FUNCTIONS

We give an introduction to FUNCTIONS and some of their PROPERTIES

MAPS

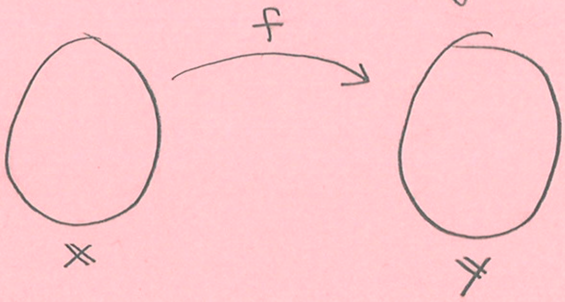
WE WILL LOOK AT THREE PROPERTIES
- INJECTIVITY
- SURJECTIVITY
- BIJECTIVITY

TO DEFINE A FUNCTION WE START WITH TWO SETS OF ELEMENTS

⇝ WE DEFINE THEN A FUNCTION WHICH MAPS ELEMENTS FROM ONE SET 'INTO' ELEMENTS IN THE OTHER SET

Consider the bubble diagrams — THE TWO SETS ARE $\mathbb{X}$ AND $\mathbb{Y}$

THE FUNCTION f
- f maps the elements of $\mathbb{X}$ 'into' the elements of $\mathbb{Y}$



$$\boxed{f : \mathbb{X} \to \mathbb{Y}}$$

THE RANGE OF f, $f(\mathbb{X})$

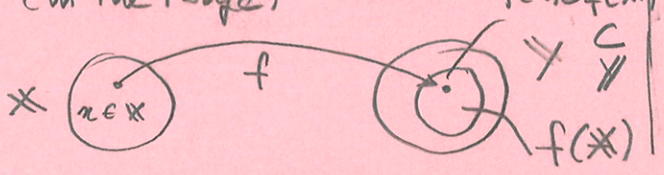
$$f(\mathbb{X}) = \{ f(x) \mid x \in \mathbb{X} \}$$

IT IS A SET OF ELEMENTS

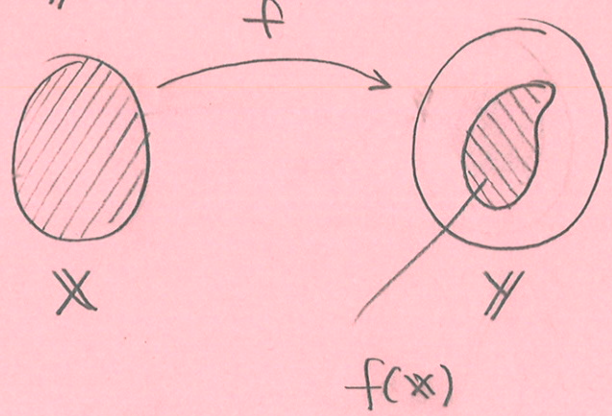
- FACTS :

* In general, the range is not equal to the codomain

* About $f : X \to Y$ means that for all elements $x \in \mathbb{X}$, f assigns a unique value $f(x)$ to that (in the range)



WHAT DOES 'INTO' MEAN? IT MEANS THAT IF WE TAKE ALL THE ELEMENTS OF $\mathbb{X}$ AND MAP THEM THROUGH f, WE ARE GOING TO GET A SUBSET OF $\mathbb{Y}$

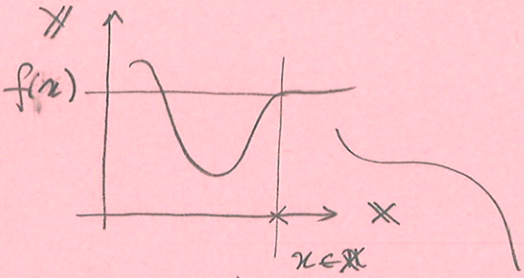


$\mathbb{X}$: the domain of f

$\mathbb{Y}$: the codomain of f

$f(\mathbb{X})$: the range of f

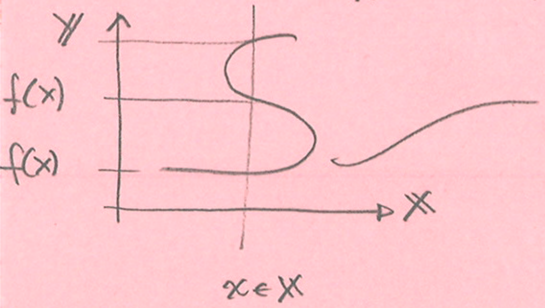
The given definition allows us the following



→ THE DOMAIN IS THE REAL LINE, $\mathbb{X} = \mathbb{R}$

→ THE CODOMAIN IS THE REAL LINE $\mathbb{Y} = \mathbb{R}$

This is a function



This is NOT a function

We now discuss three properties of a function — INJECTIVITY, SURJECTIVITY, BIJECTIVITY

WHAT ARE THESE PROPERTIES ?

