

## Input-output representation

UFC/DC  
SA (CK0191)  
2018.1

Representation and analysis

Homogeneous equation and modes

Force-free evolution

Modes

Aperiodic

Pseudo-periodic

Impulse response

Forced evolution

# Input-output representation

## Stochastic algorithms

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# Input-output models

We concentrate on single-input single-output (SISO) systems

- Input-output (IO) representation
- Linear and stationary systems

Linear ordinary differential equations w/ constant coefficients

- Direct integration of the ODEs in time

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# Input-output models (cont.)

The analysis consists of determining the output signal for a given model

~> **Force-free** and **forced evolution**

~> Decomposition by linearity

We study the homogeneous equation associated to the model equation

~> A definition of the **system modes**

~> They characterise this evolution

The force-free evolution is given by a linear combination of modes

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# Input-output models (cont.)

We study the forced response of the system to the unit impulse

- It is a **canonical regime**

~> Full characterisation

The forced evolution to any input is given as a convolution

- The input and the response to the unit impulse
- The **Duhamel integral**

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# Representation and analysis

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## Representation and analysis

Consider a SISO system represented by a linear, time-invariant IO model

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \quad (1)$$

The **independent variable**

↪ Time,  $t \in \mathcal{R}$

The **dependent variables**

↪ The input,  $u(t) : \mathcal{R} \rightarrow \mathcal{R}$

↪ The output,  $y(t) : \mathcal{R} \rightarrow \mathcal{R}$

The **parameters**

↪  $a_i \in \mathcal{R}$ , with  $i = 0, \dots, n$

↪  $b_i \in \mathcal{R}$ , with  $i = 0, \dots, m$

The order of the system is the highest order of derivation of the output

- We suppose that the system is proper ( $n \geq m$ )

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## Representation and analysis (cont.)

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

### The problem

The fundamental problem of analysis for an IO model representation

↪ Calculate the solution of the differential equation  $y(t)$

↪ From a given initial time  $t_0$  ( $t \geq t_0$ )

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## Representation and analysis (cont.)

This corresponds to determine the evolution of output  $y(t)$ , for  $t \geq t_0$

### Initial conditions

$$\begin{aligned} y(t) \Big|_{t=t_0} &= y_0 \\ \frac{dy(t)}{dt} \Big|_{t=t_0} &= y'_0 \\ &\dots = \dots \\ \frac{d^{n-1}y(t)}{dt^{n-1}} \Big|_{t=t_0} &= y_0^{(n-1)} \end{aligned} \quad (2)$$

The values of the output and its derivatives at the initial time  $t_0$

### Input signal

$$u(t), \quad \text{for } t \geq t_0 \quad (3)$$

The value of the input at the initial time  $t_0$

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## Representation and analysis (cont.)

We overview standard solution methods of ordinary differential equations

↪ And, some less standard methods will be introduced

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## Representation and analysis (cont.)

The past of the system for  $t \in (-\infty, t_0]$  is summarised by the state  $\mathbf{x}(t_0)$

- The initial state is not given/available in the IO representation
  - We have initial conditions for the output and its derivatives
- ↪ The information is equivalent

Initial state and initial conditions are univocally related<sup>1</sup>

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<sup>1</sup>This is strictly true only for observable systems.

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## Representation and analysis (cont.)

### Initial state and initial conditions

If the initial state of the system is null, then all initial conditions are null

$$\mathbf{x}(t_0) = \mathbf{0} \quad \rightsquigarrow \quad y_0 = y'_0 = \dots = y_0^{(n-1)} = 0$$

If the initial state is not null, then not all initial conditions are null

$$\mathbf{x}(t_0) \neq \mathbf{0} \quad \rightsquigarrow \quad (\exists i \in \{0, 1, \dots, n-1\}) \quad y_0^{(i)} \neq 0$$

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## Representation and analysis (cont.)

### The solution (in terms of force-free and forced evolution)

We will consider the evolution of the **output** of a system

- We assumed that this is an **effect**

We assumed that the effect is due to two types of **causes**

- ↪ **Internal causes** in the system, the **initial state**
- ↪ **External causes** to the system, the **input**

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### Representation and analysis (cont.)

Consider a linear system (superposition principle)

The effect is due to the simultaneous existence of both causes

The response can be determined as the sum of effects

- Each cause is acting alone

$$\rightsquigarrow y(t) = y_u(t) + y_f(t), \quad \text{for } t \geq t_0$$

$y_u(t)$  is called the **force-free response**

- Contribution to the output that is only due to **initial state** at  $t = t_0$

$y_f(t)$  is called the **forced response**

- Contribution to the output that is only due to **input** for any  $t \geq t_0$

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### Representation and analysis (cont.)

We want to study the two terms separately and show how they are calculated

- The analysis is restricted to stationary models

We introduce a simplification that will not disrupt generality

- We will assume that the initial time is  $t_0 = 0$

If  $t_0 \neq 0$ , solve for  $\tau = (t - t_0)$  to get  $y(\tau)$  for  $\tau \geq 0$

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### Representation and analysis (cont.)

#### Force-free and forced response

$$y(t) = y_u(t) + y_f(t), \quad \text{for } t \geq t_0$$

$y_u(t) \rightsquigarrow$  can be defined as the system response for an input  $u(t)$  that is identically null for  $t \geq t_0$  and for given initial conditions

$y_f(t) \rightsquigarrow$  can be defined as the system response for a given input  $u(t)$  for  $t \geq t_0$  and for initial conditions that are identically null

## Homogeneous equation and modes

### Input-output representation

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## Homogeneous equation and modes

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We study a simplified form of this differential equation

- The **homogeneous equation** (RHS is null)

$$\rightsquigarrow a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

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## Homogeneous equation and modes (cont.)

### Definition

#### Homogeneous equation

Consider the differential equation of a IO model

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

Suppose that we let the RHS of the IO representation be zero

Define the **homogenous equation** associated to it

$$\rightsquigarrow a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

$$\rightsquigarrow t \in \mathcal{R}$$

$$\rightsquigarrow y : \mathcal{R} \rightarrow \mathcal{R}$$

$$\rightsquigarrow a_i \in \mathcal{R}, \text{ with } i = 0, \dots, n$$

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## Homogeneous equation and modes (cont.)

This form allow us to introduce the fundamental concept of **system mode**

System modes are functions that characterises the system evolution

↪ The number of modes equals the system's order

Linear combinations of modes solve the homogeneous equations

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## Homogeneous equation and modes (cont.)

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

The homogeneous equation is a simplified form of the differential equation

It is possible to associate a polynomial to any homogenous equation

↪ **Characteristic polynomial**

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## Homogeneous equation and modes (cont.)

### Definition

#### Characteristic polynomial

Consider the homogeneous differential equation

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

The **characteristic polynomial** of a homogenous differential equation is a  $n$ -order polynomial in the variable  $s$  whose coefficients correspond to the coefficients  $\{a_0, a_1, \dots, a_n\}$  of the homogeneous equation

$$\rightsquigarrow P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = \sum_{i=0}^n a_i s^i \quad (4)$$

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## Homogeneous equation and modes (cont.)

Consider any polynomial of order  $n$  with real coefficients

- It has  $n$  real or complex-conjugate roots

The roots are solutions of the **characteristic equation**

$$\rightsquigarrow P(s) = \sum_{i=0}^n a_i s^i = 0$$

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## Homogeneous equation and modes (cont.)

In general, there are  $r \leq n$  **distinct roots**  $p_i$ , each with **multiplicity**  $\nu_i$

$$\rightsquigarrow \overbrace{p_1 \dots p_1}^{\nu_1} \overbrace{p_2 \dots p_2}^{\nu_2} \dots \overbrace{p_r \dots p_r}^{\nu_r}$$

$\rightsquigarrow$  If  $i \neq j$ , then  $p_i \neq p_j$

$\rightsquigarrow \sum_{i=1}^r \nu_i = n$

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## Homogeneous equation and modes (cont.)

Consider the particular case in which all roots have multiplicity equal one

$$\rightsquigarrow p_1 \quad p_2 \quad \dots \quad p_{n-1} \quad p_n$$

$\rightsquigarrow$  If  $i \neq j$ , then  $p_i \neq p_j$

$\rightsquigarrow \nu_i = 1$ , for every  $i$

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## Homogeneous equation and modes (cont.)

### Definition

#### Modes

Let  $p$  be a root with multiplicity  $\nu$  of the characteristic polynomial

The **modes** associated to that root are the  $\nu$  functions of time

$$\rightsquigarrow e^{pt}, te^{pt}, t^2 e^{pt}, \dots, t^{\nu-1} e^{pt}$$

A system with a  $n$ -order characteristic polynomial has  $n$  modes



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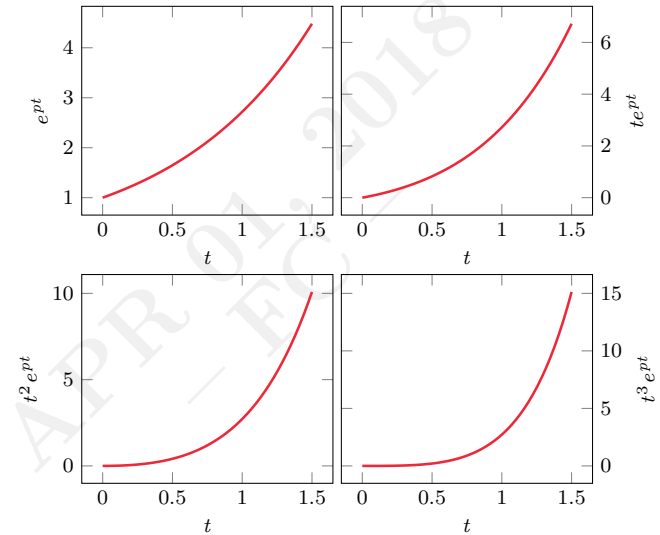
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Let  $p = 1$



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## Homogeneous equation and modes (cont.)

### Example

Consider the following homogenous differential equation

$$3 \frac{d^4 y(t)}{dt^4} + 21 \frac{d^3 y(t)}{dt^3} + 45 \frac{d^2 y(t)}{dt^2} + 39 \frac{dy(t)}{dt} + 12 y(t) = 0$$

The associated characteristic polynomial

$$P(s) = 3s^4 + 21s^3 + 45s^2 + 39s + 12 = 3(s+1)^3(s+4)$$

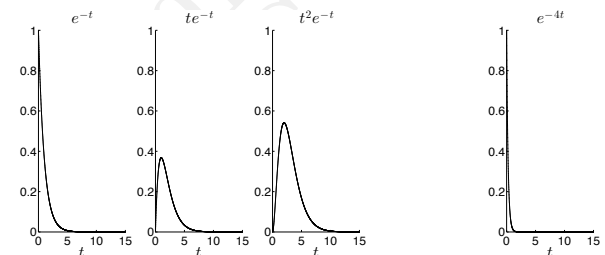
Its roots

$$\rightsquigarrow \begin{cases} p_1 = -1, & \text{multiplicity } \nu_1 = 3 \\ p_2 = -4, & \text{multiplicity } \nu_2 = 1 \end{cases}$$

## Homogeneous equation and modes (cont.)

Such a system has four modes

$$\begin{aligned} p_1 = -1, \quad (\nu_1 = 3) &\rightsquigarrow \begin{cases} e^{-t} \\ te^{-t} \\ t^2 e^{-t} \end{cases} \\ p_2 = -4, \quad (\nu_2 = 1) &\rightsquigarrow \begin{cases} e^{-4t} \end{cases} \end{aligned}$$



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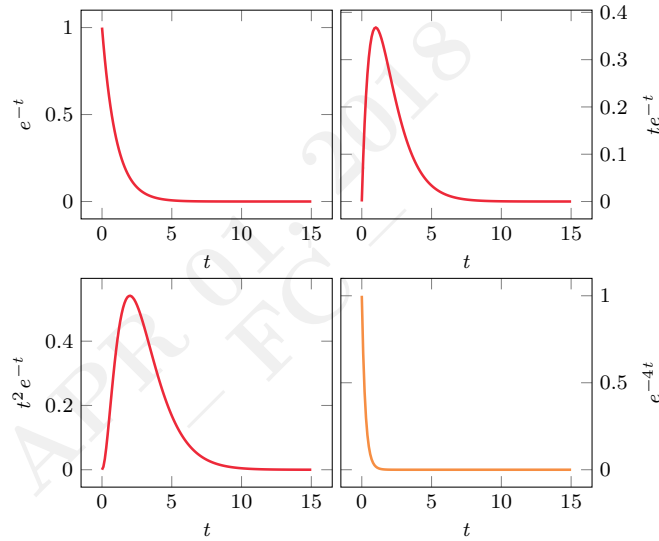
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## Homogeneous equation and modes (cont.)

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As modes are functions, their linear combinations are a family of functions

- The family is parameterised by the coefficients of the combination

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## Homogeneous equation and modes (cont.)

