  $n_{\mathcal{A}}, n_{\mathcal{B}} \in \{0, 1, \dots, \bar{n}\}$ .
- compare the result from (b) with the empirical distribution for the number of molecules of  $\mathcal{A}$  and  $\mathcal{B}$ . For this, use  $1 \times 10^4$  realisations of the system with the Gillespie algorithm. Again, it is enough to use a single time  $t$ , namely,  $t = 20$ .

# Deterministic versus stochastic modelling

**Problem 3.** (Lotka-Volterra) Consider the model for two interacting species:



in a container of volume  $V$ . Here,  $\mathcal{Y}_1$  represents a prey species and  $\mathcal{Y}_2$  represents a predator species.

(a) consider the following set of ordinary differential equations (ODEs) for the system:

$$d \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \kappa_1 y_1(t) - \frac{\kappa_2}{V} y_1(t) y_2(t) \\ \frac{\kappa_2}{V} y_1(t) y_2(t) - \kappa_3 y_2(t) \end{bmatrix} dt, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix},$$

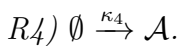
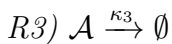
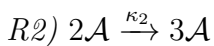
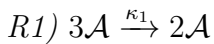
where  $y_1(t)$  and  $y_2(t)$  are (approximate) number of molecules of  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$ . Solve this set of ODEs numerically for the scenario  $\kappa_1 = 1, \kappa_2 = 0.005, \kappa_3 = 0.6, V = 1$  and  $t \in [0, 15]$ .

(b) draw one realizations from the Gillespie algorithm for the first 15 seconds of the reaction and plot it against the solution to the ODE model from (a).

(c) draw 200 realizations from the Gillespie algorithm for the first 15 seconds of the reaction and estimate the stochastic mean. Does the deterministic model predict the behaviour shown by your realization from the stochastic model? Note that the each realisation comes with a specific draw of reaction times exponentially distributed. You might need to store the number of molecules at fixed times  $(t_i), i = 0, 1, \dots$ .

## Homework

**Problem 4.** (System with multiple favourable states) Consider a chemical species  $\mathcal{A}$  being subjected to four chemical reactions in a container of volume  $V$ :



The following ODE approximates the evolution of the average number  $\overline{n_{\mathcal{A}}}$  of molecules of  $\mathcal{A}$ :

$$d\overline{n_{\mathcal{A}}}(t) = -\frac{\kappa_1}{6V^2} \overline{n_{\mathcal{A}}}^3(t) + \frac{\kappa_2}{2V} \overline{n_{\mathcal{A}}}^2(t) - \kappa_3 \overline{n_{\mathcal{A}}}(t) + \kappa_4 V, \quad \overline{n_{\mathcal{A}}}(0) = 0,$$

where the equation rates are in the time scale of minutes. Draw few realisations from the Gillespie algorithm for the first 40 minutes of the reaction and plot it against the solution to the ODE model above when

$$\kappa_1 = 1.5 \times 10^{-3}, \quad \kappa_2 = 0.36, \quad \kappa_3 = 37.5, \quad \kappa_4 = 2200.$$

Are there any significant differences between the stochastic and deterministic model?

Assume  $V = 0.5$  and consider the following propensity functions for the first two reactions:

$$\nu_1 = \frac{\kappa_1}{6V^2} \overline{n_{\mathcal{A}}}(\overline{n_{\mathcal{A}}} - 1)(\overline{n_{\mathcal{A}}} - 2), \quad \nu_2 = \frac{\kappa_2}{2V} \overline{n_{\mathcal{A}}}(\overline{n_{\mathcal{A}}} - 1).$$