

SINGULAR VALUE DECOMPOSITION

WE START USING THE CONCEPT OF SINGULAR VALUES TO PRESENT A USEFUL DECOMPOSITION OF MATRICES

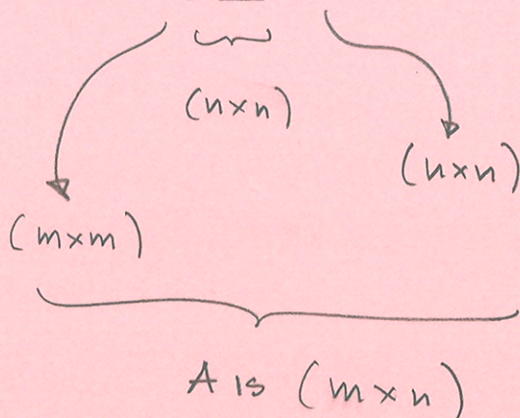
CONSIDER A MATRIX $A \in \mathbb{C}^{m \times n}$ WITH RANK $r \leq \min(m, n)$

\leadsto Theorem there exist unitary matrices ($U^*U = UU^* = I$)

$$U \in \mathbb{C}^{m \times m}$$

$$V \in \mathbb{C}^{n \times n}$$

$$\text{s.t. } A = U \Sigma V^*$$



Σ is a diagonal matrix

$$\leadsto \Sigma = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array} \right] \downarrow r$$

WITH Σ_r IS A SQUARE DIAGONAL MATRIX WHOSE DIAGONAL ELEMENTS ARE THE NONZERO SINGULAR VALUES OF A

$$\Sigma_r = \text{diag}(\lambda_1, \dots, \lambda_r)$$

THE STRUCTURE OF U AND V

$$U = [U_1 \mid U_2]$$

▷ The first r columns of the matrix U
 U_1 is $(m \times r)$

the remaining $(n-r)$ columns of matrix U
 U_2 is $m \times (n-r)$

SIMILARLY FOR V

$$V = [V_1 \mid V_2]$$

WITH V_1 A $(n \times r)$

V_2 A $(n \times (n-r))$

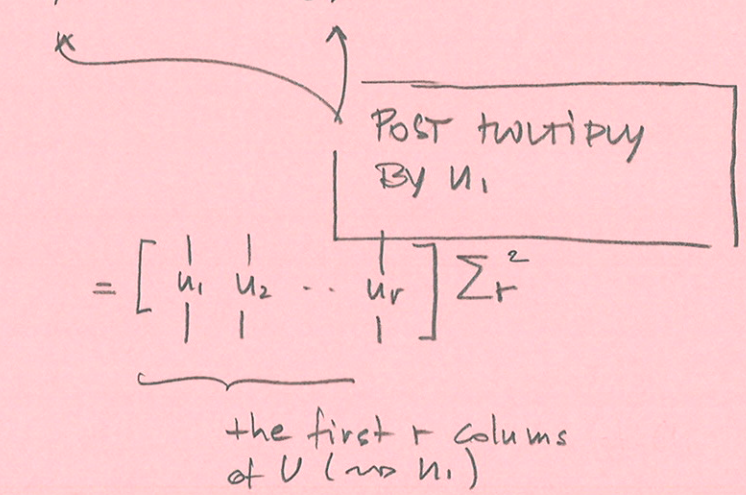
WE CAN NOW WRITE $A = U \Sigma V^*$ CAN BE WRITTEN AS $\leadsto \leadsto$

$$A = \underbrace{[U_1 \mid U_2]}_U \underbrace{\begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}}_\Sigma \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{V^*} = U_1 \Sigma_r V_1^*$$

WE HAVE THAT $U^*U = I_m \rightarrow U_i^*U_i = I_r$
 $V^*V = I_r \rightarrow V_i^*V_i = I_r$

Now considering matrix $AA^* = (U_1 \Sigma_r V_1^*) (U_1 \Sigma_r V_1^*)^*$
 $= (U_1 \Sigma_r V_1^*) (V_1 \Sigma_r^* U_1^*)$
 $= U_1 \Sigma_r^2 U_1^*$

We then have that $AA^*U_i = U_i \Sigma_r^2 U_i^* U_i = U_i \Sigma_r^2$



$$AA^* \begin{bmatrix} | & | & & | \\ U_1 & U_2 & \dots & U_r \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ U_1 & U_2 & \dots & U_r \\ | & | & & | \end{bmatrix} \Sigma_r^2$$

THIS IS SOME KIND OF EIGEN RELATIONSHIP

↳ U_i COLUMNS ARE SOME SORT OF EIGENVECTORS

↳ THEY ARE CALLED THE 'LEFT SINGULAR VECTORS' OF THE MATRIX A

SIMILARLY, WE CAN LOOK AT A^*A

$$\begin{aligned} A^*A &= (U_1 \Sigma_r V_1^*)^* (U_1 \Sigma_r V_1^*) \\ &= (V_1 \Sigma_r \underbrace{U_1^*}_{I}) (U_1 \Sigma_r V_1^*) \\ &= V_1 \Sigma_r^2 V_1^* \end{aligned}$$

Then $A^*A V_1 = V_1 \Sigma_r^2$ WITH $V_1 = \begin{bmatrix} | & | & \dots & | \\ v_1 & v_2 & \dots & v_r \\ | & | & \dots & | \end{bmatrix}$

$\rightsquigarrow A^*A v_i = \sigma_i^2 v_i \rightsquigarrow$ 'RIGHT SINGULAR VECTORS'
OF THE MATRIX A