### Definition

#### Linear combinations of modes

A linear combination of the  $n$  modes of a system is a function  $h(t)$

- A sum of the modes, each weighted by some coefficient

Each root  $p_i$  with multiplicity  $\nu_i$  is associated to a combination of  $\nu_i$  terms

$$\begin{aligned} & \rightsquigarrow A_{i,0}e^{p_i t} + A_{i,1}te^{p_i t} + \dots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_i t} \\ & = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_i t}}_{\text{root } p_i} \quad (5) \end{aligned}$$

## Homogeneous equation and modes (cont.)

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$$A_{i,0}e^{p_i t} + A_{i,1}te^{p_i t} + \dots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_i t} = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_i t}}_{\text{root } p_i}$$

There is a total of  $r$  distinct roots

The complete linear combination of modes

$$\begin{aligned} h(t) &= \underbrace{\sum_{k=0}^{\nu_1-1} A_{1,k}t^k e^{p_1 t}}_{\text{root } p_1} + \underbrace{\sum_{k=0}^{\nu_2-1} A_{2,k}t^k e^{p_2 t}}_{\text{root } p_2} + \dots + \underbrace{\sum_{k=0}^{\nu_r-1} A_{r,k}t^k e^{p_r t}}_{\text{root } p_r} \quad (6) \\ &\rightsquigarrow = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_i t} \end{aligned}$$

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## Homogeneous equation and modes (cont.)

Consider the case in which all roots ( $n$ ) have multiplicity equal to one

$$\rightsquigarrow h(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t} = \sum_{i=1}^n A_i e^{p_i t}$$

(We have omitted the second subscript of coefficients  $A$ )

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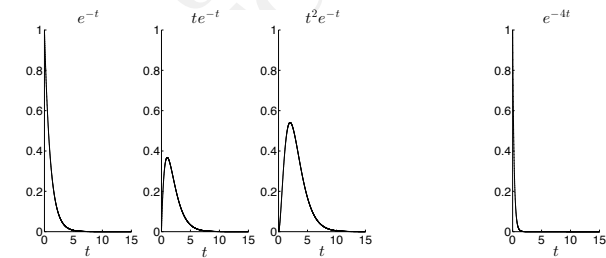
## Homogeneous equation and modes (cont.)

### Example

Consider a system with homogeneous differential equation

$$3 \frac{d^4 y(t)}{dt^4} + 21 \frac{d^3 y(t)}{dt^3} + 45 \frac{d^2 y(t)}{dt^2} + 39 \frac{dy(t)}{dt} + 12 y(t) = 0$$

- Two roots  $p_1 = -1$  ( $\nu_1 = 3$ ) and  $p_2 = -4$  ( $\nu_2 = 1$ )
- Four modes  $e^{-t}$ ,  $te^{-t}$ ,  $t^2 e^{-t}$  and  $e^{-4t}$



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## Homogeneous equation and modes (cont.)

A linear combination of the modes

$$h(t) = \underbrace{A_{1,0} e^{-t} + A_{1,1} t e^{-t} + A_{1,2} t^2 e^{-t}}_{\text{root } p_1} + \underbrace{A_2 e^{-4t}}_{\text{root } p_2}$$

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## Homogeneous equation and modes (cont.)

The modes are known through the characteristic polynomial

The coefficients of their linear combination are parameters

$$h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

The equation is a parametric form of a family of functions

The actual coefficients determine the force-free evolution

↪ From every possible initial condition

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## Homogeneous equation and modes (cont.)

### Theorem

#### *Solution of the homogeneous equation*

Consider the homogeneous equation

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

A real function  $h(t)$  is a solution of the homogeneous equation if and only if it is a linear combination of its modes

$$\rightsquigarrow h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

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## Homogeneous equation and modes (cont.)

### Proof

We demonstrate only the necessary condition

Consider the case in which all  $n$  roots have multiplicity equal to one

$$h(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t} = \sum_{i=1}^n A_i e^{p_i t}$$

For the  $k$ -th order derivative of function  $h(t)$ , we have

$$\begin{aligned} \rightsquigarrow \frac{d^k}{dt^k} h(t) &= \frac{d^k}{dt^k} (A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots + A_n e^{p_n t}) \\ &= p_1^k A_1 e^{p_1 t} + p_2^k A_2 e^{p_2 t} + \dots + p_n^k A_n e^{p_n t} \\ &= \sum_{i=1}^n p_i^k A_i e^{p_i t}, \quad \text{for } k = 0, 1, \dots, n \end{aligned}$$

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## Homogeneous equation and modes (cont.)

$$\frac{d^k}{dt^k} h(t) = \sum_{i=1}^n p_i^k A_i e^{p_i t}, \quad \text{for } k = 0, 1, \dots, n$$

We substitute the  $k$ -th order derivatives of  $h(t)$  in the homogeneous equation

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \sum_{k=0}^n a_k \frac{d^k}{dt^k} h(t) = 0$$

We have,

$$\begin{aligned} \rightsquigarrow \sum_{k=0}^n a_k \frac{d^k}{dt^k} h(t) &= \sum_{k=0}^n a_k \sum_{i=1}^n p_i^k A_i e^{p_i t} \\ &= \sum_{k=0}^n \sum_{i=1}^n a_k p_i^k A_i e^{p_i t} = \sum_{i=1}^n A_i e^{p_i t} \left( \sum_{k=0}^n a_k p_i^k \right) \end{aligned}$$

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## Homogeneous equation and modes (cont.)

$$\sum_{k=0}^n a_k \frac{d^k}{dt^k} h(t) = \sum_{i=1}^n A_i e^{p_i t} \left( \sum_{k=0}^n a_k p_i^k \right) = 0$$

For all values of  $i = 1, \dots, n$ , the term between parenthesis is equal to zero

$\rightsquigarrow$  As  $p_i$  is a root of the characteristic polynomial

$$\sum_{k=0}^n a_k p_i^k = a_n p_i^n + \dots + a_1 p_i + a_0 = P(s) \Big|_{s=p_i} = 0$$

■

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## Homogeneous equation and modes (cont.)

### Complex and conjugate roots

Consider as characteristic polynomial  $P(s)$  whose roots are complex

The modes in  $h(t)$  are complex signals

$$h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

Let  $P(s)$  be a polynomial with real coefficients and complex roots

- Let  $p_i = \alpha_i + j\omega_i$  with multiplicity  $\nu_i$  be a complex root

For each  $p_i = \alpha_i + j\omega_i$  there is a conjugate complex root  $p'_i = \alpha_i - j\omega_i$

- Multiplicity  $\nu'_i = \nu_i$

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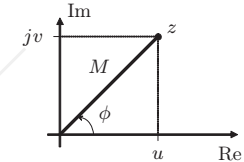
## Homogeneous equation and modes (cont.)

### Complex numbers (Cartesian representation)

Consider the set of complex numbers  $\mathcal{C} = \{u + jv \mid u, v \in \mathcal{R}\}$  ( $j = \sqrt{-1}$ )

A **complex number**

$$z = \text{Re}(z) + j\text{Im}(z) \\ = u + jv$$



It consists of two parts

- Real part**,  $\text{Re}(z) = u$
- Imaginary part**,  $\text{Im}(z) = v$

The **complex conjugate** of  $z$

$$z' = \text{Re}(z) - j\text{Im}(z)$$

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## Homogeneous equation and modes (cont.)

### The complex exponential function

Consider an imaginary number  $z = 0 + j\phi$

We have,

$$\rightsquigarrow e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

The exponential of an imaginary number is a complex number

- Real part,  $\cos(\phi)$
- Imaginary part,  $\sin(\phi)$

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## Homogeneous equation and modes (cont.)

### Proof

Let  $z \in \mathcal{C}$  be any scalar

We have (by definition),

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Let  $z = j\phi$ , for this particular case

$$\rightsquigarrow e^{j\phi} = 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \dots \\ = \left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!} \right] + j \left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} \right]$$

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## Homogeneous equation and modes (cont.)

$$e^{j\phi} = \underbrace{\left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!} \right]}_{\cos(\phi)} + j \underbrace{\left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} \right]}_{\sin(\phi)}$$

The first sum is the McLaurin expansion of the cosine function

$$\begin{aligned} \cos(\phi) &= \sum_{k=0}^{\infty} \frac{\phi^k}{k!} \left[ \frac{d^k \cos(x)}{dx^k} \right]_{x=0} \\ &= \cos(0) - \sin(0)\phi - \cos(0)\frac{\phi^2}{2!} + \sin(0)\frac{\phi^3}{3!} + \dots \\ &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \dots = \left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!} \right] \end{aligned}$$

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## Homogeneous equation and modes (cont.)

$$e^{j\phi} = \left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!} \right] + j \underbrace{\left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} \right]}_{\sin(\phi)}$$

The second sum is the McLaurin expansion of the sine function

$$\begin{aligned} \sin(\phi) &= \sum_{k=0}^{\infty} \frac{\phi^k}{k!} \left[ \frac{d^k \sin(x)}{dx^k} \right]_{x=0} \\ &= \sin(0) - \cos(0)\phi - \sin(0)\frac{\phi^2}{2!} + \cos(0)\frac{\phi^3}{3!} + \dots \\ &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \dots = \left[ \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} \right] \end{aligned}$$

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## Homogeneous equation and modes (cont.)

A pair of roots  $(p_i, p'_i)$  is associated to a linear combination of  $2\nu_i$  modes

$$\rightsquigarrow \underbrace{(A_{i,0} e^{p_i t} + A'_{i,0} e^{p'_i t})}_{k=0} + \dots + \underbrace{t^{\nu_i-1} (A_{i,\nu_i-1} e^{p_i t} + A'_{i,\nu_i-1} e^{p'_i t})}_{k=\nu_i-1} \quad (7)$$

(Pairs of terms for  $k = 0, \dots, \nu_i - 1$  have been grouped up)

$$\underbrace{(A_{i,0} e^{p_i t} + A'_{i,0} e^{p'_i t})}_{k=0} + \dots + \underbrace{t^{\nu_i-1} (A_{i,\nu_i-1} e^{p_i t} + A'_{i,\nu_i-1} e^{p'_i t})}_{k=\nu_i-1}$$

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## Homogeneous equation and modes (cont.)

$$h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

Function  $h(t)$  is a real function (must take real values for all values of  $t$ )

$$\underbrace{(A_{i,0} e^{p_i t} + A'_{i,0} e^{p'_i t})}_{k=0} + \dots + \underbrace{t^{\nu_i-1} (A_{i,\nu_i-1} e^{p_i t} + A'_{i,\nu_i-1} e^{p'_i t})}_{k=\nu_i-1}$$

Coefficients  $A_{i,k}$  and  $A'_{i,k}$  need be complex and conjugated

- For all  $k = 0, \dots, \nu_i - 1$

Then,  $A_{i,k} e^{p_i t}$  and  $A'_{i,k} e^{p'_i t}$  are complex and conjugated

- Their sum will be a real number (as desired)
- For all values of  $t$

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## Homogeneous equation and modes (cont.)

Consider a characteristic polynomial  $P(s)$  that has complex roots

It is possible to derive a *proper* parameterisation of  $h(t)$

↪ (That is, one that only contains real terms)

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## Homogeneous equation and modes (cont.)

### Proof

Consider the term  $(Ae^{pt} + A'e^{p't})$  in which  $(p, p') = \alpha \pm j\omega$

Write the coefficients  $A$  and  $A'$  in polar form

$$A = |A|e^{j\phi}$$

$$A' = |A|e^{-j\phi}$$

↪  $|A|$  denotes the magnitude of coefficient  $A$

↪  $\phi = \arg(A)$  is the phase of coefficient  $A$

We have,

$$\begin{aligned} Ae^{pt} + A'e^{p't} &= |A|e^{j\phi}e^{(\alpha+j\omega)t} + |A|e^{-j\phi}e^{(\alpha-j\omega)t} \\ &= |A|e^{\alpha t} [e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}] \\ &= 2|A|e^{\alpha t} \cos(\omega t + \phi) \text{ [Euler's formula]} \\ &= \underbrace{M}_{M=2|A| \geq 0} e^{\alpha t} \cos(\omega t + \phi) \end{aligned}$$

■

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## Homogeneous equation and modes (cont.)

### Proposition

Consider the contribution of  $(p_i, p'_i) = \alpha_i \pm j\omega_i$  a pair of conjugate complex roots with multiplicity  $\nu_i$  to the linear combination of the  $(2\nu_i)$  modes

$$\underbrace{(A_{i,0}e^{p_i t} + A'_{i,0}e^{p'_i t})}_{k=0} + \cdots + \underbrace{t^{\nu_i-1}(A_{i,\nu_i-1}e^{p_i t} + A'_{i,\nu_i-1}e^{p'_i t})}_{k=\nu_i-1}$$

This sum of terms can be re-written

$$\rightsquigarrow \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \quad (8)$$

The  $2\nu_i$  complex coefficients,  $A_{i,k}$  and  $A'_{i,k}$ , are replaced by  $2\nu_i$  real ones

↪  $M_{i,k}$

↪  $\phi_{i,k}$

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## Homogeneous equation and modes (cont.)

The linear combination of two modes  $(At^k e^{pt} + A't^k e^{p't})$

$$\rightsquigarrow M t^k e^{\alpha t} \cos(\omega t + \phi)$$

The term is denoted **pseudo-periodic mode**

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## Homogeneous equation and modes (cont.)

### Complex numbers (Polar representation)

Consider the set of complex numbers  $\mathcal{C} = \{u + jv \mid u, v \in \mathcal{R}\} \ (j = \sqrt{-1})$

The **complex number**  $z = \text{Re}(z) + j\text{Im}(z) = u + jv$

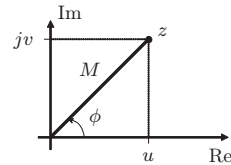
We can define

#### Module

- $M = |z| = \sqrt{u^2 + v^2}$

#### Phase

- $\phi = \arg(z) = \arctan(v/u)$



## Homogeneous equation and modes (cont.)

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The inverse formulæ hold

$$\rightsquigarrow u = M \cos(\phi)$$

$$\rightsquigarrow v = M \sin(\phi)$$

We have,

$$\begin{aligned} z &= u + jv \\ &= M \cos(\phi) + jM \sin(\phi) = M [\cos(\phi) + j \sin(\phi)] \\ &\rightsquigarrow = M e^{j\phi} \end{aligned}$$

The polar representation of a complex number

$$z = M e^{j\phi} = |z| e^{j\phi} = |z| e^{j \arg(z)}$$

The complex conjugate

$$\rightsquigarrow z' = |z| e^{-\arg(z)}$$

■

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## Homogeneous equation and modes (cont.)

