FIELDS and VECTOR SPACES

WE WANT TO ATTACH THE VECTOR SPACE STRUCTURE TO THE SETS THAT WE HAVE DEFINED

We start with the concept of a FIELD, we will use it in the definiton of a VECTOR SPACE

Def (FIELD) $\mathbb{F}$ (blackboard $\mathbb{F}$) is defined as a set of objects and two binary operations (typically addition and multiplication, classical ones for most of our cases)

$(\mathbb{F}, +, \cdot)$

An operation which we call multiplication

An operation which we call addition

THE FOLLOWING RULES MUST BE SATISFIED

RULES for ADDITION

(+) THE FIELD IS ASSOCIATIVE, IF WE CAN TAKE ANY THREE ELEMENTS OF $\mathbb{F}$ α AND β

$$\rightsquigarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$

THE FIELD IS COMMUTATIVE, IF WE CAN TAKE TWO ELEMENTS OF $\mathbb{F}$, α AND β

$$\rightsquigarrow \alpha+\beta = \beta+\alpha$$

THE FIELD HAS AN IDENTITY (UNDER ADDITION), IF THERE EXIST AN ELEMENT ZERO SUCH THAT IF WE ADD SUCH ELEMENT TO ANY ELEMENT OF $\mathbb{F}$, α

$$\rightsquigarrow \alpha+0 = \alpha \qquad [\exists \text{ identity } 0 \text{ st } \alpha+0=\alpha]$$

THE FIELD HAS AN INVERSE (UNDER ADDITION), IF THERE EXIST AN ELEMENT, $-\alpha$, SUCH THAT IF WE ADD SUCH ELEMENT TO AN ELEMENT OF $\mathbb{F}$, α, FOR ALL $\alpha \in \mathbb{F}$

$$\rightsquigarrow \alpha+(-\alpha)=0 \qquad [\forall \alpha, \exists \text{ inverse } (-\alpha) \text{ st } \alpha+(-\alpha)=0]$$

(·)

RULES FOR MULTIPLICATION

ASSOCIATIVE

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

COMMUTATIVE

$$\alpha \cdot \beta = \beta \cdot \alpha$$

IDENTITY ELEMENT (UNDER MULTIPLICATION)

$\exists$ identity 1 st $\alpha \cdot 1 = \alpha$

INVERSE ELEMENT (UNDER MULTIPLICATION)

$\exists$ inverse, $\forall \alpha \neq 0 \quad \exists\, \alpha^{-1}$ st $\alpha \cdot \alpha^{-1} = 1$

(+ and ·)

COMBINING THE RULES, WE SAY THAT **MULTIPLICATION DISTRIBUTES OVER ADDITION**

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

THIS IS THE GENERAL DEFINITION OF A FIELD

→ Examples of commonly used field:

$\mathbb{R}$

$\mathbb{C}$

* THE SET OF REAL NUMBERS, UNDER THE STANDARD DEFINITION OF (+) (·)
* THE SET OF COMPLEX NUMBERS, UNDER THE DEFINITION OF (+) AND (·) FOR COMPLEX NUMBERS
* THE FIELD OF RATIONAL FUNCTIONS, WITH COEFFICIENTS THAT ARE REAL NUMBERS (s)
  - → $\mathbb{R}(s)$
  - → $(s^2 + 2s + 1)/(s + 3)$ Example of element in $\mathbb{R}(s)$

Examples of NOT fields

* THE SET OF POLINOMIALS WHOSE COEFFICIENTS ARE REAL NUMBERS

⤳ $\mathbb{R}[s]$

⤳ This is not a field because there is no multiplicative inverse that if multiplied by an element in the set would return the identity element

(Ask yourself what's the inverse of a polynomial ?
1/ polinomial ⟶ IT'S NOT AN ELEMENT OF $\mathbb{R}[s]$
BECAUSE IT'S NOT A POLINOMIAL

FROM THE DEFINITION OF A FIELD WE CAN CONSTRUCT THE DEFINITION OF A VECTOR SPACE (OR LINEAR SPACE)

⤳ $(V, \mathbb{F}, +, \cdot)$

an associated field

a set of elements (vectors)

Two operations, now on the elements of V, the vectors

(+) VECTOR + VECTOR ⤳ VECTOR

(·) VECTOR * $\underbrace{\text{ELEMENT OF THE FIELD}}_{\text{(A SCALAR, IN SIMPLE WORDS)}}$ ⤳ VECTOR

Again, a set of rules on the operations must be satisfied

ADDITION (+) IS

ASSOCIATIVE	$(x+y)+z = x+(y+z)$, $x,y,z \in V$
COMMUTATIVE	$x+y = y+x$
IDENTITY	$\exists$ identity θ "0-vector" $x+\theta = x$
INVERSE	$\exists$ inverse, $\forall x \in V \; \exists (-x)$ st $x+(-x) = \theta$

SCALAR MULTIPLICATION $(\cdot)$

- $(\alpha \cdot \beta) \cdot x = \alpha(\beta x)$ $\qquad \alpha, \beta \in \mathbb{F}, \ x \in V$

OPERATION FROM THE FIELD

OPERATION FROM THE VECTOR SPACE

- $1 \cdot x = x$ $\qquad 1 \in \mathbb{F}, \ x \in V$
- $0 \cdot x = \theta$ $\qquad 0 \in \mathbb{F}, \ x, \theta \in V$

DISTRIBUTIVE LAWS

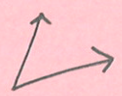
$$(\alpha + \beta) \cdot x = \alpha x + \beta x$$

THE VECTOR SPACE IS CLOSED UNDER VECTOR ADDITION AND SCALAR MULTIPLICATION

IF THE OPERATIONS ARE PERFORMED ON THE ELEMENTS OF THE VECTOR SPACE, THE RESULT WILL BE AN ELEMENT OF THAT SPACE

We can consider some familiar? examples of vector spaces

* $V = \mathbb{R}^N$
  $\mathbb{F} = \mathbb{R}$
  with standard definition of (+) and $(\cdot)$ for actual vectors



* $V = \mathbb{C}^N$
  $\mathbb{F} = \mathbb{C}$
  with standard definition of (+) and $(\cdot)$ for complex vectors

* $V = \mathbb{C}^n$
  $\mathbb{F} = \mathbb{R}$
  → THIS IS STILL A VECTOR SPACE

* $V = \mathbb{R}^N$
  $\mathbb{F} = \mathbb{C}$
  → THIS IS NOT A VECTOR SPACE
  (Not closed under scalar multip)

One more example, not so standard

⇝ WHEN VECTORS ARE FUNCTIONS

⇝ A FUNCTION SPACE

The function space $\mathcal{F}(D, V)$

a set of functions that maps a set D into a set V, with V a vectorspace with scalar field $\mathbb{F}$

$(V, \mathbb{F})$

$$\left(\mathcal{F}(D, V), \mathbb{F}\right) \text{ is a vector space}$$

- the set of functions that map a set D onto a set V, where V is also a vector space with field $\mathbb{F}$, is a vector space with the same field