Linear algebra

Stochastic algorithms

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Linear algebra

We overview fundamental concepts in linear algebra

- Matrix and vectors, definitions
- Main matrix operators
- Matrix determinant and rank
- Systems of linear equations
- Matrix inverse
- Eigenvalues and eigenvectors

Matrices and vectors

Linear algebra

Matrices and vectors

Definition

A matrix

A matrix $\mathbf{A}$ of dimension $(m \times n)$ is a table of elements

- *m rows and n columns*

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{bmatrix}$$

We use the notation $\mathbf{A} = \{a_{i,j}\}$ to denote that matrix $\mathbf{A}$ has elements $a_{i,j}$

- *At the intersection of row i with column j*

Matrices and vectors (cont.)

We consider real matrices, in which element $a_{i,j} \in \mathcal{R}$

To indicate a matrix, we use upper-case bold letters

$$\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$$

$\mathbf{A}^{m \times n}$ indicates a matrix $\mathbf{A}$ of dimension $(m \times n)$

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Matrices and vectors (cont.)

Example

Consider the (2×3) matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

The elements of the matrix

- $a_{1,1} = 1$
- $a_{1,2} = 3.5$
- $a_{1,3} = 2$
- $a_{2,1} = 0$
- $a_{2,2} = 1$
- $a_{2,3} = 3$

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Matrices and vectors

Definition

A scalar and a vector

A scalar is a matrix of dimension (1×1)

A vector is a matrix in which one of the dimensions is one

- *Row-vector, a* $(m \times 1)$ *matrix (a column)*
- *Column-vector,* $(1 \times n)$ *matrix (a row)*

To indicate a vector, we use lower-case bold letters

$$\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$$

$\mathbf{x} \in \mathcal{R}^m$ indicates a column-vector $\mathbf{x}$ of dimension $(m \times 1)$

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Matrices and vectors (cont.)

Example

Consider the 2 vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 2 & 3 & 0 & 1.4 \end{bmatrix}$$

The type of vectors

- Vector $\mathbf{x}$ has dimension (3×1), a column-vector with 3 components
- Vector $\mathbf{y}$ has dimension (1×4), a row-vector with 4 components

Matrices and vectors (cont.)

A $(m \times n)$ matrix is understood as consisting of n $(m \times 1)$ column-vectors

$$\rightsquigarrow \quad \mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & | & \mathbf{a}_2 & | & \cdots & | & \mathbf{a}_n \end{bmatrix}$$

$\rightsquigarrow$ $\mathbf{a}_i$ is the i-th column

A $(m \times n)$ matrix is understood as consisting of m $(1 \times n)$ row-vectors

$$\rightsquigarrow \quad \mathbf{A} = \begin{bmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_n \end{bmatrix}$$

$\rightsquigarrow$ $\mathbf{a}'_i$ is the i-th row

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Matrices and vectors (cont.)

Example

Consider the (2×3) matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

As component columns

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3.5 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

As component rows

$$\mathbf{a}'_1 = \begin{bmatrix} 1 & 3.5 & 2 \end{bmatrix}, \quad \mathbf{a}'_2 = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix}$$

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Matrices and vectors (cont.)

Definition

A square matrix

A matrix $\mathbf{A}$ is said to be a square matrix if its dimension is $(n \times n)$

- *The number of rows equals the number of columns*

The diagonal of a square matrix $\mathbf{A}$ of order n is the set of elements

$$\{a_{1,1}, a_{2,2}, \cdots, a_{n,n}\}$$

They have the same row- and column-number

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Matrices and vectors (cont.)

Example

Consider the order 4 square matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 4 & 3 \\ 3 & 2 & 6 \end{bmatrix}$$

The diagonal

$$\{1, 4, 6\}$$

Matrices and vectors

Definition

Square matrices

Diagonal

- *All off-diagonal elements are zero*

Block-diagonal

- *All elements are zero except for some square blocks along the diagonal*

Lower- (upper-) triangular

- *All elements above (below) the diagonal are zero*

Lower- (upper-) block-triangular

- *All elements above (below) the diagonal are zero except for some square blocks along the diagonal*

Identity matrix

- *A diagonal matrix whose diagonal elements are equal to one,* $\mathbf{I}$ *or* $\mathbf{I}_n$

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Matrices and vectors (cont.)

Example

Consider the order 4 square matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 6 & 0 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 2 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\leadsto$ Matrix $\mathbf{A}$ is diagonal

$\leadsto$ Matrix $\mathbf{B}$ is lower-triangular

$\leadsto$ Matrix $\mathbf{C}$ is upper-triangular

$\leadsto$ Matrix $\mathbf{I}$ is an identity of order 3

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Matrices and vectors (cont.)

Matrix $\tilde{\mathbf{A}}$ is block-diagonal

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{A}}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{A}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Three blocks, $\tilde{\mathbf{A}}_1$, $\tilde{\mathbf{B}}_2$ and $\tilde{\mathbf{B}}_3$, one of order 2 and 2 of order 1

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Matrices and vectors (cont.)