Def (INJECTIVE) A function is said to be injective, or '1-to-1', if and only if ($\Leftrightarrow$) the following property holds

$$\underbrace{\left[f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2 \right]}$$

ONE x TO ONE $f(x)$ (Definition of a function)

ONE $f(x)$ FROM ONE x (Definition of injectivity)

Alternatively $\left[x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2) \right]$

Def (SURJECTIVE) A function is said to be surjective, or 'onto', if and only if for all the elements in the codomain, there is an element in the domain that the element in the codomain come from

→ THE RANGE COVERS THE WHOLE CODOMAIN

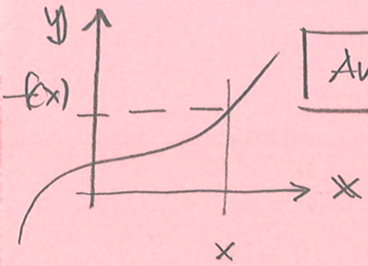
$$\left[\forall y \in \mathbb{Y},\ \exists x \in \mathbb{X} \text{ such that } y = f(x) \right]$$

Def (BIJECTIVE) A function is said to be bijective when it is both injective and surjective

$$\left[\forall y \in \mathbb{Y},\ \exists!\, x \in \mathbb{X} \text{ such that } y = f(x)\right]$$

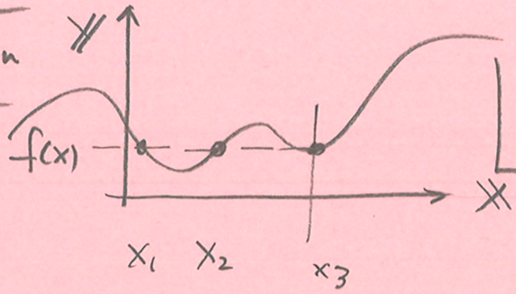
'there exists a unique'
(due to injectivity)

Consider an injective function (one 'f(x)' for each 'x')



An injective function

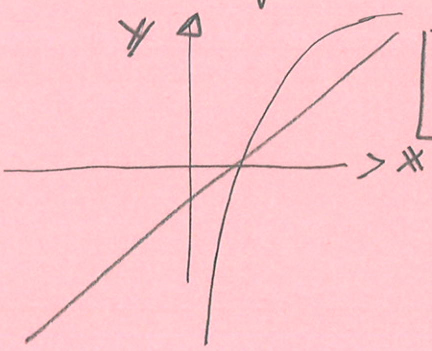
f(x) ⇝ a single x



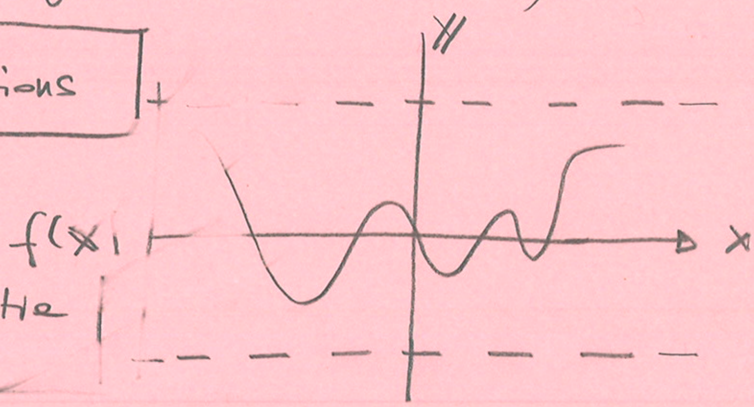
A NOT injective function

f(x) ⇝ three different 'x'
(x_1, x_2, and x_3

Consider a surjective function (the range covers the codomain);