### Euler's formula

Relationships to write a periodic function as sum of exponential functions

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

### Proof

$$\begin{aligned} \frac{e^{j\phi} + e^{-j\phi}}{2} &= \frac{[\cos(\phi) + j \sin(\phi)] + [\cos(-\phi) + j \sin(-\phi)]}{2} \\ &= \frac{[\cos(\phi) + j \sin(\phi)] + [\cos(\phi) - j \sin(\phi)]}{2} = \frac{2 \cos(\phi)}{2} = \cos(\phi) \end{aligned}$$

$$\begin{aligned} \frac{e^{j\phi} - e^{-j\phi}}{2} &= \frac{[\cos(\phi) + j \sin(\phi)] - [\cos(-\phi) + j \sin(-\phi)]}{2} \\ &= \frac{[\cos(\phi) + j \sin(\phi)] - [\cos(\phi) - j \sin(\phi)]}{2} = \frac{2j \sin(\phi)}{2} = \sin(\phi) \end{aligned}$$

■

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## Homogeneous equation and modes (cont.)

We can define an alternative structure of the linear combination of modes

- The structure will be equivalent to the form in A

$$\sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

Pairs of conjugate complex roots are expressed using the form in M and  $\phi$

$$\sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

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## Homogeneous equation and modes (cont.)

Let  $R$  be the number of distinct real roots  $p_i$

- Multiplicity  $\nu_i$  ( $i = 1, \dots, R$ )

$$\rightsquigarrow p_1, p_2, \dots, p_i, \dots, p_R$$

Let  $S$  be the number of pairs of distinct complex conjugate roots  $(p_i, p'_i)$

- Multiplicity  $\nu_i$  ( $i = R+1, \dots, R+S$ )

$$\rightsquigarrow (p_{R+1}, p'_{R+1}), (p_{R+2}, p'_{R+2}), \dots, (p_i, p'_i), \dots, (p_{R+S}, p'_{R+S})$$

Clearly, the total number of roots

$$\rightsquigarrow n = \sum_{i=1}^R \nu_i + 2 \sum_{i=R+1}^{R+S} \nu_i$$

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## Homogeneous equation and modes (cont.)

$$n = \sum_{i=1}^R \nu_i + 2 \sum_{i=R+1}^{R+S} \nu_i$$

We consider a particular representation of the linear combination of modes

We distinguish modes associated with real and conjugate complex roots

$$\rightsquigarrow h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \quad (9)$$

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## Homogeneous equation and modes (cont.)

Consider the case in which all roots have multiplicity equal to one

$$n = R + 2S$$

We have,

$$\rightsquigarrow h(t) = \sum_{i=1}^R A_i e^{p_i t} + \sum_{i=R+1}^{R+S} M_i e^{\alpha_i t} \cos(\omega_i t + \phi_i) \quad (10)$$

(We have omitted the second subscript of the coefficients  $A$ ,  $M$  and  $\phi$ )

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## Homogeneous equation and modes (cont.)

### Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 2 \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} = 0$$

The characteristic polynomial without constant term

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

Its roots, from  $P(s) = 0$

$$\rightsquigarrow \begin{cases} p_1 = 0, & (\nu_1 = 1) \\ p_2 = \alpha_2 + j\omega_2 = -1 + j2, & (\nu_2 = 1) \\ p'_2 = \alpha_2 - j\omega_2 = -1 - j2, & (\nu'_2 = 1) \end{cases}$$

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## Homogeneous equation and modes (cont.)

We can write a linear combination of the modes

$$h(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p_2')} = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

■

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## Homogeneous equation and modes (cont.)

We can define yet another structure of the linear combination of modes

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## Homogeneous equation and modes (cont.)

### Proposition

Consider the contribution of  $(p_i, p'_i) = \alpha_i \pm j\omega_i$  a pair of conjugate complex roots with multiplicity  $\nu_i$  to the linear combination of the  $(2\nu_i)$  modes

$$\underbrace{(A_{i,0} e^{p_i t} + A'_{i,0} e^{p'_i t})}_{k=0} + \dots + \underbrace{t^{\nu_i-1} (A_{i,\nu_i-1} e^{p_i t} + A'_{i,\nu_i-1} e^{p'_i t})}_{k=\nu_i-1}$$

This sum of terms can be re-written

$$\rightsquigarrow \sum_{k=0}^{\nu_i-1} [B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t)] \quad (11)$$

The  $2\nu_i$  complex coefficients,  $A_{i,k}$  and  $A'_{i,k}$ , are replaced by  $2\nu_i$  real ones

$$\rightsquigarrow B_{i,k}$$

$$\rightsquigarrow C_{i,k}$$

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## Homogeneous equation and modes (cont.)

### Proof

Consider the term  $(Ae^{p_i t} + A'e^{p'_i t})$  in which  $(p_i, p'_i) = \alpha + j\omega$

Write the coefficients  $A$  and  $A'$  in cartesian form

$$A = u + jv$$

$$A' = u - jv$$

We have,

$$\begin{aligned} Ae^{p_i t} + A'e^{p'_i t} &= (u + jv)e^{\alpha t} [\cos(\omega t) + j \sin(\omega t)] \\ &\quad + (u - jv)e^{\alpha t} [\cos(\omega t) - j \sin(\omega t)] \\ &= 2ue^{\alpha t} \cos(\omega t) - 2ve^{\alpha t} \sin(\omega t) \\ &= \underbrace{B}_{B=2u} e^{\alpha t} \cos(\omega t) + \underbrace{C}_{C=-2v} e^{\alpha t} \sin(\omega t) \end{aligned}$$

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## Homogeneous equation and modes (cont.)

We distinguish modes associated with real and conjugate complex roots

$$\rightsquigarrow h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} \left[ B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \right] \quad (12)$$

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The equations

$$h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

$$\left( \rightsquigarrow \sum_{i=1}^R A_i e^{p_i t} + \sum_{i=R+1}^{R+S} M_i e^{\alpha_i t} \cos(\omega_i t + \phi_i) \right)$$

The equations

$$h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} \left[ B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \right]$$

$$\left( \rightsquigarrow \sum_{i=1}^R A_i e^{p_i t} + \sum_{i=R+1}^{R+S} \left[ B_i e^{\alpha_i t} \cos(\omega_i t) + C_i e^{\alpha_i t} \sin(\omega_i t) \right] \right)$$

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SA (CK0191)  
2018.1

Representation and analysis

Homogeneous equation and modes

Force-free evolution

Modes

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Pseudo-periodic

Impulse response

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## Homogeneous equation and modes (cont.)

Consider the case in which all roots have multiplicity equal to one

$$\rightsquigarrow n = R + 2S$$

We have,

$$\rightsquigarrow h(t) = \sum_{i=1}^R A_i e^{p_i t} + \sum_{i=R+1}^{R+S} \left[ B_i e^{\alpha_i t} \cos(\omega_i t) + C_i e^{\alpha_i t} \sin(\omega_i t) \right] \quad (13)$$

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## Homogeneous equation and modes (cont.)

They provide the parametric structure of the linear combination

$\rightsquigarrow$  They are all equivalent

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## Homogeneous equation and modes (cont.)

### Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 2 \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} = 0$$

Characteristic polynomial  $P(s)$  w/o constant term and the roots of  $P(s) = 0$

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

$$\rightsquigarrow \begin{cases} p_1 = 0, & (\nu_1 = 1) \\ p_2 = \alpha_2 + j\omega_2 = -1 + j2, & (\nu_2 = 1) \\ p'_2 = \alpha_2 - j\omega_2 = -1 - j2, & (\nu'_2 = 1) \end{cases}$$

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## Homogeneous equation and modes (cont.)

This problem can be solved in two equivalent ways

$$\rightsquigarrow h(t) = \underbrace{A_1}_{\text{root } p_1} + \underbrace{B_2 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t)}_{\text{root } (p_2, p'_2)}$$

$$\rightsquigarrow h(t) = \underbrace{A_1}_{\text{root } p_1} + \underbrace{M_2 e^{-t} \cos(2t + \phi_2)}_{\text{root } (p_2, p'_2)}$$

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## Homogeneous equation and modes (cont.)

The two coefficients  $A$  and  $A'$  in the complex plane

$$A = (M/2)e^{+j\omega} = B/2 - jC/2$$

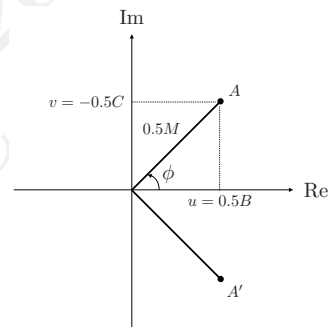
$$A' = (M/2)e^{-j\omega} = B/2 + jC/2$$

$$M = 2|A| = \sqrt{B^2 + C^2}$$

$$\phi = \arg(A) = \arctan(-C/B)$$

$$B = +M \cos \phi = +2u$$

$$C = -M \sin \phi = -2v$$



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**Force-free evolution**  
**Input-output representation**

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## Force-free evolution

The **force-free response** is a particular contribution to the output

It is due to the fact that the system is NOT initially at rest

- (This is the cause due to the non-zero state at  $t_0$ )

$$y(t) = \underbrace{y_u(t)}_{\text{force-free response}} + y_f(t), \quad \text{for } t \geq t_0$$

We study how to characterise it

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## Force-free evolution (cont.)

$$a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

The force-free response  $y_u(t)$  for  $t \geq 0$  is equal to the solution of the associated homogeneous differential equation, for some given initial conditions

$$\begin{cases} y_0 = y(t)|_{t=t_0} \\ y'_0 = \frac{dy(t)}{dt}|_{t=t_0} \\ \dots = \dots \\ y_0^{(n-1)} = \frac{d^{n-1}y(t)}{dt^{n-1}}|_{t=t_0} \end{cases}$$

$h(t)$  solves the homogeneous equation iff it is linear combination of modes

~> Thus,  $y_u(t)$  can be expressed as a linear combination of the modes

- (The  $n$  coefficients are still unknown)



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## Force-free evolution (cont.)

### Proposition

#### Free-force response

Consider a SISO system represented by a linear, time-invariant IO model

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

The **free-force response**  $y_u(t)$  is a linear combination of the modes

#### Proof

Let the input  $u(t)$  be always zero for  $t \geq 0$

- Then, also its derivatives are zero

$$\rightsquigarrow a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

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## Force-free evolution (cont.)

The coefficients of the force-free response depend on the initial conditions

~> So, does its evolution

The force-free response  $y_u(t)$  is a particular linear combination of the modes

- The  $n$  coefficients are determined from initial conditions

$$\begin{cases} y_0 = y(t)|_{t=t_0} \\ y'_0 = \frac{dy(t)}{dt}|_{t=t_0} \\ \dots = \dots \\ y_0^{(n-1)} = \frac{d^{n-1}y(t)}{dt^{n-1}}|_{t=t_0} \end{cases}$$

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## Force-free evolution (cont.)

### Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 8 \frac{d^2 y(t)}{dt^2} + 21 \frac{dy(t)}{dt} + 18y(t) = 0$$

We are interested in the force-free response  $y_u(t)$ , for  $t \geq 0$

- The initial conditions

$$\begin{aligned} y_0 &= 2 \\ y'_0 &= 1 \\ y''_0 &= -20 \end{aligned}$$

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## Force-free evolution (cont.)

The characteristic polynomial

$$P(s) = s^3 + 8s^2 + 21s + 18 = (s+2)(s+3)^2$$

Its roots from  $P(s) = 0$  are all real

$$\begin{cases} p_1 = -2, & \text{multiplicity } \nu_1 = 1 \\ p_2 = -3, & \text{multiplicity } \nu_2 = 2 \end{cases}$$

The force-free response

$$\rightsquigarrow y_u(t) = \underbrace{A_1 e^{-2t}}_{\text{root } p_1} + \underbrace{A_{2,0} e^{-3t} + A_{2,1} t e^{-3t}}_{\text{root } p_2}$$

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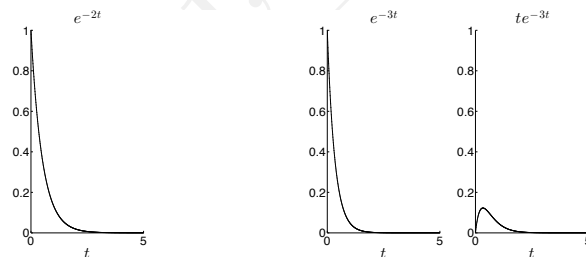
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## Force-free evolution (cont.)

$$y_u(t) = \underbrace{A_1 e^{-2t}}_{\text{root } p_1} + \underbrace{A_{2,0} e^{-3t} + A_{2,1} t e^{-3t}}_{\text{root } p_2}$$

The three modes to be combined to get the force-free response



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## Force-free evolution (cont.)

The force-free response

$$y_u(t) = A_1 e^{-2t} + A_{2,0} e^{-3t} + A_{2,1} t e^{-3t}$$

Its first- and second-order derivatives

$$\begin{aligned} \frac{dy_u(t)}{dt} &= -2A_1 e^{-2t} - 3A_{2,0} e^{-3t} + A_{2,1} (e^{-3t} - 3t e^{-3t}) \\ \frac{d^2 y_u(t)}{dt^2} &= 4A_1 e^{-2t} + 9A_{2,0} e^{-3t} + A_{2,1} (-6e^{-3t} + 9t e^{-3t}) \end{aligned}$$

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## Force-free evolution (cont.)