Matrix $\tilde{\mathbf{A}}$ is upper-block-triangular

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{B}}_1 & \tilde{\mathbf{B}}_3 \\ \mathbf{0} & \tilde{\mathbf{B}}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Two diagonal blocks, $\tilde{\mathbf{B}}_1$ and $\tilde{\mathbf{B}}_2$, both of order 2

Matrix operators

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Transposition

Matrix operators

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Transposition

Definition

Matrix transposition

Consider a matrix $\mathbf{A} = \{a_{i,j}\}$ *of dimension* $(m \times n)$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

The ***transpose*** *of* $\mathbf{A}$ *is the matrix* $\mathbf{A}^T = \{a'_{i,j} = a_{j,i}\}$ *of dimension* $(n \times m)$

$$\mathbf{A}^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}$$

- *On the* j*-th row of* $\mathbf{A}^T$, *the elements of the* j*-th column of* $\mathbf{A}$
- *On the* i*-th column of* $\mathbf{A}^T$, *the elements of the* j*-th row of* $\mathbf{A}$

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Transposition (cont.)

Example

Consider the (2×3) matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Its transpose

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 \\ 3.5 & 1 \\ 2 & 3 \end{bmatrix}$$

Transposition (cont.)

The following properties hold

- If $\mathbf{D}$ is a diagonal matrix, we have $\mathbf{D} = \mathbf{D}^T$
- If $\mathbf{A}$ is lower-triangular, then $\mathbf{A}^T$ is upper-triangular
- If $\mathbf{A}$ is upper-triangular, then $\mathbf{A}^T$ is lower-triangular
- If $\mathbf{A}$ is a row-vector, $\mathbf{A}^T$ is a column-vector
- If $\mathbf{A}$ is a column-vector, $\mathbf{A}^T$ is a row-vector
- If $\mathbf{B} = \mathbf{A}^T$, we have $\mathbf{B}^T = (\mathbf{A}^T)^T$

Sum and difference

Matrix operators

Sum and difference

Definition

Matrix sum and difference

Consider two matrices $\mathbf{A} = \{a_{i,j}\}$ *and* $\mathbf{B} = \{b_{i,j}\}$ *both of dimension* $(m \times n)$

Define the ***sum*** *of* $\mathbf{A}$ *and* $\mathbf{B}$ *as the* $(m \times n)$ *matrix* $\mathbf{C} = \{c_{i,j} = a_{i,j} + b_{i,j}\}$

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$= \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdots & a_{1,j}+b_{1,j} & \cdots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} & \cdots & a_{2,j}+b_{2,j} & \cdots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1}+b_{i,1} & a_{i,2}+b_{i,2} & \cdots & a_{i,j}+b_{i,j} & \cdots & a_{i,n}+b_{i,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{m,2}+b_{m,2} & \cdots & a_{m,j}+b_{m,j} & \cdots & a_{m,n}+b_{m,n} \end{bmatrix}$$

- *Element* $c_{i,j}$ *is equal to the sum of elements* $a_{i,j}$ *and* $b_{i,j}$

Define the ***difference*** *of* $\mathbf{A}$ *and* $\mathbf{B}$ *as the* $(m \times n)$ *matrix*

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \{d_{i,j} = a_{i,j} - b_{i,j}\}$$

Sum and difference (cont.)

Example

Consider the two (2×3) matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Their sum

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 2.5 & 5 \\ 4 & 6 & 9 \end{bmatrix}$$

Their difference

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} 0 & 1.5 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

Matrix-scalar product

Matrix operators

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Matrix-scalar product

Definition

Matrix-scalar product

Consider a number $s \in \mathcal{R}$ *and a* $(m \times n)$ *matrix* $\mathbf{A} = \{a_{i,j}\}$

*Define **matrix-scalar product** of* $\mathbf{A}$ *and* s *as the* $(m \times n)$ *matrix* $\mathbf{B} = s\mathbf{A}$

$$\mathbf{B} = s\mathbf{A} = \begin{bmatrix} s \cdot a_{1,1} & \cdots & s \cdot a_{1,j} & \cdots & s \cdot a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{i,1} & \cdots & s \cdot a_{i,j} & \cdots & s \cdot a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{m,1} & \cdots & s \cdot a_{m,j} & \cdots & s \cdot a_{m,n} \end{bmatrix}$$

- *Element* $b_{i,j}$ *is equal to the product of* s *and element* $a_{i,j}$

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Matrix-scalar product (cont.)

Example

Let $s = 4$ and let $\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

We have,

$$s\mathbf{A} = 4 \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 14 & 8 \\ 0 & 4 & 12 \end{bmatrix}$$

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Matrix-matrix product

Matrix operators

Matrix-matrix product

Definition

Matrix-matrix product

Let $\mathbf{A} = \{a_{i,j}\}$ *be a* $(m \times n)$ *matrix and let* $\mathbf{B} = \{b_{i,j}\}$ *be a* $(n \times p)$ *matrix*

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product between $\mathbf{A}$ *and* $\mathbf{B}$ *is defined as a* $(m \times p)$ *matrix* $\mathbf{C} = \{c_{i,j}\}$

$$\mathbf{C} = \{c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}\}$$

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Matrix-matrix product (cont.)

