CHANGE of BASIS

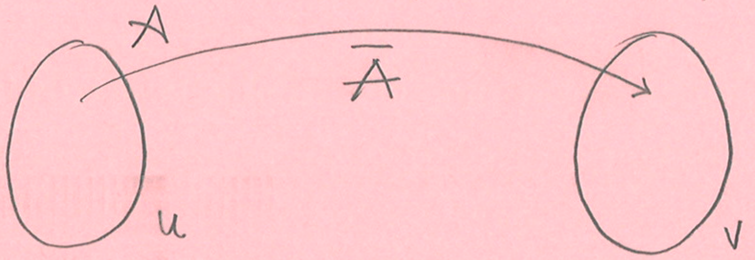
HOW A MATRIX REPRESENTATION OF A LINEAR MAP CHANGES AS THE BASIS OF THE DOMAIN AND/OR THE CODOMAIN ARE CHANGED

→ WE ALREADY KNOW HOW TO CONSTRUCT THE MATRIX REPRESENTATION OF THE LINEAR MAP BETWEEN FINITE DIMENSIONAL VECTOR SPACES (GIVEN THE CHOICE of BASIS)

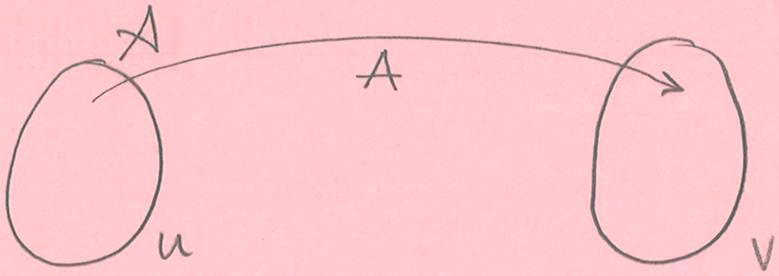
WE NOW WE WANT TO KNOW HOW TO DETERMINE THE NEW MATRIX REPRESENTATION IF THE BASIS ARE CHANGED
– (without having to reconstruct it from scratch)

CONSIDER THE TWO SPACES, THEIR BASIS, AND THE MAP BETWEEN THEM

a)



b)



a) and b) are two copies of the same sets

The diagram will support the explaination

STARTING FROM b), WE CHOSE THE BASES AND COORDINATES FOR U AND V

– $\{u_i\}$ and $\{\xi_i\}$ for U

– $\{v_i\}$ and $\{\eta_i\}$ for V

BASED ON THIS CHOICE OF BASES → A IS THE MATRIX REPRESENTATION

LET US CONSIDER NOW a) AND DEFINE A DIFFERENT CHOICE OF BASES AND COORDINATES

– $\{\bar{u}_i\}$ and $\{\bar{\xi}_i\}$ for U

– $\{\bar{v}_i\}$ and $\{\bar{\eta}_i\}$ for V

THE CORRESPONDING MATRIX REPR. CAN BE CALLED $\bar{A}$

SO THE QUESTION WE WANT TO ANSWER BECOMES "given A, how do we construct $\bar{A}$, knowing $\{u_i\}$ and $\{v_i\}$, $\{\bar{u}_i\}$ and $\{\bar{v}_i\}$ ... "?

WE CAN SHOW THE CONSTRUCTION OF THE CHANGE of BASIS FOR VECTOR SPACE IN $\mathbb{R}^m$ or $\mathbb{R}^n$

Consider a vector $x \in \mathcal{U}$, its coordinate representation wrt to $\{u_i\}$ and $\{\bar{u}_i\}$ as

$$\nabla \quad x = \xi_1 u_1 + \xi_2 u_2 + \text{—} + \xi_n u_n = \sum_{i=1}^{n} \xi_i u_i =$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \text{—} & u_n \\ | & | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \xi_1 & \xi_2 & \text{—} & \xi_n \end{bmatrix}}_{\xi}$$

$$\Delta \quad x = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \text{—} & \bar{u}_n \end{bmatrix} \bar{\xi}$$

By equating, we get $[u_1\ u_2\ \text{—}\ u_n]\,\xi = [\bar{u}_1\ \bar{u}_2\ \text{—}\ \bar{u}_n]\,\bar{\xi}$

Because the basis vectors are indipendent, matrices $[u_1 \text{—} u_n]$ and $[\bar{u}_1 \text{—} \bar{u}_n]$ are invertible

