

INDUCED NORMS

WE DISCUSSED THE CONCEPT OF A NORM - A map on a vector space with specific properties

(We discussed vector norms and some matrix norms)

In linear systems theory, those norms are not used (in general)

It is more common to use a set of induced norms

INDUCED NORMS : These are norms that are induced from a linear operator (a matrix form of a linear operator)

CONSIDER THE LINEAR OPERATOR $A : (U, \mathbb{F}) \rightarrow (V, \mathbb{F})$

- A is continuous
- A is a linear operator (OKAY, AGAIN)

Suppose that U and V are normed spaces

- ASSOCIATED TO U WE HAVE THE NORM $\| \cdot \|_U$
- ASSOCIATED TO V WE HAVE THE NORM $\| \cdot \|_V$

WE DEFINE THE INDUCED NORM OF THE MAP A AS

$$\|A\|_i = \sup_{\substack{u \in U \\ u \neq 0_U}} \frac{\|Au\|_V}{\|u\|_U} \quad (\text{How big } u \text{ can get after it is transformed using } A)$$

WE MEASURE THE SIZE OF THE OPERATOR BY HOW IT OPERATES ON VECTORS

$\|A\|_i$ CAN NOW BE APPLIED TO THE MATRIX REPRESENTATION OF THE OPERATOR (Induced norms applied to matrices)

$$\|A\|_i = \sup_{\substack{u \in U \\ u \neq 0}} \frac{\|Au\|_r}{\|u\|_u} \quad \left. \begin{array}{l} \text{By specializing the vector norms} \\ \text{in } U \text{ and } V, \text{ we specialize the} \\ \text{induced norm of the operator} \end{array} \right\}$$

WE CAN, FOR EXAMPLE, DEFINE THE INDUCED 1-NORM

$$-\|A\|_{i,1} = \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|Au\|_1}{\|u\|_1} \quad \left(\text{of course the 1-norm} \right.$$

must be defined in both
 U and V)

WE NOW STUDY WHAT THE INDUCED NORM CONCRETELY MEANS WHEN WE SPECIALISE IT TO BE THE 1-NORM, THE 2-NORM,

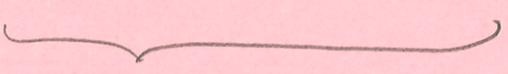
Consider an operator $\mathcal{X}: \mathbb{F}^n \rightarrow \mathbb{F}^m$, we use $\|\cdot\|_p$ to denote the p -norm

and THE INDUCED p -NORM ON THE OPERATOR \mathcal{X}

$$\|A\|_{i,p} = \sup_{u \neq 0} \frac{\|Au\|_{V,p}}{\|u\|_{U,p}}$$

$$p=1 \Rightarrow \|A\|_{i,1} = \sup_{u \neq 0} \frac{\|Au\|_{V,1}}{\|u\|_{U,1}} \quad \text{WE CAN SHOW THAT}$$

$$= \max_{j=1, \dots, n} \left\{ \sum_{i=1}^m |a_{ij}| \right\}$$



MAX COLUMN SUM

$$p=2 \rightsquigarrow \|A\|_{i,2} = \sup_{u \neq 0_n} \frac{\|Au\|_{2,v}}{\|u\|_{2,u}} \quad \text{WE CAN SHOW THAT}$$

$$= \max_{j=1, \dots, n} \left\{ \underbrace{\lambda_j(A^* A)}_{\text{the } j\text{-th eigenvalue}} \right\}^{1/2} \quad \text{Square the matrix}$$

$$p=\infty \rightsquigarrow \|A\|_{i,\infty} = \sup_{u \neq 0_n} \frac{\|Au\|_{\infty,v}}{\|u\|_{\infty,u}} \quad \text{WE CAN SHOW THAT}$$

$$= \max_{i=1, \dots, m} \left\{ \underbrace{\sum_{j=1}^m |a_{ij}|}_{\text{MAX ROW SUM}} \right\}$$

We can try to understand how to go from the induced norm definition to one of the given concrete form

WE PROVE THE EQUALITY FOR THE 1-norm CASE

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \text{with} \quad x = [x_1, \dots, x_n] \quad \begin{matrix} \text{the } j\text{-th element} \\ \text{of vector} \\ Ax \end{matrix}$$

$$\begin{aligned} \text{WE CONSIDER} \quad & \|A\|_{i,1} = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \frac{\sum_{i=1}^n |(Ax)_i|}{\sum_{i=1}^n |x_i|} \\ & = \sup_{x \neq 0} \frac{\sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right|}{\sum_{i=1}^n |x_i|} \leq \sup_{x \neq 0} \frac{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}| |x_j|}{\sum_{i=1}^n |x_i|} \\ & = \sup_{x \neq 0} \frac{\sum_{j=1}^n \left(|x_j| \sum_{i=1}^n |a_{ij}| \right)}{\sum_{i=1}^n |x_i|} \end{aligned}$$

$$\therefore \leq \max_{j=1, \dots, N} \sum$$