ORTHOGONALITY AND ADJOINTS

We defined the inner product, we can use the concept to develop some derivatives and geometrical interpretation

WE START BY LOOKING AT THE CONCEPT OF ORTHOGONALITY

→ We start with an HILBERT SPACE $(H, \mathbb{F}, \langle \cdot, \cdot \rangle)$
(a vector space H over the field $\mathbb{F}$, endowed with the inner product)

WE DEFINE ORTHOGONALITY BETWEEN TWO VECTORS IN H

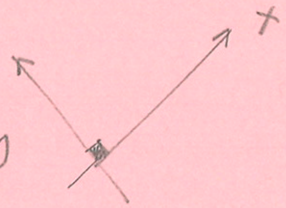
$x, y \in H$ are ortogonal $x \perp y$ iff $\langle x, y \rangle = 0$

▷ the standard inner product is the dot product between those two vectors

WE CAN NOW DEFINE SUBSPACE THAT ARE NOW ORTHOGONAL TO EACH OTHER

in $\mathbb{R}^2$

ORTHOGONALITY $\equiv$ PERPENDICULARITY



IF $M \subseteq H$, WE DEFINE $M^{\perp}$

AS $\underbrace{M^{\perp} = \{ y \in H \mid \langle x, y \rangle = 0 \ \forall x \in M \}}$

the set of vectors in H that are orthogonal to every vector in M

▷ THE ORTHOGONAL COMPLEMENT OF M

- THE ONLY INTERSECTION BETWEEN M AND ITS ORTHOGONAL COMPLEMENT $M^{\perp}$ IS THE ZERO VECTOR

$$M \cap M^{\perp} = \{ 0_H \}$$

We can prove that $M \cap M^{\perp} = \{\theta\}$ by assuming the existence of some $x \neq \theta$ in $M \cap M^{\perp}$

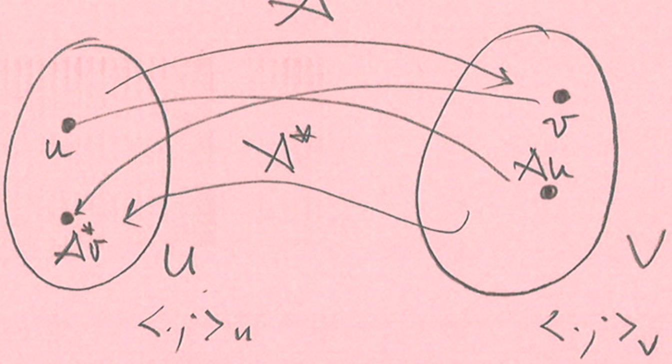
⇝ by definition that means that $\langle x, y \rangle = 0$ for all $y \in M$

⇝ but since x is in the intersection in is also in M, then we must have that also $\langle x, x \rangle = 0$

ONLY FOR $x = \theta$
BY DEFINITION

WE NOW DISCUSS THE CONCEPT OF ADJOINT MAP (defined in terms of an inner product)

LET $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$



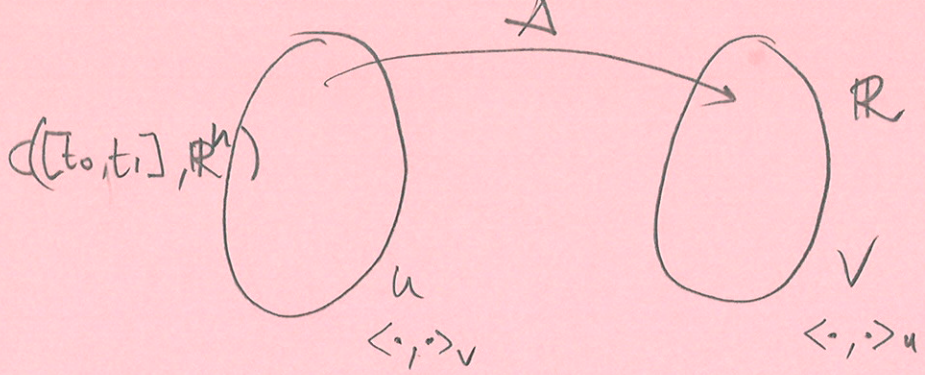
Let $A: U \to V$ continuous and linear, then the adjoint of A, A^* is defined as

$A^*: V \to U$ such that

$$\langle v, Au \rangle_V = \langle A^* v, u \rangle_U$$

<u>Example</u> Define U as the space of vector valued functions

$C([t_0, t_1]) = U$ and $V \equiv \mathbb{R}$



$A: C([t_0, t_1], \mathbb{R}^n) \to \mathbb{R}$

$A: f \mapsto \langle \underbrace{g}_{\text{a given function in } U}(\cdot), f(\cdot) \rangle$

$A^*: \mathbb{R} \to C([t_0, t_1], \mathbb{R}^n)$

WHAT IS A^*?

WE HAVE,

$$\langle v, \mathcal{A} f(\cdot) \rangle_v = v^* \langle g(\cdot), f(\cdot) \rangle_u$$
$$= v^* \int_{t_0}^{t_1} g^*(t) f(t)\, dt$$

REARRANGING,

$$\rightsquigarrow \int_{t_0}^{t_1} v g^*(t) f(t)\, dt$$
$$= \langle v g(\cdot), f(\cdot) \rangle_u$$
$$= \langle \underbrace{\mathcal{A}^* v}, f(\cdot) \rangle_u$$

$\boxed{\text{▹ } \mathcal{A}^* \text{ thus simply take } v \text{ and multiplies it by function } g(\cdot)}$

$\mathcal{A}^* : v \longrightarrow v g(\cdot)$ WITH $g(\cdot)$ THE FUNCTION CHOSEN FOR THE DEFINITION OF THE MAP $\mathcal{A}$