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,j} & \cdots & c_{1,p-1} & c_{1,p} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,j} & \cdots & c_{2,p-1} & c_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{i,1} & c_{i,2} & \cdots & c_{i,j} & \cdots & c_{i,p-1} & c_{i,p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{m-1,1} & c_{m-1,2} & \cdots & c_{m-1,j} & \cdots & c_{m-1,p-1} & c_{m-1,p} \\ c_{m,1} & c_{m,2} & \cdots & c_{m,j} & \cdots & c_{m,p-1} & c_{m,p} \end{bmatrix}$$

Element $c_{i,j}$ of matrix $\mathbf{C}$ is given by the scalar product between $\mathbf{a}'_i$ and $\mathbf{b}_j$

$$c_{i,j} = \mathbf{a}'_i \mathbf{b}_j = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,k} & \cdots & a_{i,n} \end{bmatrix} \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{k,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

$$= a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

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Matrix-matrix product (cont.)

Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

We have,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} 1 \cdot 1 + 3.5 \cdot 3 + 2 \cdot 5 & 1 \cdot 2 + 3.5 \cdot 4 + 2 \cdot 6 \\ 0 \cdot 1 + 1 \cdot 3 + 3 \cdot 5 & 0 \cdot 2 + 1 \cdot 4 + 3 \cdot 6 \\ 0 \cdot 1 + 0 \cdot 3 + 1 \cdot 5 & 0 \cdot 2 + 0 \cdot 4 + 1 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 21.5 & 28 \\ 18 & 22 \\ 5 & 6 \end{bmatrix}$$

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Matrix-matrix product (cont.)

For every $(m \times n)$ matrix $\mathbf{A}$, we have

$$\underbrace{\mathbf{I}_m}_{(m \times m)} \underbrace{\mathbf{A}}_{(m \times n)} = \underbrace{\mathbf{A}}_{(m \times n)} \underbrace{\mathbf{I}_n}_{(n \times n)} = \underbrace{\mathbf{A}}_{(m \times n)}$$

Right- and left-multiplication of matrix $\mathbf{A}$ by an identity matrix

Matrix-matrix product (cont.)

Matrix product is not necessarily commutative, $\mathbf{AB} \neq \mathbf{BA}$

$$\underbrace{\mathbf{A}}_{(m\times n)}\underbrace{\mathbf{B}}_{(n\times p)} = \underbrace{\mathbf{C}}_{(m\times p)}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product $\mathbf{BA}$ is not defined

$\mathbf{A}$ and $\mathbf{B}$ must be both square and of the same order

- (necessary condition)

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Matrix-matrix product (cont.)

A $(n \times n)$ diagonal matrix $\mathbf{D}$ commutes with any $(n \times n)$ matrix $\mathbf{A}$

$$\mathbf{DA} = \mathbf{AD}$$

$$\underbrace{\mathbf{D}}_{(n\times n)}\underbrace{\mathbf{A}}_{(n\times n)} = \underbrace{\mathbf{C}}_{(n\times n)}$$

$$= \begin{bmatrix} d_{1,1} & \cdots & d_{1,k} & \cdots & d_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{i,1} & \cdots & d_{i,k} & \cdots & d_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{n,1} & \cdots & d_{n,k} & \cdots & d_{n,n} \end{bmatrix} \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,j} & \cdots & a_{k,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

$$\rightsquigarrow \quad c_{ij} = \cancel{d_{i,1}}\, a_{1,j} + \cdots + d_{i,k}\, a_{k,j} + \cdots + \cancel{d_{i,n}}\, a_{n,j} = d_{i,k}\, a_{k,j}$$

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Matrix-matrix product (cont.)

$$\underbrace{\mathbf{A}}_{(n\times n)}\underbrace{\mathbf{D}}_{(n\times n)} = \underbrace{\mathbf{C}}_{(n\times n)}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,j} & \cdots & a_{k,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} d_{1,1} & \cdots & d_{1,k} & \cdots & d_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{i,1} & \cdots & d_{i,k} & \cdots & d_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{n,1} & \cdots & d_{n,k} & \cdots & d_{n,n} \end{bmatrix}$$

$$\rightsquigarrow \quad c_{ij} = a_{k,1}\, \cancel{d_{1,k}} + \cdots + a_{k,j}\, d_{i,k} + \cdots + a_{k,n}\, \cancel{d_{n,k}} = a_{k,j}\, d_{i,k}$$

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Matrix-matrix product (cont.)

Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$

We have,

$$\mathbf{AB} = \begin{bmatrix} 6 & 6 \\ 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 2 & 10 \end{bmatrix} = \mathbf{BA}$$

■

Matrix-matrix product (cont.)

Proposition

Let $\mathbf{A}$ *be a* $(m \times n)$ *matrix*

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_m \end{bmatrix}$$

Let $\mathbf{B}$ *be a* $(n \times p)$ *matrix*

$$\mathbf{B} = \big[\mathbf{b}_1 | \mathbf{b}_2 | \cdots | \mathbf{b}_p\big]$$

Let $\mathbf{S}$ *and* $\mathbf{Z}$ *be order* m *and order* p *diagonal matrices*

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & s_m \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & z_p \end{bmatrix}$$

We can state a number of identities

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Matrix-matrix product (cont.)