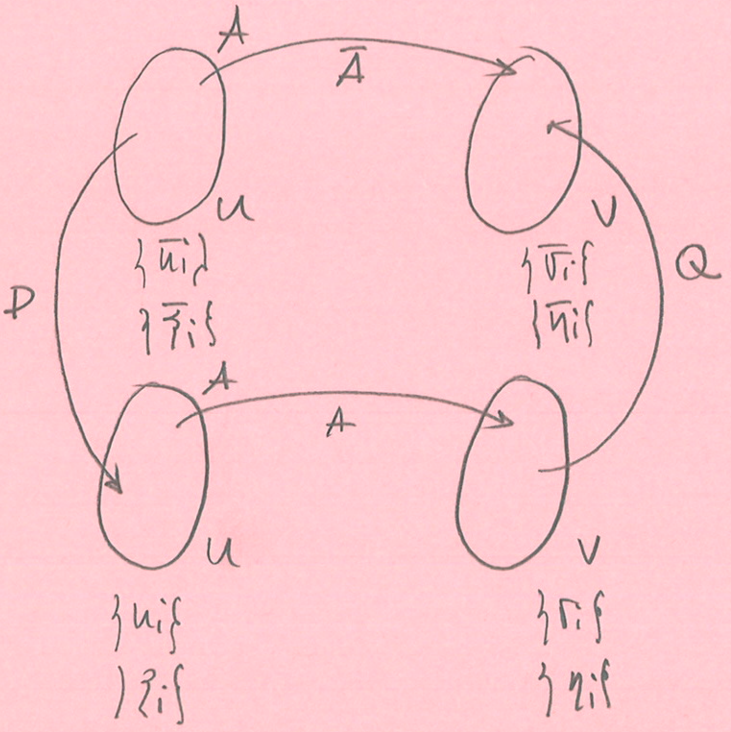
$$\rightsquigarrow \quad \xi = P\bar{\xi} = [u_1 \text{—} u_n]^{-1} [\bar{u}_1 \text{—} \bar{u}_n]\, \bar{\xi}$$

We can proceed similarly, for a vector in the codomain

$$\rightsquigarrow \quad \bar{\eta} = Q\eta = [\bar{v}_1 \text{—} \bar{v}_m]^{-1} [v_1 \text{—} v_m]\, \eta$$

In order to construct a relashionship between the coordinates wrt to one bases and the coordinates with respect to the other basis, all that is needed is a bijective relationship between the coordinate vectors

$\rightsquigarrow$ P needs to be bijective for example (Q, actually)



WE NOW HAVE A RULE TO TRANSFORM THE MATRIX REPRESENTATION A and $\bar{A}$

Given A, want $\bar{A}$?

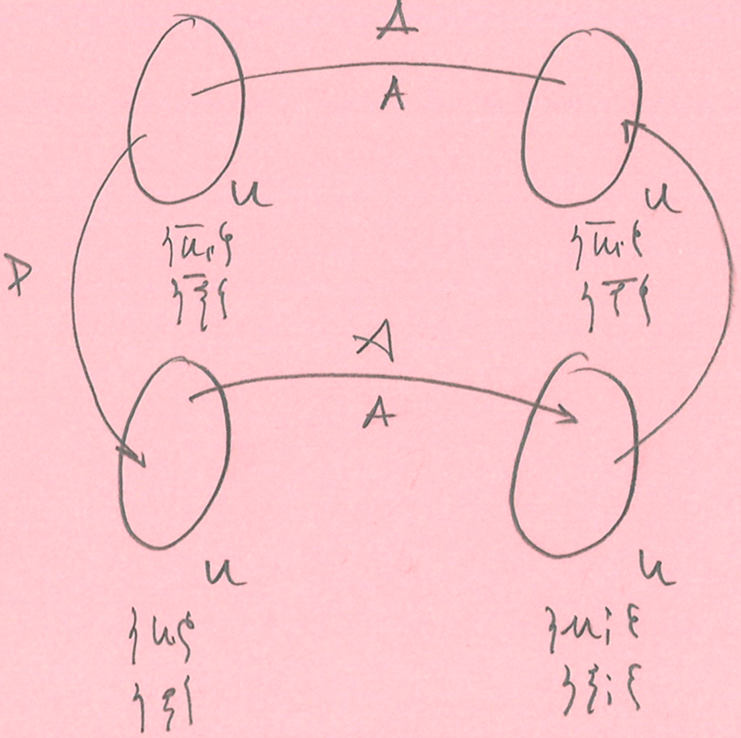
From earlier

$\eta = A\xi$, with $\xi = P\bar{\xi}$ $\rightsquigarrow$ $\eta = AP\bar{\xi}$

We also have that $\bar{\eta} = Q\eta$ $\rightsquigarrow$ $\bar{\eta} = \underbrace{QAP}_{\bar{A}}\bar{\xi}$

$P \to A \to Q$ equals $\bar{A}$

If the codomain is equal to the domain, that is $V = U$, then



$Q = P^{-1}$

$$\rightsquigarrow \boxed{\bar{A} = P^{-1}AP}$$

SIMILARITY TRANSFORM (BETWEEN BASIS)

Example $A : \mathbb{R}^3 \to \mathbb{R}^3$

Consider two basis: $B = \{b_1, b_2, b_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and another

$$C = \{c_1, c_2, c_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We consider the map A and let's say that

$$\left.\begin{aligned} A(b_1) &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ A(b_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ A(b_3) &= \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \end{aligned}\right\}$$

WHAT IS THE MATRIX REPRESENTATION OF THE MAP A, WITH RESPECT TO ?

SAY $A : \mathbb{R}^3_B \to \mathbb{R}^3_B$, then $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ (the construction rule, only)

Now suppose: $A : \mathbb{R}^3_B \to \mathbb{R}^3_C$, then $\bar{A}$? Try using the diagram