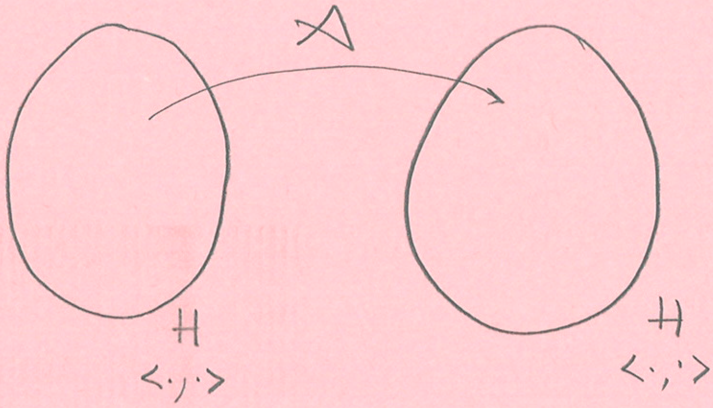
HERMITIAN MATRICES

WE DISCUSS A NUMBER OF PROPERTIES OF MATRICES THAT EMERGED FROM DEFINING INNER PRODUCTS AND ADJOINT MAPS

BASED ON THE DEFINITION OF ADJOINTS, WE CAN EXPLORE A NUMBER OF SPECIAL CASES

1) MAPS $\mathcal{A}$ THAT ARE SELF-ADJOINT

We define a self-adjoint maps, we consider $(\mathcal{H}, \mathbb{F}, \langle \cdot, \cdot \rangle)$ AND WE consider a linear and continuous map $\mathcal{A}$



$\mathcal{A}$ linear + continuos

$\mathcal{A} : \mathcal{H} \to \mathcal{H}$

$\mathcal{A}^* : \mathcal{H} \to \mathcal{H}$

We say that map $\mathcal{A}$ is self-adjoint if it is equal to its adjoint

$\leadsto$ $\mathcal{A} = \mathcal{A}^*$ or equivalently, from the definition of adjoint map

$$\leadsto \langle x, \mathcal{A}y \rangle = \langle \mathcal{A}x, y \rangle \quad \forall\, x, y \in \mathcal{H}$$

A typical example when $\mathcal{A}$ is matrix multiplication (when $\mathcal{H}$ are finite dimensional vector spaces so we can represent $\mathcal{A}$ in terms of matrix multiplication)

$\to$ WHAT IS A MAP IN THIS CONTEXT THAT IS ~~SELF~~-ADJOINT ?

$\leadsto$ HERMITIAN MATRICES

$\mathcal{A}$ is represented by a matrix $\underbrace{A}_{A \in \mathbb{F}^{n \times n}} = (a_{ij})$ with $\left\{ \begin{array}{l} i = 1, \ldots, n \\ j = 1, \ldots, n \end{array} \right.$

Then, the map $\mathcal{A}$ is self-adjoint iff matrix A is hermitian

⇝ $\underbrace{A = A^*}$ with $*$ meaning $\underbrace{a_{ij} = \overline{a}_{ji}}$

The matrix is equal to its complex conjugate transpose

COMPLEX CONJUGATE TRANSPOSE

WE CAN DEFINE A UNITARY MATRIX U

⇝ A matrix is unitary iff $U^* U = U U^* = I$

⇝ The colums or rows of U form an orthonormal basis

⇝ For real matrices, we simply call U an orthoganal matrix

WE CAN DEFINE THE SINGULAR VALUE DECOMPOSITION

⇝ First, the notion of SINGULAR VALUES

We consider a matrix $A \in \mathbb{F}^{m \times n}$, then $AA^* \in \mathbb{F}^{m \times m}$

* WE LOOK AT THE EIGENVALUES OF AA^* $(\lambda_i)_{i=1}^{m}$
  - THEY ARE ALWAYS REAL FOR HERMITIAN MATRICES
  - THEY ARE ALWAYS NON-NEGATIVE

We can sort them, from the largest to the smallest

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \underbrace{\lambda_r}_{} \geq \cdots \geq \lambda_m \geq 0$$

↳ let r be the rank of A (and of AA^*)

Then the eigenvalues $\lambda_1 \to \lambda_r$ are all positive and the rest $(\lambda_{r+1} \to \lambda_m)$ are all zero

WE DEFINE THE SINGULAR VALUES OF MATRIX A TO BE THE SQRT OF THE EIGENVALUES OF A

⇝ there are r non-zero singular values

Example Consider a 2×3 matrix, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the rank is one and we want to compute its singular values

$$\lambda_i(AA^*) = \lambda_i\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \lambda_i\left(\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\right) \rightsquigarrow \begin{matrix} \lambda_1 = \sqrt{5} \\ \lambda_2 = \sqrt{0} \end{matrix}$$

The induced 2-norm of a matrix $\|A\|_{2,i} = \max_j \left(\lambda_i(\underbrace{A^*A}_{})\right)^{1/2}$

$\boxed{A^*A \text{ not } AA^*}$

The idea of singular values becomes interesting in the singular value decomposition of a matrix