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_m \end{bmatrix} \mathbf{B} = \begin{bmatrix} \mathbf{a}'_1\mathbf{B} \\ \mathbf{a}'_2\mathbf{B} \\ \vdots \\ \mathbf{a}'_m\mathbf{B} \end{bmatrix}$$

$$= \mathbf{A}\big[\mathbf{b}_1 | \mathbf{b}_2 | \cdots | \mathbf{b}_p\big] = \big[\mathbf{A}\mathbf{b}_1 | \mathbf{A}\mathbf{b}_2 | \cdots | \mathbf{A}\mathbf{b}_p\big]$$

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Matrix-matrix product (cont.)

$$\mathbf{SA} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & s_m \end{bmatrix} \begin{bmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \vdots \\ \mathbf{a}'_m \end{bmatrix} = \begin{bmatrix} s_1\mathbf{a}'_1 \\ s_2\mathbf{a}'_2 \\ \vdots \\ s_m\mathbf{a}'_m \end{bmatrix}$$

$$\mathbf{BZ} = \big[\mathbf{b}_1 | \mathbf{b}_2 | \cdots | \mathbf{b}_p\big] \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & z_p \end{bmatrix} = \big[z_1\mathbf{b}_1 | z_2\mathbf{b}_2 | \cdots | z_p\mathbf{b}_p\big]$$

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Matrix powers

Definition

Powers of a matrix

Let $\mathbf{A}$ *be a square matrix of order* n

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

The k*-th power of matrix* $\mathbf{A}$ *is defined as matrix* $\mathbf{A}^k$ *of order* n

$$\mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k \text{ times}}$$

Special cases,

- $\rightsquigarrow$ $\mathbf{A}^{k=0} = \mathbf{I}$
- $\rightsquigarrow$ $\mathbf{A}^{k=1} = \mathbf{A}$

Matrix powers (cont.)

Example

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

We have,

$$\mathbf{A}^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A}^1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A}^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A}^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$$
$$\cdots = \cdots$$

Matrix exponential

Matrix operators

The matrix exponential

Definition

The matrix exponential

Let z *be some scalar, by definition its exponential is a scalar*

$$\rightsquigarrow \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

The series always converges

Let $\mathbf{A}$ *be a* $(n \times n)$ *matrix, by definition its exponential is a* $(n \times n)$ *matrix*

$$\rightsquigarrow \quad e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

The series always converges

The matrix exponential (cont.)

Proposition

The matrix exponential of block-diagonal matrices

Consider a block-diagonal matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_q \end{bmatrix}$$

We have,

$$\rightsquigarrow \quad e^{\mathbf{A}} = \begin{bmatrix} e^{\mathbf{A}_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{A}_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & e^{\mathbf{A}_q} \end{bmatrix}$$

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The matrix exponential (cont.)

Proof

For all $k \in \mathcal{N}$, we have

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{A}_1^k & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^k & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_q^k \end{bmatrix}$$

Thus,

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\mathbf{A}_1^k}{k!} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sum_{k=0}^{\infty} \frac{\mathbf{A}_2^k}{k!} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sum_{k=0}^{\infty} \frac{\mathbf{A}_q^k}{k!} \end{bmatrix}$$

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The matrix exponential (cont.)

Proposition

The matrix exponential of diagonal matrixes

Consider a diagonal $(n \times n)$ *matrix* $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

We have,

$$\rightsquigarrow \quad e^{\mathbf{A}} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

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The result is a special case of the previous proposition

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The matrix exponential (cont.)

Example

Consider the (3×3) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

We are interested in its matrix exponential

We have,

$$e^{\mathbf{A}} = \begin{bmatrix} e^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{0.5} \end{bmatrix}$$

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Determinant

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Determinant (cont.)

Definition

Matrix minors

Consider a square matrix $\mathbf{A}$ *of order* $n \geq 2$

The ***minor*** (i, j) *of matrix* $\mathbf{A}$ *is a square matrix* $\mathbf{A}_{i,j}$ *of order* $(n-1)$

$\rightsquigarrow$ *From* $\mathbf{A}$ *by deleting the* i*-th row and the* j*-th column*

$$\mathbf{A}_{i,j} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cancel{a_{1,j}} & \cdots & a_{1,p} \\ a_{2,1} & a_{2,2} & \cdots & \cancel{a_{2,j}} & \cdots & a_{2,p} \\ \vdots & \vdots & \ddots & \cancel{\vdots} & \ddots & \vdots \\ \cancel{a_{i,1}} & \cancel{a_{i,2}} & \cancel{\cdots} & \cancel{a_{i,j}} & \cancel{\cdots} & \cancel{a_{i,p}} \\ \vdots & \vdots & \ddots & \cancel{\vdots} & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & \cancel{a_{m,j}} & \cdots & a_{m,p} \end{bmatrix}$$

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Determinant (cont.)

Example

Consider the (3×3) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The minors of order 2

$$\mathbf{A}_{1,1} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad \mathbf{A}_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

$$\mathbf{A}_{2,1} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}, \quad \mathbf{A}_{2,2} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

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Determinant (cont.)