We substitute the initial conditions

$$\begin{aligned} y_u(t) \Big|_{t=0} &= A_1 + A_{2,0} = 2 \\ \frac{dy_u(t)}{dt} \Big|_{t=0} &= -2A_1 - 3A_{2,0} + A_{2,1} = 1 \\ \frac{d^2 y_u(t)}{dt^2} \Big|_{t=0} &= 4A_1 + 9A_{2,0} - 6A_{2,1} = -20 \end{aligned}$$

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## Force-free evolution (cont.)

$$\begin{aligned} y_u(t) \Big|_{t=0} &= A_1 + A_{2,0} = 2 \\ \frac{dy_u(t)}{dt} \Big|_{t=0} &= -2A_1 - 3A_{2,0} + A_{2,1} = 1 \\ \frac{d^2 y_u(t)}{dt^2} \Big|_{t=0} &= 4A_1 + 9A_{2,0} - 6A_{2,1} = -20 \end{aligned}$$

We have,

$$\mathbf{Ax} = \mathbf{b} \rightsquigarrow \begin{cases} \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 1 \\ 4 & 9 & -6 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \\ \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -20 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} A_1 \\ A_{2,0} \\ A_{2,1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

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## Force-free evolution (cont.)

The solutions of the linear system of equations

- $A_1 = x_1 = 4$
- $A_{2,0} = x_2 = -2$
- $A_{2,1} = x_3 = 3$

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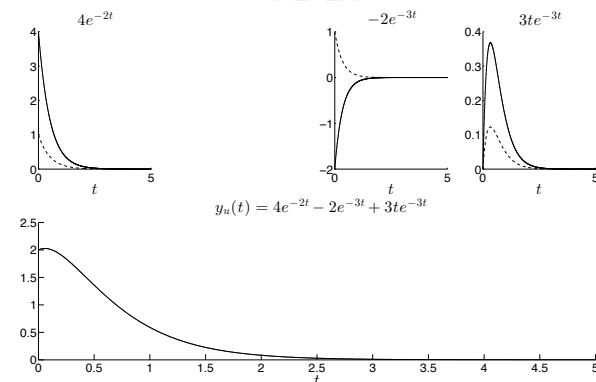
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## Force-free evolution (cont.)

We can write the complete expression of the force-free evolution  $y_u(t)$

$$\begin{aligned} y_u(t) &= A_1 e^{p_1 t} + A_{2,0} e^{p_2, t} + A_{2,1} t e^{p_2 t} \\ &= 4e^{-2t} - 2e^{-3t} + 3te^{-3t} \end{aligned}$$



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## Force-free evolution (cont.)

### Complex conjugate roots

Consider a characteristic polynomial  $P(s)$  with conjugate complex roots

$$(p_i, p'_i) = \alpha_i \pm j\omega_i$$

We want to determine an expression for force-free evolution

- We need to use a(ny) linear combination of the modes

$$\begin{aligned} \rightsquigarrow h(t) &= \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} \\ &\quad + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \\ \rightsquigarrow h(t) &= \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} \\ &\quad + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} [B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t)] \end{aligned}$$

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## Force-free evolution (cont.)

### Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 2 \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} = 0$$

We are interested in the force-free response  $y_u(t)$ , for  $t \geq 0$

- The initial condition

$$y_0 = 3$$

$$y'_0 = 2$$

$$y''_0 = 1$$

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## Force-free evolution (cont.)

The characteristic polynomial

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

Its roots from  $P(s) = 0$

$$\begin{cases} p_1 = -0, & \text{multiplicity } \nu_1 = 1 \\ p_2 = -\alpha_2 + j\omega = -1 + j2, & \text{multiplicity } \nu_2 = 1 \\ p'_2 = -\alpha_2 - j\omega = -1 - j2, & \text{multiplicity } \nu'_2 = 1 \end{cases}$$

- $R = 1$  distinct real roots
- $S = 1$  distinct pair of complex conjugate roots

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## Force-free evolution (cont.)

We first consider a parameterisation in the form

$$h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

We get the force-free response

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p'_2)} = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

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## Force-free evolution (cont.)

The force-free response and its derivatives of order 1 and order 2

$$y_u(t) = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

$$\frac{dy_u(t)}{dt} = -M_2 e^{-t} \cos(2t + \phi_2) - 2M_2 e^{-t} \sin(2t + \phi_2)$$

$$\frac{d^2 y_u(t)}{dt^2} = -2M_2 e^{-t} \cos(2t + \phi_2) + 4M_2 e^{-t} \sin(2t + \phi_2)$$

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## Force-free evolution (cont.)

$$y_u(t) \Big|_{t=0} = A_1 + M_2 \cos(\phi_2) = 3$$

$$\frac{dy_u(t)}{dt} \Big|_{t=0} = -M_2 \cos(\phi_2) - 2M_2 \sin(\phi_2) = 2$$

$$\frac{d^2 y_u(t)}{dt^2} \Big|_{t=0} = -3M_2 \cos(\phi_2) + 4M_2 \sin(\phi_2) = 1$$

The system of equations is linear in the unknowns

$$\rightsquigarrow x = M_2 \cos(\phi_2)$$

$$\rightsquigarrow y = M_2 \sin(\phi_2)$$

For consistency, we let  $z = A_1$

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## Force-free evolution (cont.)

We substitute the initial conditions

$$y_u(t) \Big|_{t=0} = A_1 + M_2 \cos(\phi_2) = 3$$

$$\frac{dy_u(t)}{dt} \Big|_{t=0} = -M_2 \cos(\phi_2) - 2M_2 \sin(\phi_2) = 2$$

$$\frac{d^2 y_u(t)}{dt^2} \Big|_{t=0} = -3M_2 \cos(\phi_2) + 4M_2 \sin(\phi_2) = 1$$

The system of equations is non-linear in the unknowns

$$\rightsquigarrow M_2$$

$$\rightsquigarrow \phi_2$$

$$\rightsquigarrow (A_1)$$

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## Force-free evolution (cont.)

$$y_u(t) \Big|_{t=0} = \underbrace{A_1}_z + \underbrace{M_2 \cos(\phi_2)}_x = 3$$

$$\frac{dy_u(t)}{dt} \Big|_{t=0} = -\underbrace{M_2 \cos(\phi_2)}_x - 2\underbrace{M_2 \sin(\phi_2)}_y = 2$$

$$\frac{d^2 y_u(t)}{dt^2} \Big|_{t=0} = -3\underbrace{M_2 \cos(\phi_2)}_x + 4\underbrace{M_2 \sin(\phi_2)}_y = 1$$

The resulting system of linear equation

$$\rightsquigarrow \begin{cases} z + x = 3 \\ -x - 2y = 2 \\ -3x + 4y = 1 \end{cases}$$

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## Force-free evolution (cont.)

The solution

- $z = 4 = A_1$
- $x = -1 = M_2 \cos(\phi_2)$
- $y = -0.5 = M_2 \sin(\phi_2)$

Thus, we get

$$\rightsquigarrow \begin{cases} A_1 = 4 \\ M_2 = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0.5^2} = 1.12 \\ \phi_2 = \arctan(y/x) = \arctan(-0.50/-1) = -2.68 \text{ [rad]} \end{cases}$$

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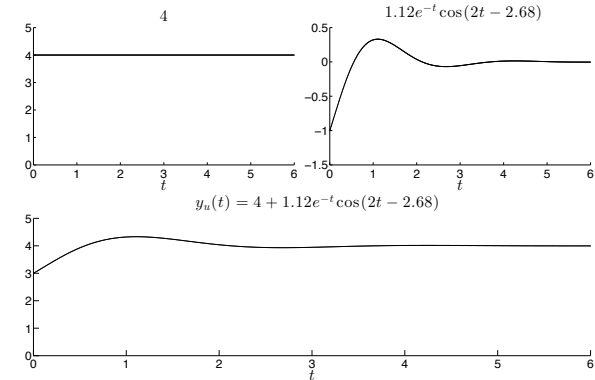
Forced evolution

## Force-free evolution (cont.)

The force-free response for  $t \geq 0$

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p_2')} = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

$$\rightsquigarrow = 4 + 1.12 e^{-t} \cos(2t - 2.68)$$



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## Force-free evolution (cont.)

We now consider a parameterisation in the form

$$h(t) = \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} [B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t)]$$

We get the force-free response

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{B_2 e^{\alpha_2 t} \cos(\omega_2 t) + C_2 e^{\alpha_2 t} \sin(\omega_2 t)}_{\text{root } (p_2, p_2')}$$

$$= A_1 + B_2 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t)$$

$$\rightsquigarrow = 4 - e^{-t} \cos(2t) + 0.5 e^{-t} \sin(2t)$$

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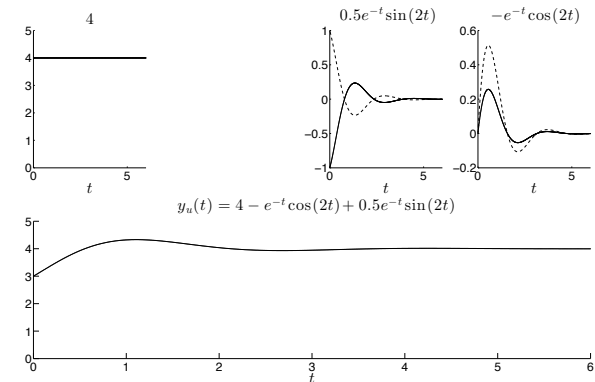
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## Force-free evolution (cont.)





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### Force-free evolution (cont.)

We can compare the different forms of the solution

$$A = (M/2)e^{+j\omega} = B/2 - jC/2$$

$$A' = (M/2)e^{-j\omega} = B/2 + jC/2$$

$$M = 2|A| = \sqrt{B^2 + C^2}$$

$$\phi = \arg(A) = \arctan(-C/B)$$

$$B = +M \cos \phi = +2u$$

$$C = -M \sin \phi = -2v$$

We get,

$$M_2 = \sqrt{B_2^2 + C_2^2}$$

$$\phi_2 = \arctan(-C_2/B_2)$$

$$M_2 = +M_2 \cos(\phi_2)$$

$$C_2 = -M_2 \sin(\phi_2)$$



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### Force-free evolution (cont.)

Initial time not equal zero

How to calculate the force-free response from an initial time  $t \neq 0$

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### Force-free evolution (cont.)

#### Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 8 \frac{d^2 y(t)}{dt^2} + 21 \frac{dy(t)}{dt} + 18y(t) = 0$$

We are interested in the force-free response, for  $t \geq t_0 \neq 0$

- The initial condition

$$y(t) \Big|_{t=t_0} = y_0 = 2$$

$$\frac{dy(t)}{dt} \Big|_{t=t_0} = y'_0 = 1$$

$$\frac{d^2 y(t)}{dt^2} \Big|_{t=t_0} = y''_0 = -20$$



#### Input-output representation

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## Classification of modes

Input-output representation

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## Classification of modes

Modes fully characterise the dynamics of a system

- It is important to study their form
- It is important to classify them

We provide an intuitive classification

↪ **Aperiodic modes**

↪ **Pseudo-periodic modes**

Aperiodic modes have no oscillatory behaviour, pseudo-periodic ones do

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## Aperiodic modes

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## Classification of modes (cont.)

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = \sum_{i=0}^n a_i s^i = 0$$

### Aperiodic modes

$$t^k e^{\alpha t}, \quad \text{for } k = 0, \dots, \nu - 1$$

Associate to real roots  $p = \alpha \in \mathcal{R}$  (multiplicity  $\nu$ )

### Pseudo-periodic modes

$$\begin{cases} t^k e^{\alpha t} \cos(\omega t) \\ t^k e^{\alpha t} \sin(\omega t) \end{cases}, \quad \text{for } k = 0, \dots, \nu - 1$$
$$t^k e^{\alpha t} \cos(\omega t + \phi_k), \quad \text{for } k = 0, \dots, \nu - 1$$

Associate to conjugate complex roots  $(p, p') = \alpha \pm j\omega \in \mathcal{C}$  (multiplicity  $\nu$ )

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## Aperiodic modes

These are the modes associated to real roots  $p = \alpha \in \mathcal{R}$ , multiplicity  $\nu$

$$t^k e^{\alpha t}, \quad k = 0, \dots, \nu - 1$$

The fundamental parameter of the generic aperiodic mode is  $\alpha \neq 0$

$$\rightsquigarrow \tau = -1/\alpha, \quad (\alpha = p \neq 0)$$

The exponent  $t/\tau$  in  $t^k e^{\alpha t} = t^k e^{-t/\tau}$  is dimensionless

- Parameter  $\tau$  has the units of time

↪ **Time-constant**

The time constant is not defined for  $\alpha = 0$

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## Aperiodic modes (cont.)

### Roots with multiplicity one

Let real root  $\alpha$  have multiplicity  $\nu = 1$ , there is only one associated mode

$$\rightsquigarrow e^{\alpha t}$$

This mode (a simple exponential) is aperiodic

**Stable** or **convergent**, if  $\alpha < 0$

$\rightsquigarrow$  As  $t$  increases, the mode  $e^{\alpha t}$  tends to 0 asymptotically

**Stability limit** or **constant**, if  $\alpha = 0$

$\rightsquigarrow$  The mode is equal to  $e^{0t} = 1$ , for any  $t \geq 0$

**Unstable** or **divergent**, if  $\alpha > 0$

$\rightsquigarrow$  As  $t$  increases, the mode  $e^{\alpha t}$  tends to  $\infty$  asymptotically

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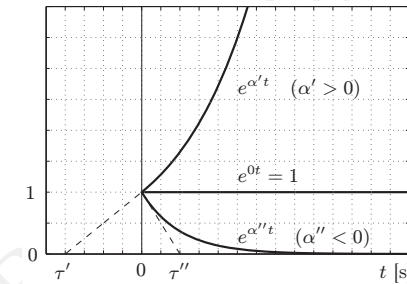
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## Aperiodic (cont.)



