SUBSPACES and BASIS

We defined a vector space, based on that concept we are going to introduce the concept of <u>subspace</u>, and an important characterization of a vector space in terms of their <u>basis elements</u>

STARTING FROM A VECTOR SPACE $(V, \mathbb{F})$ OVER THE FIELD $\mathbb{F}$,

⇝ WE NOW CONSIDER A SUBSET OF V
(V is a subset, on top of which we put some structure in terms of two binary operations)

⇝ $\underbrace{W \subseteq V}_{\text{could be equal to } V}$

WE THEN ASK THE QUESTION, IS W ITSELF A VECTOR SPACE (meaning if we take the same operations defined for V along the same field $\mathbb{F}$ and we use them on $W \subseteq V$, do we stay within W, or is W closed with respect to those operations?)

⇝ IF THIS IS TRUE, THEN W IS CALLED A SUBSPACE OF V

<u>Def</u> (SUBSPACE) W IS A SUBSPACE OF $(V, \mathbb{F})$ IF

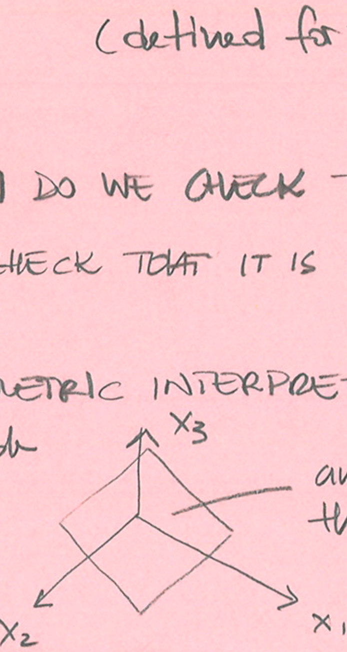
- $W \subseteq V$ (W MUST BE A SUBSET OF V)
- W IS CLOSED UNDER VECTOR ADDITION AND SCALAR MULTIPLICAT. (defined for the parent space V, over the same field $\mathbb{F}$)

HOW DO WE CHECK THAT?

- CHECK THAT IT IS CLOSED UNDER VECTOR ADDITION

GEOMETRIC INTERPRETATION: $\mathbb{R}^2$

Example



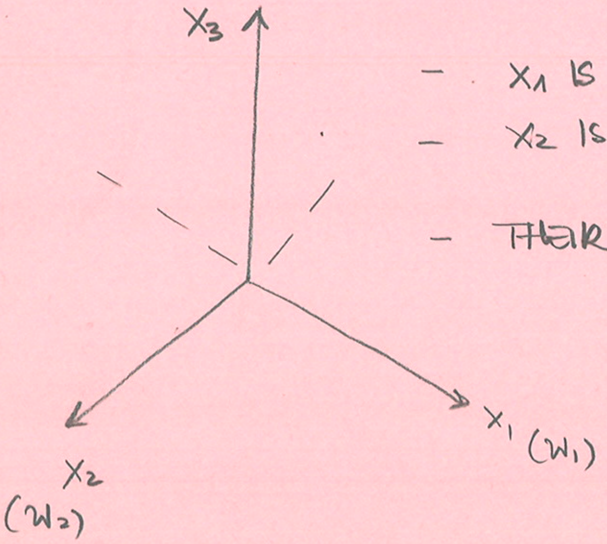
any plane that goes thru the origin is a subspace $\mathbb{R}^3$

NEEDS TO GO THRU THE ORIGINS TO ALLOW FOR THE 0 TO BE INCLUDED TO BE A VECTOR SPACE

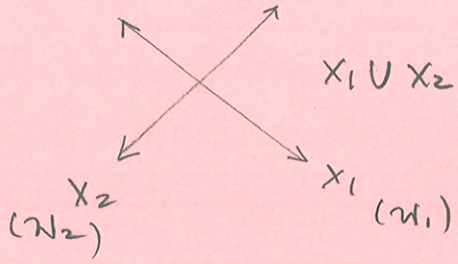
WE CAN SHOW THAT :

1) W_1 AND W_2 ARE SUBSPACES OF $(V, \mathbb{F})$ ⇝ $W_1 \cap W_2$ IS A SUBSPACE

2) " " " " ⇝ $W_1 \cup W_2$ IS NOT NECESSARILY A SUBSPACE

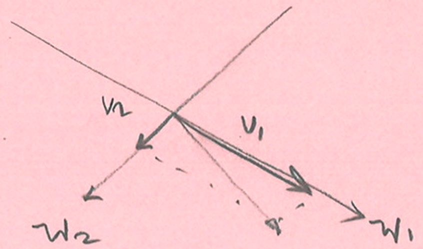


- X_1 IS A SUBSPACE
- X_2 IS A SUBSPACE
- THEIR UNION $X_1 \cup X_2$ IS NOT A SUBSPACES



NOT CLOSED TO VECTOR ADDITION

IF WE TAKE AN ELEMENT FROM W_1 AND WE ADD IT TO AN ELEMENT FROM W_2 ⇝



$V_1 + V_2 \neq W_1 \cup W_2$

LINEAR INDEPENDENCE

CONSIDER A VECTOR SPACE $(V, \mathbb{F})$, WE HAVE A SET OF p VECTORS, ELEMENTS OF V $\{v_1, v_2, \text{—}, v_p\}$, $v_i \in V$