Definition

Matrix determinant

Consider a square matrix $\mathbf{A}$ *of order* n

The ***determinant*** *of* $\mathbf{A}$ *is a real number*

$$\rightsquigarrow \quad \det(\mathbf{A}) = |\mathbf{A}|$$

- *For* $n = 1$, *let* $\mathbf{A} = [a_{1,1}]$, *we have*

$$\rightsquigarrow \quad \det(\mathbf{A}) = a_{1,1}$$

- *For* $n \geq 2$, *we have*

$$\rightsquigarrow \quad \det(\mathbf{A}) = a_{1,1}\hat{a}_{1,1} + a_{2,1}\hat{a}_{2,1} + \cdots + a_{n,1}\hat{a}_{n,1} = \sum_{i=1}^{n} a_{i,1}\hat{a}_{i,1}$$

$\hat{a}_{i,j}$, *the* ***cofactor*** *of element* (i, j), *is a scalar*

- *It is the determinant of minor* $\mathbf{A}_{i,j}$, *multiplied by* $(-1)^{i+j}$

■

Determinant (cont.)

If $\det(\mathbf{A}) = 0$, then matrix $\mathbf{A}$ is said to be **singular**

- It is otherwise said to be non-singular

This definition of determinant allows for a recursive computation

- The determinant of a matrix of order n is a function
- The determinants of matrices of order $(n-1)$
- The determinants of matrices of order $(n-2)$
- $\cdots$, $n = 1$

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Determinant (cont.)

Example

Consider a matrix $\mathbf{A}$ of order $n = 2$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

We are interested in computing its determinant

We have,

$$\mathbf{A}_{1,1} = \begin{bmatrix} a_{2,2} \end{bmatrix}, \quad \rightsquigarrow \quad \hat{a}_{1,1} = a_{2,2}$$
$$\mathbf{A}_{2,1} = \begin{bmatrix} a_{1,2} \end{bmatrix}, \quad \rightsquigarrow \quad \hat{a}_{2,1} = -a_{1,2}$$

The determinant

$$\det(\mathbf{A}) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

For $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$, we obtain $\det(\mathbf{A}) = 2 \cdot 4 - 6 \cdot 1 = 2$

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Determinant (cont.)

Example

Consider a matrix $\mathbf{A}$ of order $n = 3$

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

We are interested in computing its determinant

The cofactors of the elements along the first column

$$\hat{a}_{1,1} = \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} = a_{2,2}a_{3,3} - a_{2,3}a_{3,2}$$

$$\hat{a}_{2,1} = (-1)\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} = -(a_{1,2}a_{3,3} - a_{1,3}a_{3,2})$$

$$\hat{a}_{3,1} = \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} = a_{1,2}a_{2,3} - a_{1,3}a_{2,2}$$

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Determinant (cont.)

Sum the product of each element $a_{i,1}$ along the first column by cofactor $\hat{a}_{i,1}$

$$\rightsquigarrow \quad \det(\mathbf{A}) = a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{2,1}(a_{1,2}a_{3,3} - a_{1,3}a_{3,2}) + a_{3,1}(a_{1,2}a_{2,3} - a_{1,3}a_{2,2})$$

■

Determinant (cont.)

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

Computation of $\det(\mathbf{A})$ develops along the elements of $\mathbf{A}$'s first column

$$\det(\mathbf{A}) = a_{1,1}\hat{a}_{1,1} + a_{2,1}\hat{a}_{2,1} + \cdots + a_{n,1}\hat{a}_{n,1} = \sum_{i=1}^{n} a_{i,1}\hat{a}_{i,1}$$

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Determinant (cont.)

Analogous formulas develop along the elements of any column

For column j, we have

$$\det(\mathbf{A}) = a_{1,j}\hat{a}_{1,j} + a_{2,j}\hat{a}_{2,j} + \cdots + a_{n,j}\hat{a}_{n,j} = \sum_{i=1}^{n} a_{i,j}\hat{a}_{i,j}$$

Similarly, formulas develop along the elements of any row

For row i, we have

$$\det(\mathbf{A}) = a_{i,1}\hat{a}_{i,1} + a_{i,2}\hat{a}_{i,2} + \cdots + a_{i,n}\hat{a}_{i,n} = \sum_{j=1}^{n} a_{i,j}\hat{a}_{i,j}$$

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Determinant (cont.)

Some relationships

The determinant of a diagonal or triangular matrix $\mathbf{A}$ is equal to the product of the elements along the diagonal

$$\rightsquigarrow \quad \det(\mathbf{A}) = a_{1,1}a_{2,2}\cdots a_{n,n}$$

The determinant of a block-diagonal or block-triangular matrix $\mathbf{A}$ is equal to the product of the determinants of the blocks along the diagonal

$$\rightsquigarrow \quad \det(\mathbf{A}) = \prod_{i=1}^{q} \det(\tilde{\mathbf{A}}_i)$$

The determinant of the product of square matrices $\mathbf{C} = \mathbf{AB}$ is equal to the product of the determinants

$$\rightsquigarrow \quad \det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$$

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Rank and kernel

Definition

Matrix rank

The rank of a $(m \times n)$ matrix $\mathbf{A}$ is equal to the number of columns (or rows, equivalently) of the matrix that are linearly independent

$$\rightsquigarrow \quad rank(\mathbf{A})$$

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Rank and kernel (cont.)

