MATRIX REPRESENTATIONS OF LINEAR MAPS

WE DISCUSS HOW TO CONSTRUCT A LINEAR MAP BETWEEN FINITE-DIMENSIONAL VECTOR SPACES

WE START WITH A STATEMENT : Any linear map between finite dimensional vector spaces can be represented as matrix multiplication

⇝ We can turn quite abstract definitions of maps into matrix multiplications, as long as they are linear and between finite dimensional vector spaces

IMPORTANT FACT : THE MAP MUST BE LINEAR
THE VECTOR SPACES MUST BE FINITE-DIMENSIONAL

We derive the matrix multiplication, given some definition of the map

Let $\mathcal{A}: \mathcal{U} \to V$ such that $\mathcal{A}(u \in \mathcal{U}) = v \in V$

⇝ WE WANT TO SHOW THAT $\underbrace{Au = v}_{\text{MATRIX MULTIPLICATION}}$ (we will get back to this because not fully accurate)

A IS A MATRIX

WE REMEMBER SOME DEFINITIONS

- WE WANT TO CONSTRUCT BASES FOR BOTH THE DOMAIN AND THE CODOMAIN

- THIS IS IMPORTANT, BECAUSE THE MATRIX MULTIPLICATION DEPENDS ON THE CHOICE OF BASES

That is, if we have a basis for $\mathcal{U}$, if we have a basis for V, and if we have a description of $\mathcal{A}$, then we can construct a matrix A and perform the mapping using matrix multiplication

* IF WE CHANGE THE BASIS, MATRIX A CHANGES

Let the set of vectors $\{u_j\}_{j=1}^{n}$ be a basis for U (n-dimensional)

Let the set of vectors $\{v_j\}_{j=1}^{m}$ be a basis for V (m-dimensional)

Any vector $u \in U$ can be represented as a linear combination of the vectors in the basis of U, and similarly any vector $v \in V$ can be represented as a linear combination of the vectors in the basis of V

→ ONCE WE HAVE CHOSEN THE BASES, THE COORDINATES OF THE VECTORS u AND v ARE UNIQUE, WRT TO THE BASIS

Any $x \in U$ has a unique representation wrt $\{u_j\}_{j=1}^{n}$

$$* \exists! \; \xi = \underbrace{\{\xi_i\}_{i=1}^{n}}_{\text{SET OF COORDINATE}} \text{ such that } x = \sum_{j=1}^{n} \xi_j u_j$$

We can now operate on vector x, using A, ⇝ $A(x)$

$$* \underbrace{A(x) = A\left(\sum_{j=1}^{n} \xi_j u_j\right) = \sum_{j=1}^{n} \xi_j A(u_j)}_{A \text{ is linear, superposition holds}}$$

Each $A(u_j)$ returns a vector in V, $A(u_j) \in V$, and thus it can be written, uniquely, in terms of the basis vectors of V

$$A(u_j) = \sum_{i=1}^{m} a_{ij} v_i$$

THE COORDINATE i OF THE BASIS VECTOR u_j AFTER MAPPING IT TO V

For example, $A(u_j) = \sum_{i=1}^{m} a_{ij} v_i$

$$A(u_1) = \sum_{i=1}^{m} a_{i1} v_i = a_{11} v_1 + a_{21} v_2 + \ldots + a_{m1} v_m$$

$$A(u_2) = \sum_{i=1}^{m} a_{i2} v_i = a_{12} v_1 + a_{22} v_2 + \ldots + a_{m2} v_m$$

$$\vdots$$

NOW THAT WE HAVE THIS RELATIONSHIP, WE CAN USE IT TO CONSTRUCT THE MATRIX REPRESENTATION OF THIS LINEAR MAP

WE CAN WRITE
$$A(x) = \sum_{j=1}^{n} \xi_j A(u_j)$$
$$= \sum_{j=1}^{n} \xi_j \sum_{i=1}^{m} a_{ij} v_i$$

FINITE SUMS, WE CAN REARRANGE THEM

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \xi_j \right) v_i$$
$$= \sum_{i=1}^{m} \eta_i v_i$$

$\leftarrow$ COORDINATE REPRESENTATION IN V, UNIQUE

$$\rightsquigarrow \eta_i = \sum_{j=1}^{n} a_{ij} \xi_j \qquad \forall i = 1, \ldots, m \quad \forall j = 1, \ldots, n$$

Thus, we can rewrite this equation in matrix form

$$\left.\begin{aligned} \eta_1 &= a_{11}\xi_1 + a_{12}\xi_2 + \ldots + a_{1n}\xi_n \\ \eta_2 &= a_{21}\xi_1 + a_{22}\xi_2 + \ldots + a_{2n}\xi_n \\ &\vdots \\ \eta_m &= a_{m1}\xi_1 + a_{m2}\xi_2 + \ldots + a_{mn}\xi_n \end{aligned}\right\} \quad \underbrace{\eta = A\xi}_{\text{MATRIX MULTIPLICATION}}$$

NOTE THAT THE MATRIX MULTIPLICATION IS NOT BETWEEN THE VECTORS IN $\mathcal{U}$ AND V, BUT BETWEEN THEIR COORDINATES DEFINED WITH RESPECT TO THE CHOSEN BASES