**Unstable** ( $\alpha > 0$ )

$\rightsquigarrow$  The time-constant takes negative values  $\tau < 0$

**Stable** ( $\alpha < 0$ )

$\rightsquigarrow$  The time-constant takes positive values  $\tau > 0$

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## Aperiodic (cont.)

$\tau$  is geometrically understood as the (below) tangent to mode at  $t = 0$

The value of the tangent where it intersects the abscissa

$$\rightsquigarrow \left. \frac{d}{dt} e^{\alpha t} \right|_{t=0} = \alpha e^{\alpha t} \Big|_{t=0} = \alpha$$

The line tangent to  $e^{\alpha t}$  in  $t = 0$  is  $f(t) = at + b$  with slope  $a = \alpha$

- The intercept (at  $t = 0$ ) is  $b = f(0) = 1$
- $f(t) = \alpha t + 1 = 0$  when  $t = -1/\alpha = \tau$

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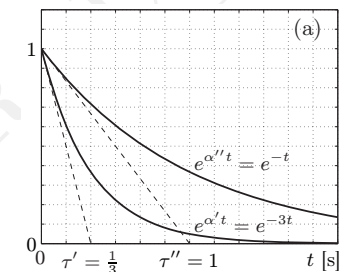
Forced evolution

## Aperiodic (cont.)

$\tau$  is also the time after which the mode has lost  $\approx 63\%$  of its initial value

$t$	0	$\tau$	$2\tau$	$3\tau$	$4\tau$	$5\tau$
$e^{\alpha t} = e^{-t/\tau}$	1	0.37	0.14	0.05	0.02	0.01

The smaller the time-constant  $\tau = -1/\alpha$ , the faster a (stable) mode vanishes



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## Aperiodic (cont.)

### Roots with multiplicity larger than one

Let real root  $\alpha$  have multiplicity  $\nu > 1$ , there are  $\nu$  associated modes

$$\rightsquigarrow e^{\alpha t}, te^{\alpha t}, t^2 e^{\alpha t}, \dots, t^k e^{\alpha t}, \dots, t^{\nu-1} e^{\alpha t}$$

We consider only modes in the form  $t^k e^{\alpha t}$ , with  $k > 0$

**Stable**, if  $\alpha < 0$  and  $k \geq 1$

$\rightsquigarrow$  As  $t$  increases, the mode  $t^k e^{\alpha t}$  tends to 0 asymptotically

**Unstable**, if  $\alpha \geq 0$  and  $k \geq 1$

$\rightsquigarrow$  As  $t$  increases, the mode  $t^k e^{\alpha t}$  tends to  $\infty$  asymptotically

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## Aperiodic (cont.)

### Case with $\alpha < 0$ and $k > 0$ ( $k \geq 1$ )

Consider the case in which  $\alpha < 0$  and  $k > 0$

$$\rightsquigarrow t^k e^{\alpha t}$$

If we study the asymptotic behaviour of the mode, we get

$$\rightsquigarrow \lim_{t \rightarrow \infty} t^k e^{\alpha t} = \lim_{t \rightarrow \infty} \frac{t^k}{e^{-\alpha t}} = \infty / \infty$$

The undetermined form is solved by differentiating  $k$  times (de l'Hospital)

$$\rightsquigarrow \lim_{t \rightarrow \infty} t^k e^{\alpha t} = \lim_{t \rightarrow \infty} \frac{t^k}{e^{-\alpha t}} = \lim_{t \rightarrow \infty} \frac{k!}{(-\alpha)^k e^{-\alpha t}} = 0 \quad (\text{stable})$$

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## Aperiodic (cont.)

### Case with $\alpha \geq 0$ and $k \geq 1$

Consider the case in which  $\alpha \geq 0$  and  $k \geq 1$

$$t^k e^{\alpha t}$$

If the root is null ( $\alpha = 0$ ), the mode is  $t^k$  and it grows

- $\rightsquigarrow k = 1$ , a line
- $\rightsquigarrow k = 2$ , a parabola
- $\rightsquigarrow k = 3$ , a cubic

For a positive root ( $\alpha > 0$ ), the mode grows faster

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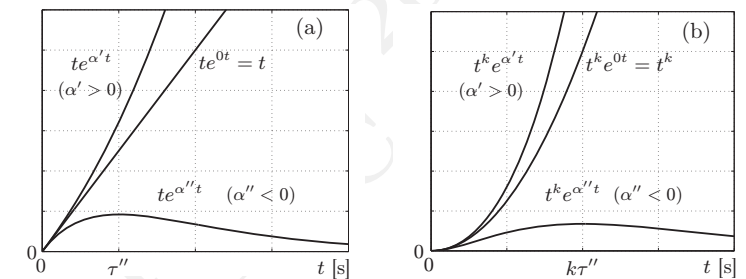
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## Aperiodic (cont.)

(a) For  $k = 1$ , the tangent to the mode has unit slope in  $t = 0$

(b) For  $k > 1$ , the tangent to the mode has zero slope in  $t = 0$



Stable modes have one maximum at  $t = k\tau$

$$\rightsquigarrow \tau = -1/\alpha$$

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## Aperiodic (cont.)

Consider a stable (decreasing) mode of the form  $t^k e^{\alpha t}$  and  $k \geq 1$

Still, the smaller the time-constant  $\tau = -1/\alpha$ , the faster a mode vanishes

- Different geometrical interpretation compared to the case  $k = 0$
- $\rightsquigarrow t = k\tau$  is the value of  $t$  where the mode has its single maximum

To appreciate this fact, we can differentiate the mode

$$\frac{d}{dt} t^k e^{\alpha t} = k t^{k-1} e^{\alpha t} + \alpha t^k e^{\alpha t} = t^{k-1} e^{\alpha t} (k + \alpha t)$$

The derivative is zero for  $t > 0$  and  $\alpha < 0$  at  $t = -k/\alpha = k\tau$

- Curve  $e^k e^{\alpha t}$  for  $\alpha < 0$  has a maximum at  $t = k\tau$

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## Pseudo-periodic modes

### Modes

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## Pseudo-periodic modes

These are the modes associated to conjugate complex roots  $(p, p') = \alpha \pm j\omega$

Pseudo-periodic modes can take various forms

- We restrict our presentation to one type

$$t^k e^{\alpha t} \cos(\omega t), \quad \text{with } k = 0, \dots, \nu - 1$$

The other cases (phased) are not considered

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## Pseudo-periodic modes (cont.)

$$t^k e^{\alpha t} \cos(\omega t), \quad (k = 0, \dots, \nu - 1)$$

The parameters that characterise the generic pseudo-periodic mode

$\rightsquigarrow$  **Time-constant**

$$\tau = -\frac{1}{\alpha}, \quad \alpha \neq 0$$

$\rightsquigarrow$  **Natural pulsation**

$$\omega_n = \sqrt{\alpha^2 + \omega^2}$$

$\rightsquigarrow$  **Damping coefficient**

$$\zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

### Input-output representation

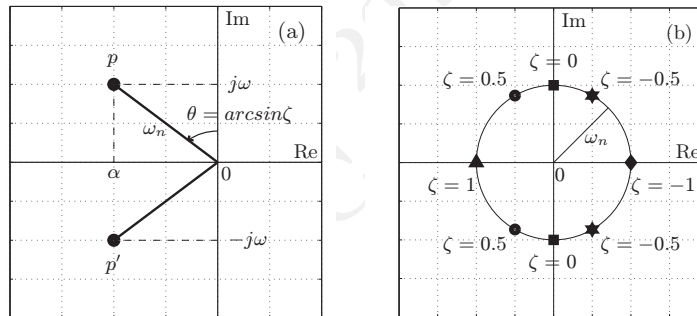
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## Pseudo-periodic modes (cont.)

### Natural pulsation

We can represent the pair of roots  $(p, p') = \alpha \pm j\omega$  on the complex plane



Suppose that  $p = \alpha + j\omega$  is a pole on the positive imaginary half-plane

- $\omega_n$  is the module of the vector that connects pole  $p$  ( $p'$ ) and origin

$$\leadsto \omega_n = \sqrt{\alpha^2 + \omega^2}$$

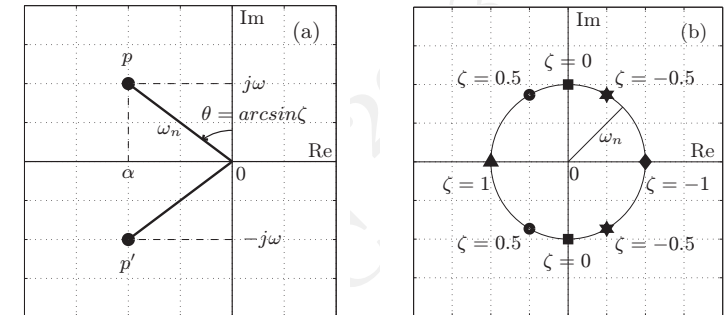
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## Pseudo-periodic modes (cont.)

### Damping coefficient



$\zeta$  is the sine of the angle  $\theta$  between the vector connecting  $p$  with the origin and the positive imaginary half-axis (counterclock-wise = positive)

$$\leadsto \zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

- Negative  $\alpha$ ,  $\leadsto$  positive  $\theta$
- Null  $\alpha$ ,  $\leadsto$  null  $\theta$
- Positive  $\alpha$ ,  $\leadsto$  negative  $\theta$

### Input-output representation

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## Pseudo-periodic modes (cont.)

### Roots with multiplicity one

Consider a pair of conjugate complex roots  $(p, p') = \alpha \pm j\omega$  with  $\nu = 1$

The corresponding pseudo-periodic mode

$$\leadsto e^{\alpha t} \cos(\omega t)$$

Such a mode has an oscillatory behaviour

- This is due to the cosine factor

The mode envelopes  $\cos(\omega t)$  with functions  $-e^{\alpha t}$  and  $e^{\alpha t}$

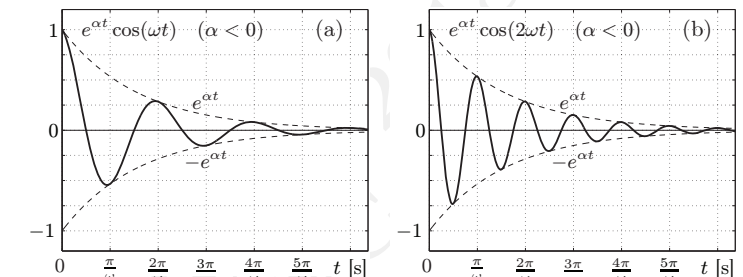
$$\leadsto e^{\alpha t} \cos(\omega t) = \begin{cases} -e^{\alpha t}, & t = (2h+1)\frac{\pi}{\omega}, \quad h \in \mathcal{N} \\ e^{\alpha t}, & t = 2h\frac{\pi}{\omega}, \quad h \in \mathcal{N} \end{cases}$$

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## Pseudo-periodic modes (cont.)



**Stable** ( $\alpha < 0$ )

$\leadsto$  As  $t$  increases the envelopes tend to 0 asymptotically

Case (a) has a larger damping factor than case (b)

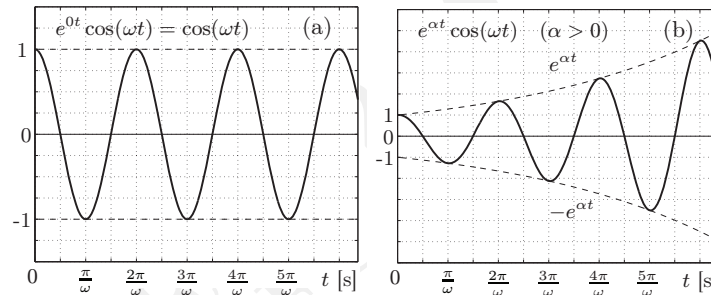
- Time constant is equal (same  $\alpha$ )

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## Pseudo-periodic modes (cont.)



**Stability limit** ( $\alpha = 0$ )

- ~> The mode becomes equal to  $\cos(\omega t)$  and it is periodic
- ~> As functions of  $t$ , the envelopes are constant  $\pm 1$  curves

**Unstable** ( $\alpha > 0$ )

- ~> As  $t$  increases the envelopes tend to  $\pm\infty$  asymptotically

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## Pseudo-periodic modes (cont.)

### The time constant

Again, the time constant indicates the velocity of the mode (envelops)

- (As in the aperiodic case)

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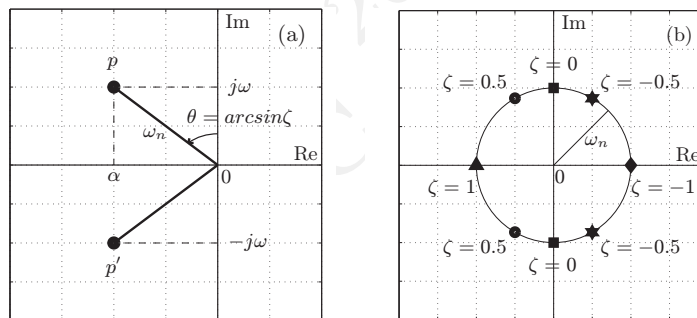
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## Pseudo-periodic modes (cont.)

