SINGULAR VALUE DECOMPOSITION

WE START USING THE CONCEPT OF SINGULAR VALUES TO PRESENT A USEFUL DECOMPOSITION OF MATRICES

CONSIDER A MATRIX $A \in \mathbb{C}^{m \times n}$ WITH RANK $r \leq \min(m, n)$

→ **Theorem** there exist unitary matrices ($U^*U = UU^* = I$)

$U \in \mathbb{C}^{m \times m}$

$V \in \mathbb{C}^{n \times n}$ s.t. $A = U \Sigma V^*$

$$\underbrace{\underset{(m \times m)}{U} \; \underset{(n \times n)}{\Sigma} \; \underset{(n \times n)}{V^*}}_{A \text{ is } (m \times n)}$$

Σ is a diagonal matrix

$$\Sigma = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

WITH Σ_r IS A SQUARE DIAGONAL MATRIX WHOSE DIAGONAL ELEMENTS ARE THE NONZERO SINGULAR VALUES OF A

$\Sigma_r = \mathrm{diag}(\lambda_1 - \lambda_r)$

THE STRUCTURE OF U AND V

$U = [\, U_1 \mid U_2 \,]$

The first r columns of the matrix U

U_1 is $(m \times r)$

the remaining $(r - m)$ columns of matrix U

U_2 is $m \times (m - r)$

SIMILARLY FOR V

$V = [\, V_1 \mid V_2 \,]$

WITH V_1 A $(n \times r)$

V_2 A $(n \times (n - r))$

WE CAN NOW WRITE $A = U \Sigma V^*$ CAN BE WRITTEN AS →

$$A = \underbrace{\left[U_1 \mid U_2 \right]}_{U} \underbrace{\left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right]}_{\Sigma} \underbrace{\left[\begin{array}{c} V_1 \\ \hline V_2 \end{array}\right]}_{V^*} = U_1 \Sigma_r V_1^*$$

WE HAVE THAT $U^* U = I_m \longrightarrow U_1^* U_1 = I_r$

$V^* V = I_r \longrightarrow V_1^* V_1 = I_r$

Now considering matrix $AA^* = (U_1 \Sigma_r V_1^*)(U_1 \Sigma_r V_1^*)^*$

$= (U_1 \Sigma_r V_1^*)(V_1 \Sigma_r^* U_1^*)$

$= U_1 \Sigma_r^2 U_1^*$

We then have that $AA^* \underbrace{U_1}_{r} = U_1 \Sigma_r^2 \underbrace{U_1^* U_1}_{r} = U_1 \Sigma_r^2$

$\boxed{\text{POST MULTIPLY BY } U_1}$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{bmatrix}}_{\substack{\text{the first } r \text{ columns} \\ \text{of } U \ (\leadsto U_1)}} \Sigma_r^2$$

$$\underbrace{AA^* \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{bmatrix} \Sigma_r^2}_{\substack{\text{THIS IS SOME KIND OF} \\ \text{EIGEN RELATIONSHIP}}}$$

$\leadsto$ u_i COLUMNS ARE SOME SORT OF EIGENVECTORS

$\leadsto$ THEY ARE CALLED THE 'LEFT SINGULAR VECTORS' OF THE MATRIX A

SIMILARLY, WE CAN LOOK AT A^*A

$$
\begin{aligned}
A^*A &= (U_1 \Sigma_r V_1^*)^* (U_1 \Sigma_r V_1^*) \\
&= (V_1 \Sigma_r \underbrace{U_1^*)(U_1}_{} \Sigma_r V_1^*) \\
&= V_1 \Sigma_r^2 V_1^*
\end{aligned}
$$

Then $A^*A V_1 = V_1 \Sigma_r^2$ WITH $V_1 = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \text{---} & v_r \\ | & | & & | \end{bmatrix}$

$\rightsquigarrow A^*A v_i = \sigma_i^2 v_i \rightsquigarrow$ 'RIGHT SINGULAR VECTORS' OF THE MATRIX A