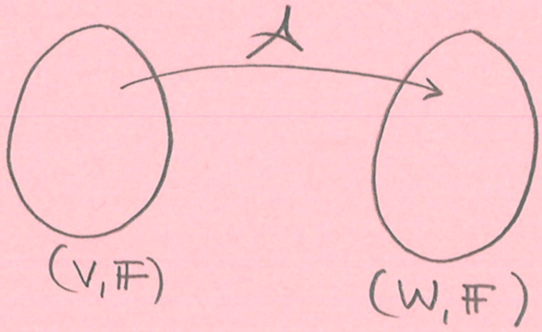
LINEAR MAPS

We discuss LINEAR MAPS and their properties

WE CONSIDER MAPS / FUNCTIONS BETWEEN TWO VECTOR SPACES



- TWO VECTOR SPACES OVER THE SAME FIELD

- A MAP $\mathcal{A} : V \longrightarrow W$

V — domain, W — codomain

Def (LINEAR MAP)

$\mathcal{A}$ is said to be a liner map or fuction iff the following property holds

$$\underbrace{\mathcal{A}(\alpha_1 v_1 + \alpha_2 v_2)}_{} = \underbrace{\alpha_1 \mathcal{A}(v_1) + \alpha_2 \mathcal{A}(v_2)}_{}, \quad \begin{cases} \forall \alpha_1, \alpha_2 \text{ in } \mathbb{F} \\ \forall v_1, v_2 \text{ in } V \end{cases}$$

$\mathcal{A}$ operates on a linear combination of elements in the domain — still an element of the domain

We obtain a linear combination of elements in the codomain — still an element in the codomain

SUPERPOSITION

Examples

* Suppose you are given a map $\mathcal{A}$ defined as follows:
  - IT TAKES ELEMENTS WHICH ARE POLYNOMIALS $as^2 + bs + c$ AND IT RETURNS POLYNOMIALS WITH a AND c COEFFICIENTS THAT ARE SWITCHED, $cs^2 + bs + a$

$$\mathcal{A} : as^2 + bs + c \longrightarrow cs^2 + bs + a$$

IS THIS MAP LINEAR?

$A: as^2 + bs + c \longrightarrow cs^2 + bs + a$ IS LINEAR IF THE SUPERPOSITION PROPERTY HOLDS

WE CAN CHECK THIS: LET $V_1 = a_1 s^2 + b_1 s + c_1$
$V_2 = a_2 s^2 + b_2 s + c_2$

WE NEED TO SHOW THAT $A(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 A(V_1) + \alpha_2 A(V_2)$

- $\alpha_1 V_1 = \alpha_1 a_1 s^2 + \alpha_1 b_1 s + \alpha_1 c_1$
  $\alpha_2 V_2 = \alpha_2 a_2 s^2 + \alpha_2 b_2 s + \alpha_2 c_2$

$\rightsquigarrow \alpha_1 V_1 + \alpha_2 V_2 = (\alpha_1 a_1 s^2 + \alpha_1 b_1 s + \alpha_1 c_1) + (\alpha_2 a_2 s^2 + \alpha_2 b_2 s + \alpha_2 c_2)$

$= (\alpha_1 a_1 + \alpha_2 a_2) s^2 + (\alpha_1 b_1 + \alpha_2 b_2) s + (\alpha_1 c_1 + \alpha_2 c_2)$

$\alpha_1 A(v_1) = \alpha_1 (c_1 s^2 + b_1 s + a_1) = \alpha_1 c_1 s^2 + \alpha_1 b_1 s + \alpha_1 a_1$
$\alpha_2 A(v_2) = \alpha_2 (c_2 s^2 + b_2 s + a_2) = \alpha_2 c_2 s^2 + \alpha_2 b_2 s + \alpha_2 a_2$

$\rightsquigarrow \alpha_1 A(v_1) + \alpha_2 A(v_2) = (\alpha_1 c_1 s^2 + \alpha_1 b_1 s + \alpha_1 a_1) + (\alpha_2 c_2 s^2 + \alpha_2 b_2 s + \alpha_2 a_2)$

$= (\alpha_1 c_1 + \alpha_2 c_2) s^2 + (\alpha_1 b_1 + \alpha_2 b_2) s + (\alpha_1 a_1 + \alpha_2 a_2)$

WHICH EQUALS $A(\alpha_1 V_1 + \alpha_2 V_2) = \underbrace{(\alpha_1 c_1 + \alpha_2 c_2) s^2 + (\alpha_1 b_1 + \alpha_2 b_2) s + (\alpha_1 a_1 + \alpha_2 a_2)}_{\text{SWITCH OF } a\text{'s WITH } c\text{'s}}$

Example

LET $A: as^2 + bs + c \longrightarrow \int_0^s (bt + a)\, dt$ IS THIS MAP LINEAR?