### The dumping factor $\zeta$

The dumping factor is a real number in the interval  $[-1, 1]$

- ~> As it is equal to  $\sin(\theta)$



$$\zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

## Input-output representation

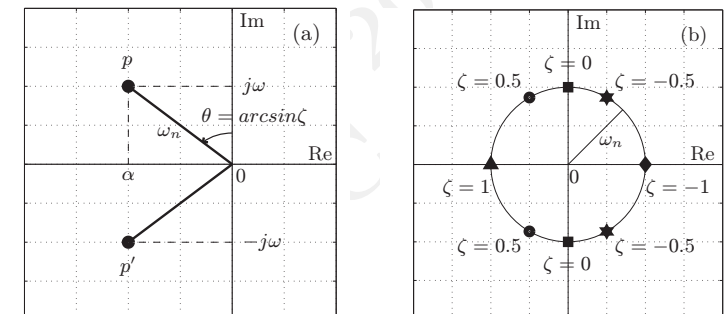
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## Pseudo-periodic modes (cont.)

We study pairs of roots with the same natural pulsation (same  $\omega_n$ )

- Dumping coefficient is changed (different  $\alpha$  and  $\omega$ )



These roots lie on the complex plane

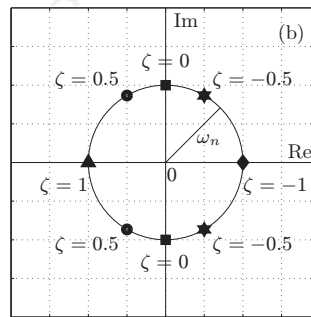
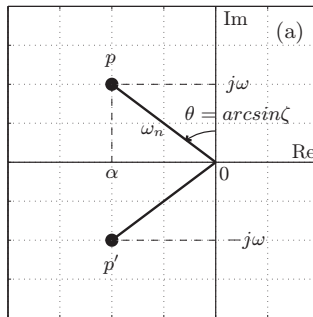
- Along a circle, radius  $\omega_n$

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## Pseudo-periodic modes (cont.)



$\zeta = +1$ , if  $\alpha = -\omega_n < 0$  and  $\omega = 0$

- Two complex roots coinciding with a negative real root, multiplicity 2
- The associated modes are  $e^{-\omega_n t}$  and  $t e^{-\omega_n t}$

$\zeta = -1$ , if  $\alpha = -\omega_n > 0$  and  $\omega = 0$

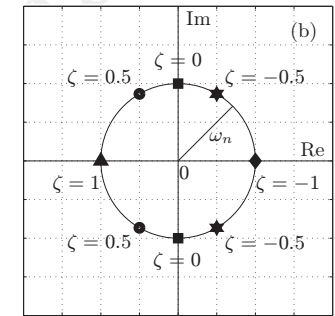
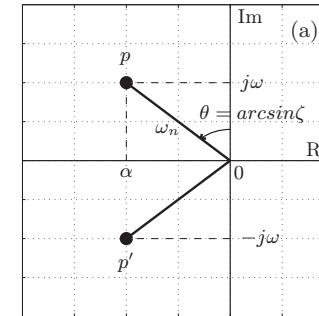
- Two complex roots coinciding with a positive real root, multiplicity 2
- The associated modes are  $e^{\omega_n t}$  and  $t e^{\omega_n t}$

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## Pseudo-periodic modes (cont.)



$\zeta = 0$ , if  $\alpha = 0$  and  $\omega = \omega_n$

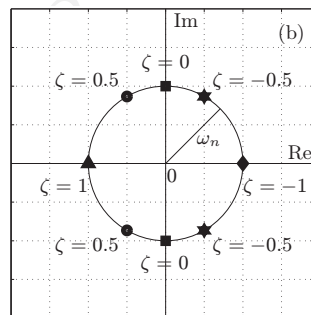
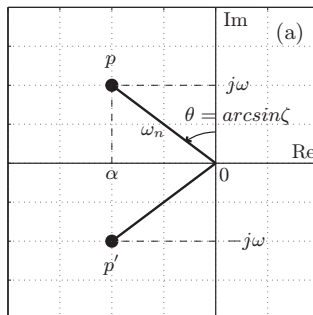
- Two conjugate imaginary roots
- The associated mode is at the stability limit

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## Pseudo-periodic modes (cont.)



$\zeta \in (0, 1)$ , if  $\alpha < 0$  and  $\omega > 0$

- The two complex roots have a negative real part
- The associated mode is stable

$\zeta \in (-1, 0)$ , if  $\alpha > 0$  and  $\omega > 0$

- The two complex roots have a positive real part
- The associated mode is unstable

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## Pseudo-periodic modes (cont.)

### Roots with multiplicity larger than one

Consider a pair of conjugate complex roots  $(p, p') = \alpha \pm j\omega$  with  $\nu > 1$

The corresponding pseudo-periodic modes

$$e^{\alpha t} \cos(\omega t), t e^{\alpha t} \cos(\omega t), t^2 e^{\alpha t} \cos(\omega t), \dots, t^k e^{\alpha t} \cos(\omega t), \dots, t^{\nu-1} e^{\alpha t} \cos(\omega t)$$

We consider the modes in the form  $t^k e^{\alpha t} \cos(\omega t)$  with  $k > 0$



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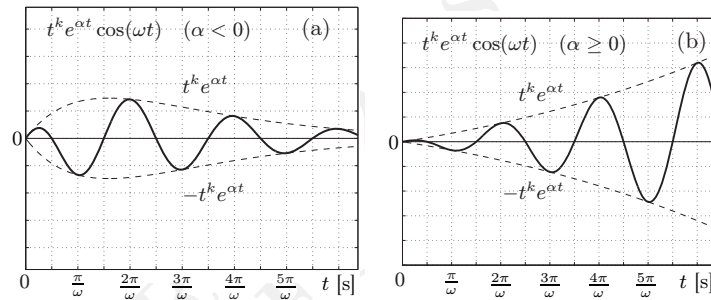
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## Pseudo-periodic modes (cont.)



**Stable** ( $\alpha < 0$  and  $k > 0$ )

↪ As  $t$  increases the mode tends to 0 asymptotically

**Unstable** ( $\alpha \geq 0$  and  $k \geq 1$ )

↪ As  $t$  increases the mode tends to  $\infty$  asymptotically

The mode envelops  $\cos(\omega t)$  with functions  $\pm t^k e^{\alpha t}$

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## Impulse response

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## Impulse response

We will study the general forced response of a system due to arbitrary inputs

We start by studying a particular forced response

↪ **Impulse response**

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## Impulse response (cont.)

### Definition

#### Impulse response

The **impulse response**  $w(t)$  is the forced evolution of a system subjected to an input  $u(t) = \delta(t)$  applied at time  $t = 0$

The impulse response is an important function, as it is a **canonical regime**

What do we get from its knowledge?

↪ The forced evolution of the system under any input

↪ The force-free evolution for any initial condition

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## Impulse response (cont.)

### Unit impulse and unit step

The **unit impulse**  $\delta(t)$  is the *derivative* of the unit step  $\delta_{-1}(t)$

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t)$$

The **unit step** is the Heaviside function

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

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## Structure of the impulse response (cont.)

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

Let  $\nu_i$  be the multiplicity of root  $p_i$  of the characteristic polynomial

$$h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

The impulsive term is present iff the system is not strictly proper

$$A_0 = \begin{cases} b_n/a_n, & m = n \\ 0, & m < n \end{cases}$$

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## Structure of the impulse response

### Proposition

#### Structure of the impulse response

Consider a linear, stationary and proper SISO system

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

For  $t < 0$ , the impulse response  $w(t)$  is null

$$\rightsquigarrow w(t) = 0$$

For  $t \geq 0$ , the impulse response  $w(t)$  can be parameterised as linear combination  $h(t)$  of the  $n$  modes of the system and, possibly, an impulsive term

$$\rightsquigarrow w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

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## Structure of the impulse response (cont.)

### Proof

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

In a causal/proper ( $n \geq m$ ) system, the effect cannot precede the cause

When subjected to impulse  $\delta(t)$  at  $t = 0$ , the response is null for  $t < 0$

- This is imposed by  $\delta_{-1}(t)$  in  $w(t)$

Moreover, an impulsive input  $u(t) = \delta(t)$  is (definition) null for  $t > 0$

The system is assumed initially at rest in  $t = 0^-$

- At time  $t = 0^+$ , it is in a non-null initial state

Because of the action due to the impulsive input

After  $t = 0^+$  the input is null

The evolution is a particular force-free response

- Unknown coefficients  $A_{i,k}$  to be determined
- This is given by  $h(t)$  in  $w(t)$



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## Structure of the impulse response (cont.)

### Example

Consider an instantaneous system with the model

$$3y(t) = 2u(t)$$

We are interested in the force-free response to a unit impulse

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

The system has  $m = n = 0$  (non strictly proper)

- $a_n = 3$
- $b_n = 2$

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## Structure of the impulse response (cont.)

$$3y(t) = 2u(t)$$

Model is an algebraic equation, characteristic polynomial has order  $n = 0$

↪ A system represented by this model does not have any mode

The impulsive response for an input  $u(t) = \delta(t)$

$$\begin{aligned} w(t) &= A_0 \delta(t) + h(t) \delta_{-1}(t) \\ &\rightsquigarrow (2/3) \delta(t) \end{aligned}$$

↪  $h(t) = 0$

↪  $A_0 = b_n/a_n$

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## Structure of the impulse response (cont.)

Consider a characteristic polynomial of the system

- $R$  distinct real roots
- $S$  distinct pairs of conjugate complex roots

We can re-write

$$h(t) = \sum_{i=1}^r \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t}$$

We can use one of the forms where the pseudo-periodic modes are explicit

$$\begin{aligned} \rightsquigarrow h(t) &= \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} \\ &+ \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} (B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t)) \end{aligned}$$

$$\begin{aligned} \rightsquigarrow h(t) &= \sum_{i=1}^R \sum_{k=0}^{\nu_i-1} A_{i,k} t^k e^{p_i t} \\ &+ \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i-1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \end{aligned}$$

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## Structure of the impulse response (cont.)

Unknown coefficients in the expression of  $h(t)$  in the impulse response  $w(t)$

- We used the symbols  $A$ ,  $M$ ,  $\phi$ ,  $B$  and  $C$
- As in force-free responses

In the force-free case, coefficients can take an infinity of arbitrary values

- They depend on the initial conditions

In the impulse case, coefficients depend univocally only on the system

We study a technique/algorithm to find their value

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## Calculation of the impulse response

### Computing the impulse response

A complicated technique to calculate the impulse response in time-domain

The algorithm is based on the knowledge of the impulse response  $w(t)$

- We know that  $w(t)$  has a known parameterisation

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

As such,  $w(t)$  must satisfy the model

$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^m b_i \frac{d^i}{dt^i} u(t)$$

↪ For a given impulse input  $u(t) = \delta(t)$

↪ For any value of  $t$

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## Calculation of the impulse response (cont.)

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

In the parameterisation of  $w(t)$  there are  $(n+1)$  unknown coefficients

- The  $n$  coefficients associated to the modes
- The coefficient  $A_0$  of the impulsive term

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## Calculation of the impulse response (cont.)

The impulse response  $w(t)$  must satisfy the model for all  $t$ ,  $t=0$  included

- All of the nasty things happen here

↪ Discontinuities or impulsive terms

$$\begin{aligned} a_n \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

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## Calculation of the impulse response (cont.)

We calculate derivatives<sup>2</sup> of  $w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$ , up to order  $n$

$$w(t) = h(t)\delta_{-1}(t) + A_0\delta(t)$$

$$\frac{d}{dt} w(t) = \dot{h}(t)\delta_{-1}(t) + h(0)\delta(t) + A_0\delta_1(t)$$

$$\dots = \dots$$

$$\frac{d^n}{dt^n} w(t) = h^{(n)}(t)\delta_{-1}(t) + h^{(n-1)}(0)\delta(t) + h^{(n-2)}(0)\delta_1(t) + \dots + A_0\delta_n(t)$$

<sup>2</sup>In the sense of distributions,

$$\frac{d^k}{dt^k} f(t)\delta_{-1}(t) = f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)$$

and

$$\delta_k(t) = \frac{d^k}{dt^k} \delta(t) = \frac{d}{dt} \delta_{k-1}(t), \text{ with } k > 1.$$

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## Calculation of the impulse response (cont.)

Moreover, we have,

$$\begin{aligned} u(t) &= \delta(t) \\ \frac{du(t)}{dt} &= \delta_1(t) \\ &\dots = \dots \\ \frac{d^m u(t)}{dt^m} &= \delta_m(t) \end{aligned}$$

Thus,

$$\begin{aligned} \rightsquigarrow a_n \frac{d^n w(t)}{dt^n} + \dots + a_1 \frac{dw(t)}{dt} + a_0 w(t) \\ = b_m \delta_m(t) + \dots + b_1 \delta_1(t) + b_0 \delta(t) \end{aligned}$$

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## Calculation of the impulse response (cont.)

We can now substitute for the expressions of  $w(t)$  and its derivatives

- We solve after imposing equality between the coefficients
- Those that multiply the terms  $\delta(t), \delta_1(t), \dots, \delta_m(t)$

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## Calculation of the impulse response (cont.)