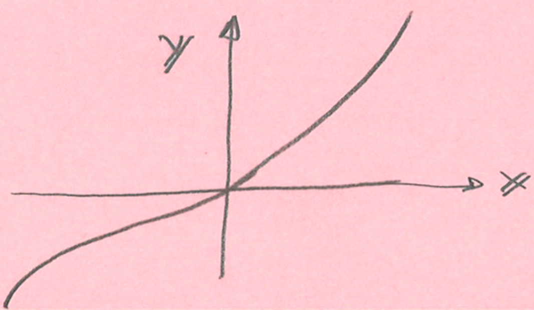
Surjective functions



A NOT surjective function

A consider a bijective function (BOTH INJECTIVE AND SURJECTIVE)

⇝ KIND OF MONOTONIC THAT COVERS THE ENTIRE CODOMAIN



We consider an example of function

We want to consider the 'inverse map of some function' that takes the elements from some set and maps them onto or into the elements of some other set (say $f: \mathbb{X} \to \mathbb{Y}$)

Intuitively, the inverse of that map goes in the opposite direction (the inverse takes elements from $\mathbb{Y}$ and maps them back to $\mathbb{X}$)

<u>Def</u> (Left Inverse) let $f: \mathbb{X} \to \mathbb{Y}$

Define the identity map on the set $\mathbb{X}$, $\mathbb{1}_x$

$\mathbb{1}_x : \mathbb{X} \to \mathbb{X}$ (maps $\mathbb{X}$ back to itself)

Now we define the left inverse, g_L, of function f as the function $\boxed{g_L : \mathbb{Y} \to \mathbb{X}}$ (the codomain back to the domain) with the property that if $\underbrace{g_L \circ f}_{g_L \text{ composed with } f} = \mathbb{1}_x$

(f first operates on an element $x \in \mathbb{X}$ to give us an element $f(x) \in \mathbb{Y}$ in the codomain, then g_L operates on that element to give us an element back in the domain, that element is the same element $x \in \mathbb{X}$)

IF f HAS A LEFT INVERSE THEN f IS INJECTIVE, AND VICEVERSA

How to prove that (in both direction)

[f has a left inverse g_L] $\Leftrightarrow$ [f is injective]

WE FIRST PROVE $\Leftarrow$

THEN WE PROVE $\Rightarrow$

PROOF of ($\Leftarrow$)

We are assuming that f is injective [$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$
or
$x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$]

We can then construct g_L as a function such that

$g_L : \mathbb{Y} \to \mathbb{X}$ such that when g_L operates on the range of $\mathbb{X}$ $(f(x))$
then $g(f(\mathbb{X})) = \mathbb{X}$

THIS IS ONLY WELL DEFINED WHEN g_L IS A FUNCTION
(THE WAY THE FUNCTION IS DEFINED IS UP TO US
AND THIS IS WHY WE DEFINED CONVENIENTLY)

So is g_L actually well defined? This requires that f is injective

PROOF OF ($\Rightarrow$)

We are now assuming that f has a left inverse g_L
By definition of left inverse, we have

$$g_L \circ f = \mathbb{1}_X$$

That is $\exists x \in \mathbb{X}$, $g_L(f(x)) = x$

We now want to prove that f is injective, we assume that it is not injective and we derive a contradiction

Suppose that f is not injective, then $\exists x_1, x_2$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$

What would happen to g_L if $f(x)$ were not injective?

$$\left.\begin{array}{l} g(f(x_1)) = x_1 \\ g(f(x_2)) = x_2 \end{array}\right\}$$ but because g_L is assumed to be a function
$\rightsquigarrow x_1 = x_2$ that contradicts the assumption

$\divideontimes$

WE CAN ALSO DEFINE/DERIVE A RIGHT INVERSE

Def (RIGHT INVERSE) $g_R : Y \to X$ such that $f \circ g_R = \mathbb{1}_Y$

[f has a right inverse] $\Leftrightarrow$ [f is surjective]

$\rightsquigarrow$ Summarizing

INJECTIVITY $\to$ LEFT INVERSE

SURJECTIVITY $\to$ RIGHT INVERSE

BIJECTIVITY $\to$ INVERSE (RIGHT + LEFT INVERSE)

$\underbrace{g = f^{-1}}_{\text{INVERSE}}$, g IS BOTH A RIGHT AND A LEFT INVERSE