

NORMS

WE TALK ABOUT THE CONCEPT OF NORM AND NORMED VECTOR SPACES

AND WE HAVE ALREADY INTRODUCED THE CONCEPT  
OF VECTOR SPACES AND MAPS BETWEEN VECTOR SPACES

→ A NORM is a special map between vector spaces

$$\| \cdot \| : (V, \mathbb{F}) \rightarrow \mathbb{R}_+$$

THIS IS THE FUNCTION / OPERATION  
DEFINED AS NORM

- IT TAKES THE ELEMENTS OF A VECTOR SPACE AND  
MAPS THEM IN THE VECTOR SPACE OF POSITIVE REAL NUMBERS

Def (NORMED VECTOR SPACE) is a vector space  $(V, \mathbb{F})$  which  
has the operation norm defined



BUT WHAT IS AN NORM?

EITHER  $\mathbb{R}$  OR  $\mathbb{C}$

THE NORM IS A MAP THAT HAS TO SATISFY A SET OF GIVEN PROPERTIES

- i)  $\| v_1 + v_2 \| \leq \| v_1 \| + \| v_2 \|, \forall v_1, v_2$  (Triangle inequality)
- ii)  $\| \alpha v \| = |\alpha| \| v \|, \forall v$
- iii)  $\| v \| = 0 \iff v = \theta_v$

Example  $(\mathbb{F}^n, \#)$  WITH  $\#^n$  either  $\mathbb{R}^n$  or  $\mathbb{C}^n$   
AND  $\mathbb{F}$  either  $\mathbb{R}$  or  $\mathbb{C}$

- The 1 or  $L_1$  NORM :  $\| x \|_1 = \sum_{i=1}^n |x_i|$  (the sum of the absolute values of the components of the vector)
- The 2 or  $L_2$  NORM :  $\| x \|_2 = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- The  $p$ -norm :  $\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$
- The  $\infty$ -norm :  $\|x\|_\infty = \max(|x_i|)$  (the maximum value of the absolute values of the components)

MATRIX NORMS are defined for vector spaces which consist of matrices and we can construct norm on those matrices

Example  $A \in \mathbb{F}^{m \times n}$  WITH  $\mathbb{F}$  either  $\mathbb{R}$  or  $\mathbb{C}$

- The  $a$ -norm :  $\|A\|_a = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|$  (the summation over the rows and the columns)
- The FROBENIUS-norm :  $\|A\|_F = \left( \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}$
- The  $b$ -norm :  $\|A\|_b = \max_{\substack{i=1, \dots, m \\ j=1, \dots, n}} (|a_{ij}|)$

THESE MATRIX NORMS OF COURSE SATISFY THE PROPERTIES OF NORMS

→ They are seldomly used in linear systems theory

→ INDUCED NORMS ARE PREFERRED

Example FUNCTION SPACES  $f(\cdot) \in C([t_0, t], \mathbb{F}^n)$

continuous functions over the interval  $[t_0, t]$

WHAT IS RETURNED BY THE FUNCTION IF A VECTOR IN  $\mathbb{F}^n$

- The  $1$ -norm :  $\|f\|_1 = \int_{t_0}^t \|f(t)\| dt$  any of the vector norms

- The  $L$ -norm :  $\|f\|_2 = \left( \int_{t_0}^{t_1} \|f(t)\|^2 dt \right)^{1/2}$

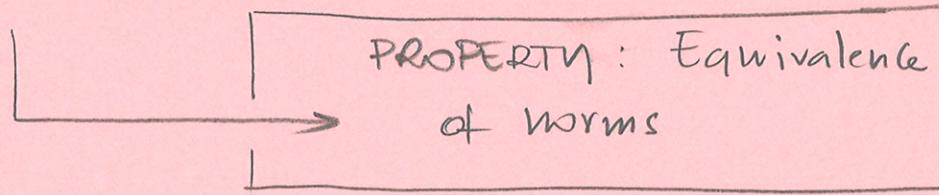
THE SPACE OF SQUARE  
INTEGRABLE FUNCTIONS

- The  $\infty$ -norm :  $\|f\|_\infty = \max (\|f(t)\|, t \in [t_0, t_1])$

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HOW ARE DIFFERENT NORMS RELATED TO EACH OTHER?

→ Or, if we have a vector that is finite in the  $1$ -norm, can we say anything about its value in terms of another norm?

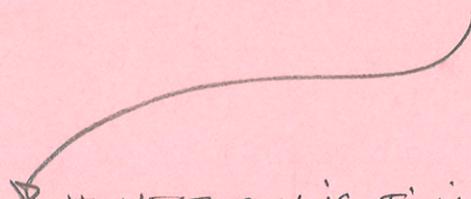


TWO NORMS  $\|\cdot\|_a$  AND  $\|\cdot\|_b$  ON THE SAME VECTOR SPACE ARE SAID TO BE EQUIVALENT IF ONE CAN BE BOUNDED WRT THE OTHER

\*  $\exists m_a, m_b \in \mathbb{R}_+$  such that

constants  
(Positive Reals)

$$m_a \|v\|_a \leq \|v\|_b \leq m_b \|v\|_a$$



IF VECTOR  $v$  IS FINITE IN THE  $b$ -NORM  
THE IT IS GOING TO BE FINITE IN THE  $a$ -NORM TOO, WITH BOUNDS

WE CAN THEN DERIVE SOME RELATIONSHIPS

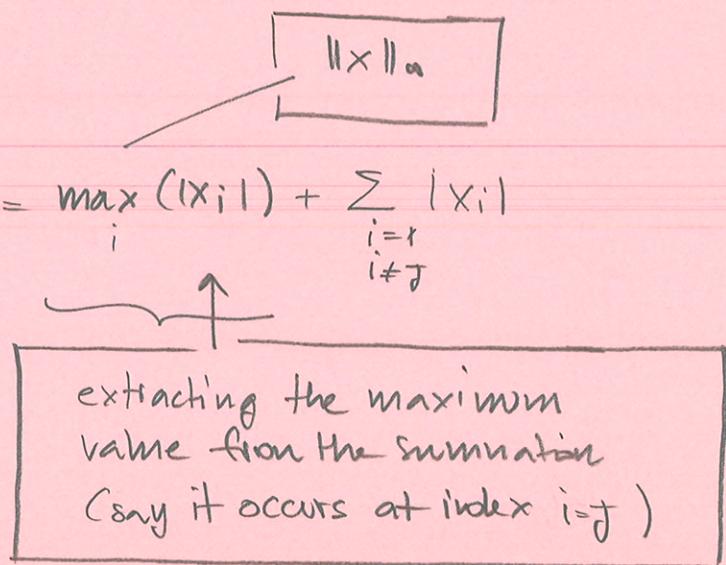
Example Consider the vector norms in  $\mathbb{F}^n$

$$\Rightarrow \|x\|_a \leq \|x\|_1 \leq n \|x\|_\infty$$

PROOF:  $\|x\|_\infty = \max(|x_i|)$

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \max_i (|x_i|) + \sum_{\substack{i=1 \\ i \neq j}} |x_i|$$

ASSUME THAT



THEN  $\|x\|_\infty \leq \|x\|_1$

each is smaller (or at most equal to  $\max(|x_i|)$ )

Moreover,  $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

$$\leq \max_i (|x_i|) + \max_i (|x_i|) + \dots + \max_i (|x_i|)$$

$$= n \max_i (|x_i|)$$

All the presented norms in  $\mathbb{F}^n$  are equivalent, as they all satisfy  $u \cdot \|x\|_a \leq u \cdot \|x\|_b \leq u \cdot \|x\|_\infty$