THE SET $\{v_i\}_{i=1}^{p}$ IS SAID TO BE <u>LINEARLY</u> <u>INDEPENDENT</u> IFF

FOR $\alpha_1 v_1 + \alpha_2 v_2 + \text{———} + \alpha_p v_p = \sum_{i=1}^{p} \alpha_i v_i = \theta$

IMPLIES THAT $\alpha_1 = \alpha_2 = - - = \alpha_p = 0$

$$\alpha_1 v_1 + \alpha_2 v_2 + \text{———} + \alpha_p v_2 = \theta$$

α_i are elements of $\mathbb{F}$
v_i are elements of V
θ is an element of V

WE CANNOT WRITE
ANY OF THE VECTORS v_i
AS A LINEAR COMBINATION OF THE OTHER ONES

$$\alpha_2 v_2 + \text{———} + \alpha_p v \neq \alpha_1 v_1$$

UNLESS ALL α_i's ARE ZEROS

THE SET $\{v_i\}_{i=1}^{p}$ IS SAID TO BE LINEARLY DEPENDENT IFF

→ There exist field elements, scalars, $\alpha_1, \dots, \alpha_p$ not all zero such that $\alpha_1 v_1 + \alpha_2 v_2 — \alpha_p v_p = \theta$

→ This is like saying that one or more of those elements are redundant

THE NOTION OF LINEAR INDEPENDENCE IS CENTRAL TO THE DEFINITION OF AN IMPORTANT SET OF VECTORS IN A VECTOR SPACE, THE BASES SET

let $(V, \mathbb{F})$ be a vector space, let be $\{b_1, b_2, \ldots, b_n\}$ be a set of vectors such that $b_i \in V$

- $\{b_1, b_2, \ldots, b_n\}$ is called a basis if:

1.) $B = \{b_1, b_2, \ldots, \underbrace{b_n}\}$ $\underbrace{\text{SPANS}}$ V
2.) B is a linearly independent set

n is the dimension of the vector space

Any vector $v \in V$ can be written as a linear combination of the basis' elements

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n$$
$$= \sum_{i=1}^{n} \alpha_i b_i$$

1. MAKES SURE THAT B IS A BIG ENOUGH SET TO REPRESENT ANY VECTOR IN V

2. MAKES SURE THAT B $\underbrace{\text{IS NOT TOO}}_{\text{(SMALL ENOUGH)}}$ BIG (that there are no redundancies)

With respect to a bases B, we can define a COORDINATE REPRESENTATION

* Once we have chosen a basis B for a vector space $(V, \mathbb{F})$, then we define the coordinates of any vector $v \in V$ to be the scalars (elements of $\mathbb{F}$) such that

$$v = \underbrace{\alpha_1} b_1 + \underbrace{\alpha_2} b_2 + \ldots + \underbrace{\alpha_n} b_n$$

$\leadsto (\alpha_1, \alpha_2, \ldots, \alpha_n)$

For a given vector space $(V, \mathbb{F})$ we can have many possible basis
(an infinite number)

→ Given a basis $\mathcal{B}$, we can take any linear combination, and that would be another basis

→ Once the basis is defined, and the number of basis is finite, then the number of elements in all possible bases is constant
* n, the dimension of the vector space

→ Once the basis is chosen, then for any vector $v \in V$, the set of coordinates is unique

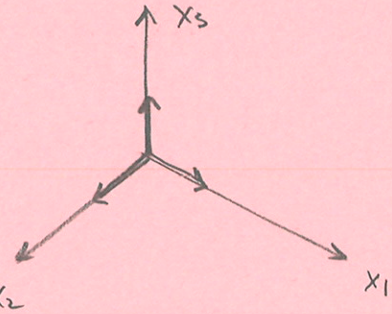
* THE COORDINATES WRT TO THE BASIS, UNIQUELY DEFINES THE ELEMENTS OF THE VECTOR SPACE
(can be proven by contradiction)

Some examples of basis

* Consider the space $(\mathbb{R}^3, \mathbb{R})$, what's a basis for this vector space?

  – THE STANDARD BASIS ARE THE AXIS

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{the unit vectors}}$$



* Consider the space $(\mathbb{R}^{3\times3}, \mathbb{R})$, the set of three-by-three matrices over the real line

$$\underbrace{\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}}_{\text{the standard basis } (n=9)}$$

FUNCTION SPACES AND BASIS FOR FUNCTION SPACES

→ Important to think about a vector space structure

→ The basis is a defining concept for the vector (function) space

→ IT ALLOWS US TO GENERATE ALL VECTORS (FUNCTIONS) IN THAT SPACE