Proposition

Define the minors of matrix $\mathbf{A}$ any matrix obtained from $\mathbf{A}$ by deleting an arbitrary number of rows and columns

$\rightsquigarrow$ $rank(\mathbf{A})$ *equals the order of the largest non-singular square minor*

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Rank and kernel (cont.)

Definition

Matrix kernel or null space

Consider a $(m \times n)$ matrix $\mathbf{A}$

We define the null space or kernel

$$\rightsquigarrow \quad ker(\mathbf{A}) = \{\mathbf{x} \in \mathcal{R}^n | \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

It is all vectors $\mathbf{x} \in \mathcal{R}^n$ that left-multiplied by $\mathbf{A}$ produce the null vector

The set is a vector space, its dimension is called the nullity of matrix $\mathbf{A}$

$$\rightsquigarrow \quad null(\mathbf{A})$$

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Rank and kernel (cont.)

The null vector always belong to $\ker(\mathbf{A})$

If the null vector is also the only element of $\ker(\mathbf{A})$, then $\text{null}(\mathbf{A}) = 0$

For a matrix $\mathbf{A}$ with n columns we have

$$\rightsquigarrow \quad \text{rank}(\mathbf{A}) + \text{null}(\mathbf{A})$$

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Systems of equations

Proposition

Consider a system of n linear equations in n unknowns

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

- $\mathbf{A}$ *is a* $(n \times n)$ *matrix of coefficients*
- $\mathbf{b}$ *is a* $(n \times 1)$ *vector of known terms*
- $\mathbf{x}$ *is a* $(n \times 1)$ *vector of unknowns*

If matrix $\mathbf{A}$ *is non-singular, the system admits one and only one solution*

If $\mathbf{A}$ *is singular, let* $\mathbf{M} = [\mathbf{A}|\mathbf{b}]$ *be a* $[n \times (n+1)]$ *matrix*

- *If* $rank(\mathbf{A}) = rank(\mathbf{M})$, *system has infinite solutions*
- *If* $rank(\mathbf{A}) < rank(\mathbf{M})$, *system has no solutions*

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Systems of equations (cont.)

Example

Consider a system of two equations and two unknowns

$$\begin{cases} 4 & = 2x_1 + x_2 \\ 14 & = 6x_1 + 4x_2 \end{cases}$$

In matrix form, $\mathbf{A}\mathbf{b} = \mathbf{x}$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The determinant of matrix $\mathbf{A}$, $\det(\mathbf{A}) = 2$

- One and only one solution

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Systems of equations (cont.)

The system can be solved by substitution

$$\begin{cases} x_1 = 2 - 1/2x_2 \\ 6x_1 + 4x_2 = 14 \end{cases} \rightsquigarrow \begin{cases} x_1 = 2 - 1/2x_2 \\ x_2 = 2 \end{cases} \rightsquigarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$$

The solution in matrix form, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Systems of equations (cont.)

Example

Consider a system of two equations and two unknowns

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases} \rightsquigarrow \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

This system of equations has not got any solution, as $\text{rank}([\mathbf{A}|\mathbf{b}]) > \text{rank}(\mathbf{A})$

- $\rightsquigarrow$ Matrix $\mathbf{A}$ is singular and rank 1
- $\rightsquigarrow$ Matrix $[\mathbf{A}|\mathbf{b}]$ is rank 2

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Systems of equations (cont.)

Example

Consider the linear system of two equation and two unknowns

$$\begin{cases} 1 = x_1 + 2x_2 \\ 2 = 2x_1 + 4x_2 \end{cases} \rightsquigarrow \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\mathbf{b}} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$$

This system of equations has infinite solutions, as $\text{rank}([\mathbf{A}|\mathbf{b}]) = \text{rank}(\mathbf{A})$

- $\rightsquigarrow$ Matrix $\mathbf{A}$ is singular and rank 1
- $\rightsquigarrow$ Matrix $[\mathbf{A}|\mathbf{b}]$ is rank 1

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Definition

Matrix inverse

Consider a square matrix $\mathbf{A}$ of order n

*Define **inverse** of $\mathbf{A}$ as the square matrix $\mathbf{A}^{-1}$ of order n*

$$\rightsquigarrow \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

The inverse of $\mathbf{A}$ exists if and only if $\mathbf{A}$ is non-singular

Inverse (cont.)

Definition

Cofactor and adjunct matrix

Consider a square matrix $\mathbf{A}$ *of order* $n \geq 2$

The cofactor matrix of $\mathbf{A}$ *is a square matrix of order* n *whose element* (i,j) *is the cofactor* $\hat{a}_{i,j}$ *of* $\mathbf{A}$

$$\rightsquigarrow \quad \hat{\mathbf{A}} = \{\hat{a}_{i,j}\}$$

The adjunct matrix of $\mathbf{A}$ *is a a square matrix of order* n *obtained by transposition of the cofactors*

$$\rightsquigarrow \quad adj(\mathbf{A}) = \{\alpha_{i,j} = \hat{a}_{j,i}\}$$

■

Inverse (cont.)

Proposition

Consider a non-singular square matrix $\mathbf{A}$ *of order* n

- *If* $n = 1$, *let* $\mathbf{A} = [a_{1,1}]$, *we have*

$$\mathbf{A}^{-1} = [a_{1,1}^{-1}]$$

- *If* $n \geq 2$, *we have*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} adj(\mathbf{A})$$

■

Systems of equations (cont.)

