State-space representation

Stochastic algorithms

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SA (CK0191)
2018.1
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Basis of generalised eigenvectors
Generalised modal matrix
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State-space representation

Analysis in time of linear stationary systems in state-space representation

- The analysis problem
- The state transition matrix
- Sylvester expansion

- Lagrange formula

- Similarity transformations
- Diagonalisation
- Jordan's form

- Modes

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Representation and analysis

Consider a linear and stationary system of order n

- Let p be the number of outputs
- Let r be the number of inputs

The **state-space** representation of the system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{1}$$

- $\mathbf{x}(t)$ is the **state vector** (n components)
- $\dot{\mathbf{x}}(t)$ is the derivative of the state vector (n components)
- $\mathbf{u}(t)$ is the **input vector** (r components)
- $\mathbf{y}(t)$ is the **output vector** (p components)

$\mathbf{A}$ $(n \times n)$, $\mathbf{B}$ $(n \times r)$, $\mathbf{C}$ $(p \times n)$ and $\mathbf{D}$ $(p \times r)$ are matrices

- The elements are not function of time

Representation and analysis

The analysis problem

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Determine the behaviour of state $\mathbf{x}(t)$ and output $\mathbf{y}(t)$ for $t \geq t_0$

- We are given the input function $\mathbf{u}(t)$, for $t \geq t_0$
- We are given the initial state $\mathbf{x}(t_0)$

The solution

- The Lagrange formula
- We discuss it at length

We first introduce the state transition matrix

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The state transition matrix

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The state transition matrix

Consider some square matrix $\mathbf{A}$

Its exponential $e^{\mathbf{A}}$ is a matrix

$$\rightsquigarrow \quad e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

The state transition matrix $e^{\mathbf{A}t}$ is a matrix exponential

$\rightsquigarrow$ Its elements are functions of time

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The state transition matrix (cont.)

The exponential function

Let z be some scalar, by definition its exponential is a scalar

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

The series always converges

The matrix exponential

Let $\mathbf{A}$ be a $(n \times n)$ matrix, by definition its exponential is a $(n \times n)$ matrix

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

The series always converges

The state transition matrix (cont.)

The scalar-matrix product

Let $s \in \mathcal{R}$ and let $\mathbf{A} = \{a_{i,j}\}$ be a $(m \times n)$ matrix

$$\mathbf{B} = s\mathbf{A} = \begin{bmatrix} s \cdot a_{1,1} & \cdots & s \cdot a_{1,j} & \cdots & s \cdot a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{i,1} & \cdots & s \cdot a_{i,j} & \cdots & s \cdot a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{m,1} & \cdots & s \cdot a_{m,j} & \cdots & s \cdot a_{m,n} \end{bmatrix}$$

The product of $\mathbf{A}$ and s is defined as the $(m \times n)$ matrix $\mathbf{B} = \{b_{i,j}\}$

$$\mathbf{B} = \{b_{i,j} = s \cdot a_{i,j}\}$$

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The state transition matrix (cont.)

The matrix product

Let $\mathbf{A} = \{a_{i,j}\}$ be a $(m \times n)$ matrix and let $\mathbf{B} = \{b_{i,j}\}$ be a $(n \times p)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product between $\mathbf{A}$ and $\mathbf{B}$ is defined as a $(m \times p)$ matrix $\mathbf{C} = \{c_{i,j}\}$

$$\mathbf{C} = \{c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}\}$$

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The state transition matrix (cont.)

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,j} & \cdots & c_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i,1} & \cdots & c_{i,j} & \cdots & c_{i,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m,1} & \cdots & c_{m,j} & \cdots & c_{m,p} \end{bmatrix}$$

Element $c_{i,j}$ of matrix $\mathbf{C}$ is given by the dot product between $\mathbf{a}_i'$ and $\mathbf{b}_j$

$$c_{i,j} = \mathbf{a}_i' \mathbf{b}_j = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,k} & \cdots & a_{i,n} \end{bmatrix} \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{k,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

$$= a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

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The state transition matrix (cont.)

For every $(m \times n)$ matrix $\mathbf{A}$, we have

$$\underbrace{\mathbf{I}_m}_{(m \times m)} \underbrace{\mathbf{A}}_{(m \times n)} = \underbrace{\mathbf{A}}_{(m \times n)} \underbrace{\mathbf{I}_n}_{(n \times n)} = \underbrace{\mathbf{A}}_{(m \times n)}$$

Right- and left-multiplication of matrix $\mathbf{A}$ by an identity matrix ($\mathbf{I}_n$ or $\mathbf{I}_m$)

The state transition matrix (cont.)

Matrix product is not necessarily commutative, $\mathbf{AB} \neq \mathbf{BA}$

$$\underbrace{\mathbf{A}}_{(m\times n)}\ \underbrace{\mathbf{B}}_{(n\times p)} = \underbrace{\mathbf{C}}_{(m\times p)}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product $\mathbf{BA}$ is not even defined

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The state transition matrix (cont.)

For $\mathbf{AB} = \mathbf{BA}$, $\mathbf{A}$ and $\mathbf{B}$ must be both square and of the same order

- (necessary condition)

A $(n \times n)$ diagonal matrix $\mathbf{D}$ commutes with any $(n \times n)$ matrix $\mathbf{A}$

$$\mathbf{DA} = \mathbf{AD}$$

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The state transition matrix (cont.)

The product of several matrices

The product of $\mathbf{A}$ and $\mathbf{B}$ is only possible when the matrixes are compatible

- Number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$

The same applies to the product of several matrixes

$$\underbrace{\mathbf{M}}_{(m\times n)} = \underbrace{\mathbf{A}_1}_{(m\times m_1)}\ \underbrace{\mathbf{A}_2}_{(m_1\times m_2)} \cdots \underbrace{\mathbf{A}_{k-1}}_{(m_{k-2}\times m_{k-1})}\ \underbrace{\mathbf{A}_k}_{(m_{k-1}\times n)}$$

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The state transition matrix (cont.)