A set of  $n + 1$  equations in  $n + 1$  unknowns coefficients of  $w(t)$

- $A_0, \{A_i\}$  and  $\{M_i\}$  and  $\{\phi_i\}$  (or,  $\{B_i\}$  and  $\{C_i\}$ )

$$\begin{aligned} b_0 &= a_0 A_0 + a_1 h(0) + \dots + a_{n-1} h^{(n-2)}(0) + a_n h^{(n-1)}(0) \\ b_1 &= a_1 A_0 + a_2 h(0) + \dots + a_n h^{(n-2)}(0) \\ &\dots = \dots \\ b_{n-1} &= a_{n-1} A_0 + a_n h(0) \\ b_n &= a_n A_0 \end{aligned}$$

The unknown coefficients  $A_0, \{A_i\}$  and  $\{M_i\}$  and  $\{\phi_i\}$  (or,  $\{B_i\}$  and  $\{C_i\}$ )

- They appear also in the expression of  $h(0), \dot{h}(0), \dots, h^{(n-1)}(0)$

- $\rightsquigarrow$  The coefficients  $a_i$  and  $b_i$  with  $i = 1, \dots, n$  are given by the model
- $\rightsquigarrow$  If we have  $n < m$ , we can set  $b_{m+1} = b_{m+2} = \dots = b_n = 0$
- $\rightsquigarrow$  Terms that multiply  $\delta_{-1}(t)$  cancel out (missing from RHS)

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## Calculation of the impulse response (cont.)

$$\begin{aligned} b_0 &= a_0 A_0 + a_1 h(0) + \dots + a_{n-1} h^{(n-2)}(0) + a_n h^{(n-1)}(0) \\ b_1 &= a_1 A_0 + a_2 h(0) + \dots + a_n h^{(n-2)}(0) \\ &\dots = \dots \\ b_{n-1} &= a_{n-1} A_0 + a_n h(0) \\ b_n &= a_n A_0 \end{aligned}$$

From  $b_n = a_n A_0$ ,

- If  $m = n$ , then  $a_n A_0 = b_n \neq 0$  and  $A_0 = b_n / a_n \neq 0$
- If  $m < n$ , then  $a_n A_0 = b_n = 0$  and  $A_0 = 0$

It thus is possible to simplify the calculation

- Determine a priori the term  $A_0$
- Treat it as constant

(Last equation of the system becomes an identity)

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## Calculation of the impulse response (cont.)

### Algorithm

- 1 Determine the characteristic polynomial  $P(s)$  of the homogeneous equation associated to the IO model and calculate its roots
- 2 Determine the  $n$  modes of the model
- 3 Write  $w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$  using a parameterisation of  $h(t)$
- 4 Calculate the derivatives of  $h(t)$ , up to the  $(n-1)$ -th order
- 5 Write the system of  $n$  equations in the  $n$  unknown coefficients of  $h(t)$

$$b_0 - a_0 A_0 = a_1 h(0) + a_2 \dot{h}(0) + \dots + a_{n-1} h^{(n-2)}(0) + a_n h^{(n-1)}(0)$$

$$b_1 - a_1 A_0 = a_2 h(0) + a_3 \dot{h}(0) + \dots + a_n h^{(n-2)}(0)$$

$$\dots = \dots$$

$$b_{n-2} - a_{n-2} A_0 = a_{n-1} h(0) + a_n \dot{h}(0)$$

$$b_{n-1} - a_{n-1} A_0 = a_n h(0)$$

- 6 Solve for the  $n$  unknown coefficients  $A_i$  of  $w(t)$

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## Calculation of the impulse response (cont.)

### Example

Consider a system described by the IO model

$$2 \frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 4y(t) = \frac{du(t)}{dt} + 3u(t)$$

We are interested in computing the the impulse response

The characteristic polynomial

$$P(s) = 2s^2 + 6s + 4$$

From  $P(s) = 0$ , two real roots, both with multiplicity one

$$\rightsquigarrow p_1 = -1, \text{ mode } e^{-p_1 t}$$

$$\rightsquigarrow p_2 = -2, \text{ mode } e^{-p_2 t}$$

A strictly proper model,  $m = 1 < n = 2$

$$\rightsquigarrow w(t) \text{ w/o the impulsive term}$$

$$\rightsquigarrow \text{Thus, } A_0 = 0$$

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## Calculation of the impulse response (cont.)

The structure of the impulse response and its derivative

$$w(t) = \underbrace{(A_1 e^{-t} + A_2 e^{-2t})}_{h(t)} \delta_{-1}(t)$$

$$\frac{dw(t)}{dt} = \underbrace{(-A_1 e^{-t} - 2A_2 e^{-2t})}_{\dot{h}(t)} \delta_{-1}(t) + \underbrace{(A_1 + A_2)}_{h(0)} \delta(t)$$

$$\begin{aligned} \frac{d^2 w(t)}{dt^2} &= \underbrace{(A_1 e^{-t} + 4A_2 e^{-2t})}_{\ddot{h}(t)} \delta_{-1}(t) + \underbrace{(-A_1 - 2A_2)}_{\dot{h}(0)} \delta(t) \\ &\quad + \underbrace{(A_1 + A_2)}_{h(0)} \delta_1(t) \end{aligned}$$

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## Calculation of the impulse response (cont.)

By substituting  $w(t)$  and its derivatives in the model and setting  $u(t) = \delta(t)$

$$\begin{aligned} &\underbrace{4(A_1 e^{-t} + A_2 e^{-2t})\delta_{-1}(t)}_{a_0 w(t)} \\ &\quad + \underbrace{6(-A_1 e^{-t} - 2A_2 e^{-2t})\delta_{-1}(t) + 6(A_1 + A_2)\delta(t)}_{a_1 \frac{d}{dt} w(t)} \\ &\quad + \underbrace{2(A_1 e^{-t} + 4A_2 e^{-2t})\delta_{-1}(t) + 2(-A_1 - 2A_2)\delta(t) + 2(A_1 + A_2)\delta_1(t)}_{a_2 \frac{d^2}{dt^2} w(t)} \\ &\quad = \underbrace{3\delta(t) + \delta_1(t)}_{b_0 \delta(t) + b_1 \delta_1(t)} \end{aligned}$$

The coefficients multiplying  $\delta_{-1}(t)$  will cancel each other out, always

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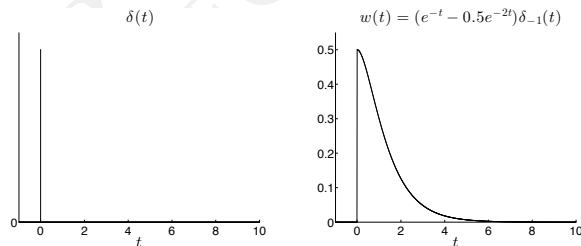
## Calculation of the impulse response (cont.)

Since  $m < n$  and thus  $A_0 = 0$ , we can write a system of two equations

$$\begin{cases} [a_1 h(0) + a_2 \dot{h}(0)] \delta(t) = b_0 \delta(t) \\ a_2 h(0) \delta_{-1}(t) = b_1 \delta_{-1}(t) \end{cases} \rightsquigarrow \begin{cases} 4A_1 + 2A_2 = 3 \\ 2A_1 + 2A_2 = 1 \end{cases} \rightsquigarrow \begin{cases} A_1 = 1 \\ A_2 = -0.5 \end{cases}$$

The resulting impulse response

$$w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$



■

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## Calculation of the impulse response (cont.)

### Example

Calculate the impulse response for the system described by the IO model

$$\frac{d^3 y(t)}{dt^3} + 2 \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} = 4 \frac{du(t)}{dt} + u(t)$$

The characteristic polynomial

$$P(s) = s^3 + 2s^2 + 5s$$

From  $P(s) = 0$ , the roots

- A real root  $p_1 = \alpha_1 = 0$  with multiplicity one  $\nu_1$
- A pair of conjugate complex roots  $p_2 = \alpha_2 \pm j\omega_2 = -1 \pm j2$  also with multiplicity one  $\nu_2 = \nu'_2 = 1$

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## Calculation of the impulse response (cont.)

A strictly proper model,  $m = 1 < n = 3$

- $w(t)$  without the impulsive term

$$\rightsquigarrow w(t) = h(t)\delta_{-1}(t) = [A_1 e^{p_1 t} + M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)]\delta_{-1}(t) = [A_1 + M_2 e^{-t} \cos(2t + \phi_2)]\delta_{-1}(t)$$

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## Calculation of the impulse response (cont.)

By differentiating  $h(t)$  two times,

$$\begin{aligned} h(t) &= A_1 + M_2 e^{-t} \cos(2t + \phi_2) \\ \dot{h}(t) &= -M_2 e^{-t} \cos(2t + \phi_2) - 2M_2 e^{-t} \sin(2t + \phi_2) \\ \ddot{h}(t) &= -3M_2 e^{-t} \cos(2t + \phi_2) + 4M_2 e^{-t} \sin(2t + \phi_2) \end{aligned}$$

We have the system of equations

$$\begin{cases} a_1 h(0) + a_2 \dot{h}(0) + a_3 \ddot{h}(0) = b_0 \\ a_2 h(0) + a_3 \dot{h}(0) = b_1 \\ a_3 h(0) = b_2 \end{cases} \rightsquigarrow \begin{cases} 5A_1 = 1 \\ 2A_1 + 5M_2 \cos(\phi_2) - 2M_2 \sin(\phi_2) = 4 \\ A_1 + M_2 \cos(\phi_2) = 0 \end{cases}$$

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## Calculation of the impulse response (cont.)

Let  $u_2 = M_2 \cos(\phi_2)$  and  $v_2 = M_2 \sin(\phi_2)$

$$\begin{cases} 5A_1 = 1 \\ 2A_1 + u_2 - 2v_2 = 4 \\ A_1 + u_2 = 0 \end{cases} \rightsquigarrow \begin{cases} A_1 = +0.2 \\ u_2 = -0.2 \\ v_2 = -1.9 \end{cases}$$

$$\rightsquigarrow \begin{cases} M_2 = \sqrt{u^2 + v^2} = 1.91 \\ \phi_2 = \arctan(u/v) = \arctan(-1.9/-0.2) = -1.68 \text{ [rad]} \end{cases}$$

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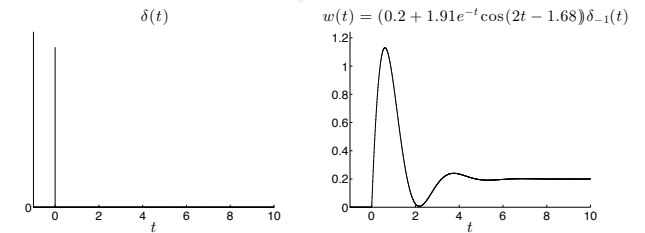
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## Calculation of the impulse response (cont.)

The impulse response

$$\rightsquigarrow w(t) = [0.2 + 1.91e^{-t} \cos(2t - 1.68)]\delta_{-1}(t)$$



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## Forced evolution and Duhamel's integral

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## Forced evolution (cont.)

We show a fundamental result in the analysis of linear IO models

- The **Duhamel's integral**

The forced evolution  $y_f(t)$  of a linear time-invariant system subjected to input  $u(t)$  is determined by its convolution with the impulse response  $w(t)$



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## Forced evolution (cont.)

### Convolution

Consider two functions  $f, g : \mathcal{R} \rightarrow \mathcal{C}$

The **convolution** of  $f$  with  $g$  is function  $h : \mathcal{R} \rightarrow \mathcal{C}$  in the real variable  $t$

$$\rightsquigarrow h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

Function  $h(t)$  is constructed using the operator **convolution integral**  $\star$

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## Forced evolution (cont.)

We start by assuming that the system is at some remote time  $t = -\infty$

- We assume that no cause has ever acted on it before

$\rightsquigarrow$  The system is therefore assumed to be at rest

At such a remote time, the system is subjected to an input  $u(t)$

- We assume that the input  $u(t)$  is known in  $(-\infty, t]$

$\rightsquigarrow$  This is needed to determine the output at time  $t$

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## Forced evolution (cont.)

### Proposition

#### Duhamel's integral

Consider a system at rest at  $t = -\infty$ , for every value of  $t \in \mathcal{R}$

We have,

$$\rightsquigarrow y(t) = \underbrace{\int_{-\infty}^t u(\tau)w(t-\tau)d\tau}_{\text{Duhamel's integral}}$$

### Proof

Let  $w_\varepsilon(t)$  be the forced response of the system due to a finite impulse  $\delta_\varepsilon(t)$

$$\delta_\varepsilon(t) = \frac{d}{dt}\delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & t \in [0, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

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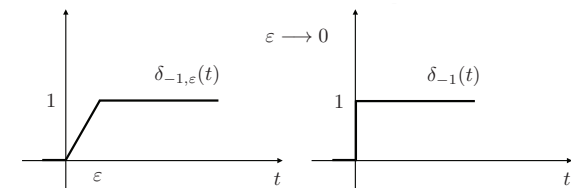
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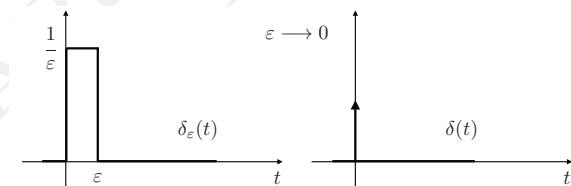
Forced evolution

## Forced evolution (cont.)



From the definition of the derivative of the unit step  $\delta_{-1}(t)$ , we have

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t) = \frac{d}{dt} \lim_{\varepsilon \rightarrow 0} \delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(t)$$



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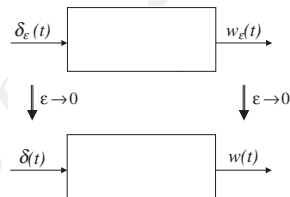
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## Forced evolution (cont.)