Proposition

Consider a system of n *linear equations in* n *unknowns* $\mathbf{Ax} = \mathbf{b}$

Suppose that matrix $\mathbf{A}$ *is non-singular*

We have,

$$\rightsquigarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Proof

Left-multiply both sides of $\mathbf{b} = \mathbf{Ax}$ by $\mathbf{A}^{-1}$

$$\mathbf{b} = \mathbf{Ax} \quad \rightsquigarrow \mathbf{A}^{-1}\mathbf{b} = \mathbf{A}^{-1}\mathbf{Ax} \quad \rightsquigarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \quad \rightsquigarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

■

Systems of equations (cont.)

Consider a non-singular diagonal matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \rightsquigarrow \mathbf{A}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n^{-1} \end{bmatrix}$$

Its inverse $\mathbf{A}^{-1}$ is obtained by inverting the diagonal elements

Systems of equations (cont.)

Consider a non-singular blockdiagonal matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix} \rightsquigarrow \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{A}_n^{-1} \end{bmatrix}$$

Its inverse $\mathbf{A}^{-1}$ is obtained by inverting the diagonal blocks

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Systems of equations (cont.)

Consider two non-singular matrices $\mathbf{A}$ and $\mathbf{B}$ of order n

We have,

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Consider a non-singular matrix $\mathbf{A}$ of order n

We have,

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{\det(\mathbf{A})}$$

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Eigenvalues and eigenvectors

Definition

Eigenvalues and eigenvectors

Let $\lambda \in \mathcal{R}$ be some scalar and let $\mathbf{v} \neq \mathbf{0}$ be a $(n \times 1)$ column vector

Consider a square matrix $\mathbf{A}$ of order n

We have,

$$\rightsquigarrow \quad \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- *The scalar λ is called an eigenvalue of $\mathbf{A}$*
- *The vector $\mathbf{v}$ is the associated eigenvector*

Eigenvalues and eigenvectors (cont.)

Proposition

Eigenvalues/eigenvectors of triangular/diagonal matrices

Let $\mathbf{A} = \{a_{i,j}\}$ be a triangular or a diagonal matrix

The eigenvalues of $\mathbf{A}$ are the n diagonal elements $\{a_{i,i}\}$, $i = 1, \dots, n$

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Eigenvalues and eigenvectors (cont.)

Example

Consider the matrices

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

We are interested in their eigenvalues

The three matrices are triangular or diagonal

We have,

- ⇝ Matrix $\mathbf{A}_1$ has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 3$
- ⇝ Matrix $\mathbf{A}_2$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$
- ⇝ Matrix $\mathbf{A}_3$ has eigenvalues $\lambda_1 = \lambda_2 = 3$ and $\lambda_3 = -2$

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Eigenvalues and eigenvectors (cont.)

Definition

Characteristic polynomial

*The **characteristic polynomial** of a square matrix $\mathbf{A}$ of order n*

The n-order polynomial in the variable s

$$\rightsquigarrow \quad P(s) = \det(s\mathbf{I} - \mathbf{A})$$

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Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

We are interested in its characteristic polynomial

We first calculate the matrix $(s\mathbf{I} - \mathbf{A})$

$$(s\mathbf{I} - \mathbf{A}) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s-2 & -1 \\ -3 & s-4 \end{bmatrix}$$

- ⇝ The elements are function of s

The determinant of the matrix

$$\rightsquigarrow \quad \det(s\mathbf{I} - \mathbf{A}) = (s-2)(s-4) - 3 = s^2 - 6s + 5$$

This is also the characteristic polynomial $P(s)$

Eigenvalues and eigenvectors (cont.)

Proposition

Eigenvalues as roots of the characteristic polynomial

The eigenvalues of a matrix $\mathbf{A}$ *of order* n *are the roots of its characteristic polynomial, that is the solutions to the equation* $P(s) = \det(s\mathbf{I} - \mathbf{A}) = 0$

Let λ *be an eigenvalue of matrix* $\mathbf{A}$

Each eigenvector $\mathbf{v}$ *associated to it is a non-trivial solution to the system*

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$

$\mathbf{0}$ *is a* $(n \times 1)$ *column-vector whose elements are all zero*

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Eigenvalues and eigenvectors (cont.)

Proof

An eigenvalue λ and an eigenvector $\mathbf{v}$ must satisfy

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$ follows from this

The non-trivial solution $\mathbf{v} \neq \mathbf{0}$ is admissible iff matrix $(\lambda\mathbf{I} - \mathbf{A})$ is singular

$$\rightsquigarrow \quad \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

Thus, λ is root to the characteristic polynomial of matrix $\mathbf{A}$ ■

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Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Its eigenvalues

$$\lambda_{1|2} = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm 4}{2} \quad \rightsquigarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 5 \end{cases}$$

We are interested in its eigenvectors

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Eigenvalues and eigenvectors (cont.)

Consider the eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

Eigenvector $\mathbf{v}_1$ corresponds to eigenvalue $\lambda_1 = 1$

- It must satisfy $(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{v}_1 = \mathbf{0}$

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_1 = \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \quad \begin{cases} 0 = -a - b \\ 0 = -3a - 3b \end{cases}$$

If the first equation is satisfied then also the second one will be

- $\rightsquigarrow$ The two equations are linearly dependent
- Always with $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$

Eigenvalues and eigenvectors (cont.)

