NORMS

WE TALK ABOUT THE CONCEPT OF NORM AND NORMED VECTOR SPACES

⤳ WE HAVE ALREADY INTRODUCED THE CONCEPT OF VECTOR SPACES AND MAPS BETWEEN VECTOR SPACES

⤳ A NORM is a special map between vector spaces

$$\underbrace{\|\cdot\|}_{} : (V, \mathbb{F}) \longrightarrow \mathbb{R}_+$$

THIS THE FUNCTION / OPERATION DEFINED AS NORM

- IT TAKES THE ELEMENTS OF A VECTOR SPACE AND MAPS THEM IN THE VECTOR SPACE OF POSITIVE REAL NUMBERS

Def (NORMED VECTOR SPACE) is a VECTOR SPACE $\underbrace{(V, \mathbb{F})}_{}$ WHICH HAS THE OPERATION NORM DEFINED

↓ BUT WHAT IS A NORM?

EITHER $\mathbb{R}$ OR $\mathbb{C}$

THE NORM IS A MAP THAT HAS TO SATISFY A SET OF GIVEN PROPERTIES

- i) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|, \forall v_1 v_2$ (Triangle inequality)
- ii) $\|\alpha v\| = |\alpha| \, \|v\|, \quad \forall v$
- iii) $\|v\| = 0 \xleftrightarrow{\text{iff}} v = \theta_v$

Example $(\mathbb{F}^n, \mathbb{F})$ WITH $\mathbb{F}^n$ either $\mathbb{R}^n$ or $\mathbb{C}^n$ AND $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$

- The 1 or L_1 NORM : $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (the sum of the absolute values of the components of the vector)
- The 2 or L_2 NORM : $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$$

- The p-norm : $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$
- The ∞-norm : $\|x\|_\infty = \max(|x_i|)$ (the maximum value of the absolute values of the componets)

MATRIX NORMS are defined for vector spaces which consist of matrices and we can construct norm on these matrices

Example $A \in \mathbb{F}^{m \times n}$ WITH $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$

- The a-norm : $\|A\|_a = \sum_{i=1}^{m} \sum_{j=1} |a_{ij}|$ (the summation over the rows and the colums)
- The FROBENIUS-norm : $\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$
- The b-norm : $\|A\|_b = \max_{\substack{i=1,\dots,m \\ j=1,\dots,n}} (|a_{ij}|)$

THESE MATRIX NORMS of COURSE SATISFY THE PROPERTIES OF NORMS

$\rightarrow$ They are seldomly use in linear systems theory

$\rightarrow$ INDUCED NORMS ARE PREFERRED

Example FUNCTION SPACES $f(\cdot) \in C([t_0, t], \mathbb{F}^n)$

Continuos functions over the interval $[t_0, t]$

WHAT IS RETURNED BY THE FUNCTION IS A VECTOR IN $\mathbb{F}^n$

- The 1-norm : $\|f\|_1 = \int_{t_0}^{t} \underbrace{\|f(t)\|}_{\text{any of the vector norms}} dt$

– The 2-norm : $\|f\|_2 = \underbrace{\left(\int_{t_0}^{t_1} \|f(t)\|^2 \, dt\right)^{1/2}}_{\text{THE SPACE OF SQUARE INTEGRABLE FUNCTIONS}}$

– The ∞-norm : $\|f\|_\infty = \max\left(\|f(t)\|,\ t \in [t_0, t_1]\right)$

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HOW ARE DIFFERENT NORMS RELATED TO EACH OTHER ?

→ Or, if we have a vector that is finite in the 1-norm, can we say anything about its value in terms of another norm ?

PROPERTY : Equivalence of norms

TWO NORMS $\|\cdot\|_a$ AND $\|\cdot\|_b$ ON THE SAME VECTOR SPACE ARE SAID TO BE EQUIVALENT IF ONE CAN BE BOUNDED WRT THE OTHER

* $\exists\ \underbrace{m_e, m_u}_{\text{constants (Positive Reals)}} \in \mathbb{R}_+$ such that

– $\underbrace{m_e \|v\|_a \leq \|v\|_b \leq m_u \|v\|_a}$

IF VECTOR v IS FINITE IN THE b-NORM THE IT IS GOING TO BE FINITE IN THE a-NORM TOO , WITH BOUNDS

WE CAN THEN DERIVE SOME RELATIONSHIPS

Example Consider the vector norms in $\mathbb{F}^n$

$\rightsquigarrow \|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$

PROOF: $\|x\|_\infty = \max(|x_i|)$

$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \underbrace{\max_i (|x_i|)}_{\boxed{\|x\|_\infty}} + \underbrace{\sum_{\substack{i=1 \\ i \neq j}} |x_i|}$

ASSUME THAT

extracting the maximum value from the summation (say it occurs at index $i=j$)

THEN $\|x\|_\infty \leq \|x\|_1$

each is smaller (or at most equal to $\max_i(|x_i|)$)

MOREOVER, $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

$$\leq \max_i(|x_i|) + \max_i(|x_i|) + \dots + \max_i(|x_i|)$$

$$= n \max_i(|x_i|)$$

All the presented norms in $\mathbb{F}^n$ are equivalent, as they all satisfy $\|\cdot\|_a\, m \leq \|\cdot\|_b \leq \|\cdot\|_a\, M$