Because  $\delta(t) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(t)$ , it is intuitive to see that  $w(t) = \lim_{\varepsilon \rightarrow 0} w_\varepsilon(t)$

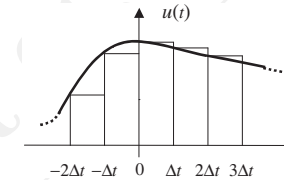


## Forced evolution (cont.)

We are interested in approximating the function  $u(t)$

We approximate  $u(t)$  with a series of rectangles

- Each rectangle is a finite impulse,  $\delta_{\Delta t}(\cdot)$



$\Delta t$  denotes the sampling time (the base of the rectangles)

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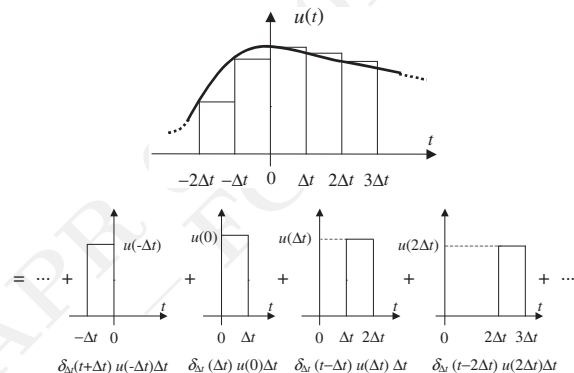
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## Forced evolution (cont.)

Each rectangle is assumed to be a finite impulse,  $\delta_{\Delta t}(t - k\Delta t)$

- Subscript  $\Delta t$  is the base of the rectangle (was  $\varepsilon$ )
- Argument  $(t - k\Delta t)$  right-shifts it by  $k\Delta t$



Each finite impulse is multiplied by the scaling factor  $u(k\Delta t)\Delta t$

- The area of a rectangle with base  $\Delta t$  and height  $u(k\Delta t)$

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## Forced evolution (cont.)

The approximation gets better as  $\Delta t$  gets smaller

Thus, we define

$$\rightsquigarrow u_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} u(k\Delta t) \delta_{\Delta t}(t - k\Delta t) \Delta t$$

We have,

$$\rightsquigarrow u(t) = \lim_{\Delta t \rightarrow 0} u_{\Delta t}(t)$$

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## Forced evolution (cont.)

The system is assumed to be linear (the superposition principle)

We approximate the total system output due to such an input  $u_{\Delta t}(t)$

- A sum of the outputs due to the component inputs

$$\rightsquigarrow y_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} u(k\Delta t)w_{\Delta t}(t - k\Delta t)\Delta t$$

Again, the approximation gets better as  $\Delta t$  gets smaller

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## Forced evolution (cont.)

The system is assumed to be proper (causes first, then effects)

- $w(t - \tau)$  is zero when  $(t - \tau) < 0$  ( $\tau \geq t$ )

We have,

$$\rightsquigarrow y(t) = \int_{-\infty}^{\infty} u(\tau)w(t - \tau)d\tau = \underbrace{\int_{-\infty}^t u(\tau)w(t - \tau)d\tau}_{\text{Duhamel's integral}}$$

■

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## Forced evolution (cont.)

We have,

$$\rightsquigarrow y(t) = \lim_{\Delta t \rightarrow 0} y_{\Delta t}(t) = \lim_{\Delta t \rightarrow 0} \sum_{k=-\infty}^{\infty} u(k\Delta t)w_{\Delta t}(t - k\Delta t)\Delta t$$

$$= \int_{-\infty}^{\infty} u(\tau)w(t - \tau)d\tau$$

As  $\Delta t \rightarrow 0$ , we let  $k\Delta t = \tau$  and  $\Delta t = d\tau$

- $\tau$  is now a real variable

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## Forced evolution (cont.)

The Duhamel's integral is a convolution integral

$$y(t) = \underbrace{\int_{-\infty}^t u(\tau)w(t - \tau)d\tau}_{\text{Duhamel's integral}}$$

The upper-limit is set to be  $t$  instead of  $+\infty$

- As the convolution of  $u(\tau)$  and  $w(\tau)$  is zero for  $\tau \geq t$

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**Forced evolution****Forced evolution (cont.)**

Convolution integrals possess the commutative property

We can write

$$y(t) = u \star w(t) = w \star u(t) \\ = \int_{-\infty}^{+\infty} u(t-\tau)w(\tau)d\tau = \int_0^{+\infty} u(t-\tau)w(\tau)d\tau$$

Moreover, for  $\tau < 0$  we have that  $w(\tau) = 0$ **Forced evolution (cont.)****Input-output representation**UFC/DC  
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**Forced evolution**

$$y(t) = \int_0^{+\infty} u(t-\tau)w(\tau)d\tau$$

Consider the contributions to  $y(t)$  at time  $t$ They are due to the value of the input  $u(t-\tau)$   $\tau$  times earlier

- Weighted by the impulse response  $w(\tau)$

Consider a system whose modes are all stable

- The impulse response  $w(t)$  tends to zero
- It is virtually zero for  $\tau > \bar{\tau}$
- $\bar{\tau}$  depends on system time-constant

The system loses memory of the input after a time  $\bar{\tau}$  from application**Input-output representation**UFC/DC  
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**Forced evolution****Forced evolution (cont.)****Decomposition in forced and force-free response**Consider some initial time  $t = t_0$ 

We decompose the Duhamel's integral

$$y(t) = \underbrace{\int_{-\infty}^{t_0} u(\tau)w(t-\tau)d\tau}_{y_u(t)} + \underbrace{\int_{t_0}^t u(\tau)w(t-\tau)d\tau}_{y_f(t)}, \quad \text{for } t \geq t_0$$

The first term  $y_u(t)$  is the contribution to the output signal at time  $t$  due to the values taken by the input before the initial time  $t_0$ 

- ↪ At time  $t_0$ , the system is a non null state
- ↪ (Non-zero initial conditions)
- ↪ Force-free evolution

The second term  $y_f(t)$  is the contribution to the output signal at time  $t$  due to the value taken by the input after the initial time  $t_0$ 

- ↪ Forced evolution

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**Forced evolution****Forced evolution (cont.)****Forced response by convolution**Consider some initial time  $t_0$ 

The forced evolution

$$\rightsquigarrow y_f(t) = \int_{t_0}^t u(\tau)w(t-\tau)d\tau = \int_0^{t-t_0} u(t-\tau)w(\tau)d\tau$$

The second formula is derived from the first one

- Change variable,  $\rho = t - \tau$

$$\rightsquigarrow \int_{t_0}^t u(\tau)w(t-\tau)d\tau \\ = \int_{t-t_0}^0 u(t-\rho)w(\rho)(-d\rho) = \int_0^{t-t_0} u(t-\rho)w(\rho)d\rho$$

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## Forced evolution (cont.)

Let  $t_0 = 0$ , we have the expression

$$\rightsquigarrow y_f(t) = \int_{t_0=0}^t u(\tau)w(t-\tau)d\tau = \int_{t_0=0}^t u(t-\tau)w(\tau)d\tau$$

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## Forced evolution (cont.)

### Example

Consider the system represented by the IO model

$$2\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{du(t)}{dt} + 3u(t)$$

We are interested in the forced evolution ( $t \geq 0$ ) due to input  $u(t) = 4\delta_{-1}(t)$

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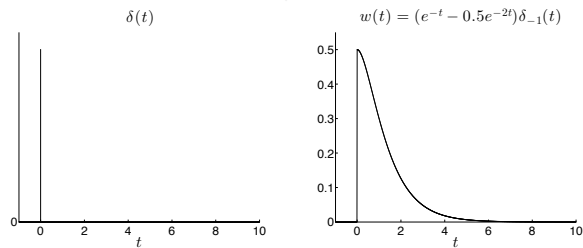
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## Forced evolution (cont.)

The impulse response of the system

$$\rightsquigarrow w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$



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## Forced evolution (cont.)

The forced response will be zero for  $t < 0$

For  $t \geq 0$ , we have

$$y_f(t) = \int_0^t u(\tau)w(t-\tau)d\tau$$

$$u(\tau) = 4, \text{ for } \tau \in [0, t]$$

$$\begin{aligned} \rightsquigarrow y_f(t) &= \int_0^t u(\tau)w(t-\tau)d\tau = \int_0^t 4[e^{-(t-\tau)} - 0.5e^{-2(t-\tau)}]d\tau \\ &= 4e^{-t} \int_0^t e^{\tau}d\tau - 2e^{-2t} \int_0^t e^{2\tau}d\tau \\ &= 4e^{-t}(e^{-t} - 1) - 2e^{-2t}(0.5e^{2t} - 0.5) \\ &= 3 - 4e^{-t} + e^{-2t} \end{aligned}$$

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## Forced evolution (cont.)

The forced response will be zero for  $t < 0$

For  $t \geq 0$ , we have

$$y_f(t) = \int_0^t u(t-\tau)w(\tau)d\tau$$

$$u(\tau) = 4, \text{ for } \tau \in [0, t]$$

$$\rightsquigarrow y_f(t) = \int_0^t 4[e^{-\tau} - 0.5e^{-2\tau}]d\tau = 4 \int_0^t e^{-\tau}d\tau - 2 \int_0^t e^{-2\tau}d\tau$$

$$= 4(e^{-t} - 1) - 2(0.5e^{-2t} - 0.5)$$

$$= 3 - 4e^{-t} + e^{-2t}$$

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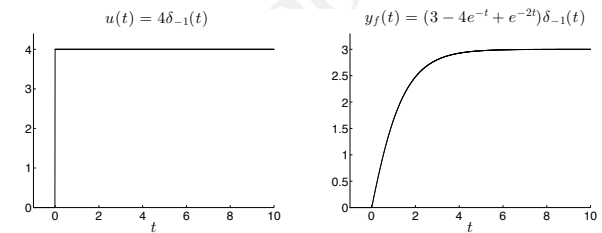
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## Forced evolution (cont.)

### Example

Consider the system represented by the IO model

$$2\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 4y(t) = \frac{du(t)}{dt} + 3u(t)$$

We are interested in the forced evolution ( $t \geq 0$ ) due to input  $u(t)$

$$u(t) = \begin{cases} 2, & t \in [1, 4) \\ 0, & \text{elsewhere} \end{cases}$$

This input can be understood as the sum of two functions

1. A step with size +2, at  $t = 1$
2. A step with size -2, at  $t = 4$

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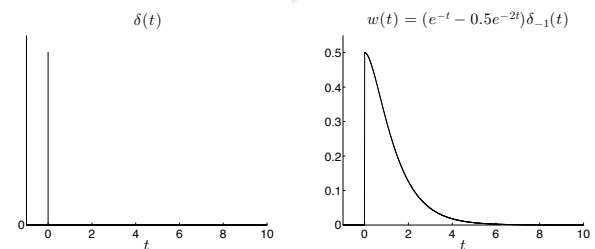
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## Forced evolution (cont.)

The impulse response of the system

$$\rightsquigarrow w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$



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## Forced evolution (cont.)

Using the Duhamels integral, we can calculate the forced response

$$y_f(t) = \int_{-\infty}^t u(\tau)w(t-\tau)d\tau$$

$$\rightsquigarrow = \begin{cases} 0, & t \in (-\infty, 1) \\ 2 \int_1^t w(t-\tau)d\tau, & t \in [1, 4) \\ 2 \int_1^4 w(t-\tau)d\tau, & t \in [4, +\infty) \end{cases}$$

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## Forced evolution (cont.)

The change of variable  $\rho = t - \tau$

For  $1 \leq t < 4$

$$\rightsquigarrow \int_0^t w(t-\tau)d\tau = \int_0^{t-1} w(\rho)d\rho = \int_0^{t-1} (e^{-\rho} - 0.5e^{-2\rho})d\rho$$

$$= 0.75 - e^{-(t-1)} + 0.25e^{-2(t-1)}$$

$$= 0.75 - 2.72e^{-t} + 1.85e^{-2t}$$

For  $t \geq 4$

$$\rightsquigarrow \int_1^4 w(t-\tau)d\tau = \int_{t-4}^{t-1} w(\rho)d\rho = \int_{t-4}^{t-1} (e^{-\rho} - 0.5e^{-2\rho})d\rho$$

$$= -e^{-(t-1)} + 0.25e^{-2(t-1)} + e^{-(t-4)} - 0.25e^{-2(t-4)}$$

$$= 51.9e^{-t} + 743e^{-2t}$$

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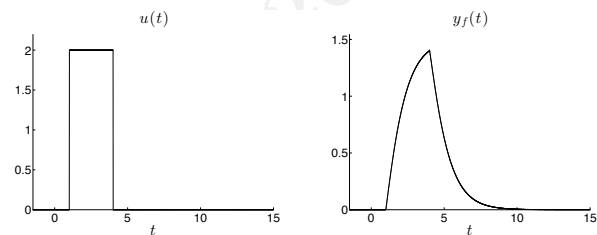
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## Forced evolution (cont.)

$$\rightsquigarrow y_f(t) = \begin{cases} 0, & t \in (-\infty, 1] \\ 1.5 - 5.44e^{-t} + 3.69e^{-2t}, & t \in [1, 4) \\ 104e^{-t} + 1487e^{-2t}, & t \in [4, +\infty) \end{cases}$$



The input signal  $u(t)$  is active only in the interval  $t \in [1, 4]$

- The response is not null for  $t \geq 4$
- At  $t = 4$  there is a non-null state

From  $t = 4$ , the evolution is force-free

