INDUCED NORMS

WE DISCUSSED THE CONCEPT OF A NORM — A map on a vector space with specific properties

(We discussed vector norms and some matrix norms)

In linear systems theory, those norms are not used (in general)

It is more common to use a set of induced norms

INDUCED NORMS : these are norms that are induced from a linear operator (a matrix form of a linear operator)

CONSIDER THE LINEAR OPERATOR $\mathcal{A} : (\mathcal{U}, \mathbb{F}) \longrightarrow (\mathcal{V}, \mathbb{F})$

— $\mathcal{A}$ IS CONTINUOUS

— $\mathcal{A}$ IS A LINEAR OPERATOR (OKAY, AGAIN)

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are normed spaces

- ASSOCIATED TO $\mathcal{U}$ WE HAVE THE NORM $\|\cdot\|_u$
- ASSOCIATED TO $\mathcal{V}$ WE HAVE THE NORM $\|\cdot\|_v$

WE DEFINE THE INDUCED NORM OF THE MAP $\mathcal{A}$ AS

$$\|\mathcal{A}\|_i = \sup_{\substack{u \in \mathcal{U} \\ u \neq \theta_u}} \frac{\|\mathcal{A}u\|_v}{\|u\|_u}$$

(How big u can get after it is transformed using $\mathcal{A}$)

WE MEASURE THE SIZE OF THE OPERATOR BY HOW IT OPERATES ON VECTORS

$\|A\|_i$ CAN NOW BE APPLIED TO THE MATRIX REPRESENTATION OF THE OPERATOR (induced norms applied to matrices)

$$\|A\|_i = \sup_{\substack{u \in U \\ u \neq \theta_U}} \frac{\|Au\|_V}{\|u\|_U}$$

By specializing the vector norms in U and V, we specialize the induced of the operator

WE CAN, FOR EXAMPLE, DEFINE THE INDUCED 1-NORM

$$\|A\|_{i,1} = \sup_{\substack{u \in U \\ u \neq \theta_u}} \frac{\|Au\|_1}{\|u\|_1}$$

(of cource the 1-norm must be defined in both U and V)

WE NOW STUDY WHAT THE INDUCED NORM CONCRETELY MEANS WHEN WE SPECIALISE IT TO BE THE 1 norm, THE 2-norm, ...

Consider an operator $A: \mathbb{F}^n \to \mathbb{F}^m$, we use $\|\cdot\|_p$ to denote the p-norm

→ THE INDUCED p-NORM ON THE OPERATOR A

$$\|A\|_{i,p} = \sup_{u \neq \theta_u} \frac{\|Au\|_{V,p}}{\|u\|_{u,p}}$$

$$p=1 \to \|A\|_{i,1} = \sup_{u \neq \theta_u} \frac{\|Au\|_{V,1}}{\|u\|_{u,1}}$$

WE CAN SHOW THAT

$$= \underbrace{\max_{j=1,\dots,n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}}_{\text{MAX COLUMN SUM}}$$

$$p=2 \rightsquigarrow \|A\|_{i,2} = \sup_{u \neq \theta_u} \frac{\|Au\|_{2,v}}{\|u\|_{2,u}}$$ WE CAN SHOW THAT

$$= \max_{j=1,\dots,n} \{ \underbrace{\lambda_j}_{\text{the } j\text{-th eigenvalue}} (\underbrace{A^* A}_{\text{Square the matrix}}) \}^{1/2}$$

$$p=\infty \rightsquigarrow \|A\|_{i,\infty} = \sup_{u \neq \theta_u} \frac{\|Au\|_{\infty,v}}{\|u\|_{\infty,u}}$$ WE CAN SHOW THAT

$$= \underbrace{\max_{i=1,\dots,m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}}_{\text{MAX ROW SUM}}$$

We can try to understand how to go from the induced norm definition to one of the given concrete form

WE PROVE THE EQUALITY FOR THE 1-NORM CASE

$\|x\|_1 = \sum_{i=1}^{N} |x_i|$ WITH $x = [x_1, \dots, x_n]$

WE CONSIDER $$\|A\|_{i,1} = \sup_{x \neq \theta} \frac{\|Ax\|_1}{\|x\|_1} = \frac{\sum_{i=1}^{M} |\underbrace{(Ax)_i}_{\text{the } j\text{-th element of vector } Ax}|}{\sum_{i=1}^{N} |x_i|}$$

$$= \sup_{x \neq \theta} \frac{\sum_{i=1}^{M} |\sum_{j=1}^{N} a_{ij} x_j|}{\sum_{i=1}^{N} |x_i|} \leq \sup_{x \neq \theta} \frac{\sum_{i=1}^{n} \sum_{j=1}^{N} |a_{ij}||x_j|}{\sum_{i=1}^{N} |x_i|}$$

$$= \sup_{x \neq \theta} \frac{\sum_{j=1}^{N} \left(|x_j| \sum_{i=1}^{n} |a_{ij}| \right)}{\sum_{i=1}^{N} |x_i|}$$

$$\vdots$$

$$\ldots \leq \max_{j=1,N} \sum$$