WE NEED TO CHECK IF THE SUPERPOSITION PROPERTY HOLDS

YES, BUT SHOW IT!

Example $\quad A : v(t) \longmapsto \int_0^1 v(t)\,dt + \text{const}$ IS THIS A LINEAR MAP?

IT TAKES FUNCTIONS OF TIME

IT RETURNS THEIR INTEGRAL PLUS A CONSTANT

NOT LINEAR, FOR A GENERAL CONSTANT, const
(IT IS AN AFFINE MAP)

WE CAN CHECK IF THE SUPERPOSITION PROPERTY HOLDS

Example $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with A being the premultiplication of a vector, in $\mathbb{R}^3$, by a 3×3 matrix

$$\underbrace{A(v)}_{\text{OPERATOR NOTATION}} = \overbrace{\underbrace{A}_{A \in \mathbb{R}^{3\times 3}}x}^{\text{MATRIX MULTIPLICATION NOTATION}}$$

IS THIS A LINEAR MAP?

WE CAN USE THE SAME PROCEDURE AND CHECK IF SUPERPOSITION HOLDS

WE NOW INTRODUCE TWO CONCEPTS FOR LINEAR MAPS $\mathcal{A}: \mathcal{U} \longrightarrow \mathcal{V}$

* THE RANGE SPACE $\mathcal{R}(\mathcal{A})$
* THE NULL SPACE $\mathcal{N}(\mathcal{A})$

<u>Def</u> ($\overbrace{\text{RANGE}}^{\text{Image}}$ SPACE of $\mathcal{A}$) THE VECTORS IN THE CODOMAIN $\overset{v}{\mathcal{V}}$ SUCH THAT $v = \mathcal{A}(u)$ WITH u AN ELEMENT IN THE DOMAIN $\mathcal{U}$, FOR ALL $u \in \mathcal{U}$

$$\mathcal{R}(\mathcal{A}) = \{ v \in \mathcal{V} \mid v = \mathcal{A}(u),\ u \in \mathcal{U} \}$$

<u>Def</u> ($\overbrace{\text{NULL SPACE}}^{\text{Kernel}}$ of $\mathcal{A}$) THE VECTORS u IN THE DOMAIN $\mathcal{U}$ SUCH THAT THEIR MAP IN THE CODOMAIN IS THE ZERO VECTOR $\theta_\mathcal{V}$

$$\mathcal{N}(\mathcal{A}) = \{ u \in \mathcal{U} \mid \mathcal{A}(u) = \theta_\mathcal{V},\ \theta_\mathcal{V} \in \mathcal{V} \}$$

WE HAVE TWO IMPORTANT THEOREMS

<u>Th(s)</u> THE RANGE SPACE $\mathcal{R}(\mathcal{A})$ OF $\mathcal{A}$ IS A SUBSPACE OF THE CODOMAIN $(\mathcal{V}, \mathbb{F})$, $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{V}$ with $(\mathcal{R}(\mathcal{A}), \mathbb{F})$

THE NULL SPACE $\mathcal{N}(\mathcal{A})$ OF $\mathcal{A}$ IS A SUBSPACE OF THE DOMAIN $(\mathcal{U}, \mathbb{F})$, $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{U}$ with $(\mathcal{N}(\mathcal{A}), \mathbb{F})$

Being subspaces, $\mathcal{R}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$ are closed with respect to vector addition and scalar multiplication

WE HAVE ANOTHER IMPORTANT THEOREM

Th LET $A: U \to V$ BE A LINEAR MAP BETWEEN TWO VECTOR SPACES AND CONSIDER AN ELEMENT b OF V, $b \in V$

THEN, THE FOLLOWING MUST HOLD

[$A(u) = b$ WITH $u \in U$ ← THIS IS A LINEAR EQUATION

HAS AT LEAST ONE SOLUTION] (A VECTOR IN THE DOMAIN THAT MAKES THE LINEAR EQUATION HOLD)

This is equivalent, by the theorem, to say that b must be in the range of A

[$b \in R(A)$]

IF $b \in R(A)$ THEN:

- $A(u) = b$ HAS A UNIQUE SOLUTION IFF $N(A) = \{\theta_U\}$
- LET $x_0 \in U$ SUCH THAT $A(x_0) = b$, THEN IF $A(u) = b$ IMPLIES THAT $u - x_0 \in N(A)$