We limit ourselves and consider only one equation

- Say, $b = -a$

The choice of the first component is arbitrary, then $b = -a$

Let $a = 1$, then we have

$$\rightsquigarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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Eigenvalues and eigenvectors (cont.)

Consider the eigenvector

$$\mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

Eigenvector $\mathbf{v}_2$ corresponds to eigenvalue $\lambda_2 = 5$

- It must satisfy $(\lambda_2 \mathbf{I} - \mathbf{A})\mathbf{v}_2 = \mathbf{0}$

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{v}_2 = \begin{bmatrix} 3 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \quad \begin{cases} 0 = 3c - d \\ 0 = -3c + d \end{cases}$$

If the first equation is satisfied then also the second one will be

- Again, the two equations are linearly dependent

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Eigenvalues and eigenvectors (cont.)

By considering only the first equation, we have $d = 3c$

As the choice of the first component is arbitrary, we set $c = 1$

$$\rightsquigarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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Eigenvalues and eigenvectors (cont.)

We have shown that the system $(\lambda \mathbf{I} - \mathbf{A})\mathbf{v}$ has an infinite number of solutions

- Eigenvectors are determined up to a multiplicative constant
- $\rightsquigarrow$ We always select the non-trivial (non-null) solution

Let $\mathbf{v}$ be the eigenvector associated to eigenvalue λ

- $\rightsquigarrow$ Then, also $\mathbf{y} = r\mathbf{v}$ is eigenvector for λ ($r \neq 0$)

$$\mathbf{A}\mathbf{y} = \mathbf{A}(r\mathbf{v}) = r(\mathbf{A}\mathbf{v}) = r(\lambda\mathbf{v}) = \lambda(r\mathbf{v}) = \lambda\mathbf{y}$$

Eigenvalues and eigenvectors (cont.)

Proposition

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be the eigenvectors of matrix $\mathbf{A}$

Suppose that the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct

It can be shown that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent

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Eigenvalues and eigenvectors (cont.)

Proposition

Let $\mathbf{A}$ be a matrix of order n with n distinct eigenvalues

It can be shown that there exists a set of n linearly independent eigenvectors

The eigenvectors are a base for $\mathcal{R}^n$

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Eigenvalues and eigenvectors (cont.)

Definition

Multiplicity

Consider a square matrix $\mathbf{A}$ or order n

Suppose that $\mathbf{A}$ has $r \leq n$ distinct eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_r$$

$\rightsquigarrow$ $\lambda_i \neq \lambda_j$, *for* $i \neq j$

The characteristic polynomial can be written in the form

$$P(s) = (s - \lambda_1)^{\nu_1}(s - \lambda_2)^{\nu_2} \cdots (s - \lambda_r)^{\nu_r}, \quad \sum_{i=1}^{r} \nu_i = n$$

$\rightsquigarrow$ $\nu_i \in \mathcal{N}^+$ (***algebraic multiplicity***)

Define the ***geometric multiplicity*** *of the eigenvalue λ_i*

- *Number ν_i of linearly independent eigenvectors associated to it*

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Eigenvalues and eigenvectors (cont.)

Proposition

Consider a square matrix $\mathbf{A}$

Let λ be an eigenvalue with algebraic multiplicity ν

The geometric multiplicity μ of the eigenvalue

$$\rightsquigarrow \quad \mu = null(\lambda \mathbf{I} - \mathbf{A}) \leq \nu$$

Proof

For each eigenvector $\mathbf{v}$ associated to λ, we have that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$

$\rightsquigarrow$ $\mathbf{v}$ belongs to the null space of $(\lambda\mathbf{I} - \mathbf{A})$

$\rightsquigarrow$ Dimension of $(\lambda\mathbf{I} - \mathbf{A})$ is $\text{null}(\lambda\mathbf{I} - \mathbf{A})$

■

Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix of order $n = 4$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The characteristic polynomial

$$P(s) = (s-2)^2(s-3)^3$$

The roots

- $\lambda_1 = 2$, algebraic multiplicity $\nu_1 = 2$
- $\lambda_2 = 3$, algebraic multiplicity $\nu_2 = 2$

We are interested in the geometric multiplicities

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Eigenvalues and eigenvectors (cont.)

The geometric multiplicity of the first eigenvalue

$$\begin{aligned} \mu_1 &= \text{null}(\lambda_1 \mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda_1 \mathbf{I} - \mathbf{A}) \\ &= 4 - \text{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right) = 4 - 3 = 1 < \nu_1 \end{aligned}$$

Each eigenvector associated to λ_1 is a linear combination of a single vector

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

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The geometric multiplicity of the second eigenvalue

$$\begin{aligned} \mu_2 &= \text{null}(\lambda_2 \mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda_2 \mathbf{I} - \mathbf{A}) \\ &= 4 - \text{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right) = 2 - 2 = 2 = \nu_2 \end{aligned}$$

Each eigenvector associated to λ_2 is a linear combination of two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$
$$\mathbf{v}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

■