Powers of a matrix

Let $\mathbf{A}$ be an order-n square matrix

The k-th power of matrix $\mathbf{A}$ is defined as the n-order matrix $\mathbf{A}^k$

$$\mathbf{A}^k = \underbrace{\mathbf{AA}\cdots\mathbf{A}}_{k \text{ times}}$$

Special cases,

$\rightsquigarrow$ $\mathbf{A}^{k=0} = \mathbf{I}$

$\rightsquigarrow$ $\mathbf{A}^{k=1} = \mathbf{A}$

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The state transition matrix

Definition

The state transition matrix

Consider the state-space model with $(n \times n)$ *matrix* $\mathbf{A}$

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

The state transition matrix is the $(n \times n)$ *matrix* $e^{\mathbf{A}t}$

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \qquad (2)$$

The state transition matrix is well defined for any square matrix $\mathbf{A}$

- *(The series always converges)*

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The state transition matrix (cont.)

Not convenient to determine the state transition matrix from its definition

- ⇝ There are more efficient procedures for the task
- ⇝ One exception, when $\mathbf{A}$ is (block-)diagonal

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The state transition matrix (cont.)

The matrix exponential of block-diagonal matrixes

Consider any block-diagonal matrix $\mathbf{A}$, we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_q \end{bmatrix} \rightsquigarrow e^{\mathbf{A}} = \begin{bmatrix} e^{\mathbf{A}_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{A}_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & e^{\mathbf{A}_q} \end{bmatrix}$$

The state transition matrix (cont.)

Proof

For all $k \in \mathcal{N}$, we have

$$\mathbf{A}^k = \begin{bmatrix} \mathbf{A}_1^k & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^k & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_q^k \end{bmatrix}$$

Thus,

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\mathbf{A}_1^k}{k!} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sum_{k=0}^{\infty} \frac{\mathbf{A}_2^k}{k!} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sum_{k=0}^{\infty} \frac{\mathbf{A}_q^k}{k!} \end{bmatrix}$$

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The state transition matrix (cont.)

The matrix exponential of diagonal matrixes

For any diagonal $(n \times n)$ matrix $\mathbf{A}$, we have

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \rightsquigarrow e^{\mathbf{A}} = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

The result is a special case of the previous proposition

The state transition matrix (cont.)

Proposition

Consider the state-space model with $(n \times n)$ diagonal matrix $\mathbf{A}$

We have,

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \rightsquigarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Proof

We have,

$$\mathbf{A}t = \begin{bmatrix} \lambda_1 t & 0 & \cdots & 0 \\ 0 & \lambda_2 t & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n t \end{bmatrix} \rightsquigarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

This matrix is diagonal, we used the result from the previous proposition

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The state transition matrix (cont.)

Example

Consider the state-space model with (2×2) diagonal matrix $\mathbf{A}$

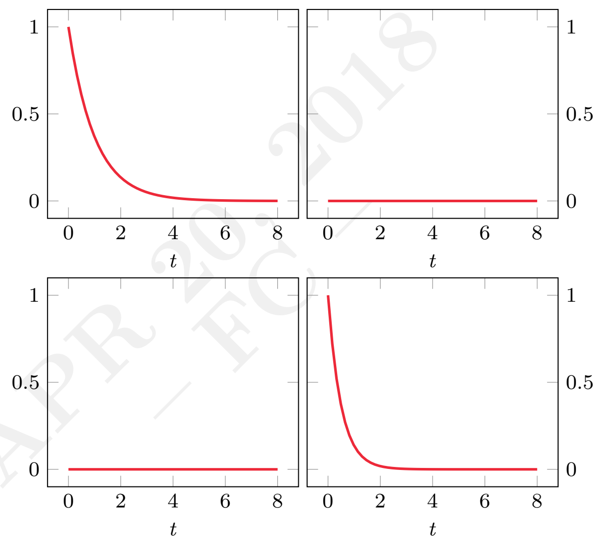
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

We are interested in the corresponding state transition matrix

We have,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{(-1)t} & 0 \\ 0 & e^{(-2)t} \end{bmatrix}$$

The state transition matrix (cont.)



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Properties

We present some fundamental results about the state transition matrix $e^{\mathbf{A}t}$

⇝ They are needed to derive Lagrange formula

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Properties (cont.)

Proposition

Derivative of the state transition matrix

Consider the state transition matrix $e^{\mathbf{A}t}$

We have,

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

Proof

To prove the first equality, we differentiate $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k t^k/k!$

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathbf{A}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{A}^k t^k}{k!} = \sum_{k=1}^{\infty}\frac{\mathbf{A}^k k t^{k-1}}{k!}$$

$$\rightsquigarrow \quad = \mathbf{A}\sum_{k=1}^{\infty}\frac{\mathbf{A}^{k-1}t^{k-1}}{(k-1)!} = \mathbf{A}\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^k}{k!} = \mathbf{A}e^{\mathbf{A}t}$$

The second equality is obtained by collecting $\mathbf{A}$ on the right

Properties (cont.)

By using the derivative property, we have that $\mathbf{A}$ commutes with $e^{\mathbf{A}t}$

$\rightsquigarrow$ That is, $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$

$\mathbf{A}$ and $e^{\mathbf{A}t}$ commute (this result is important)

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Properties (cont.)

Proposition

Composition of two state transition matrices

Consider the two state transition matrices $e^{\mathbf{A}t}$ *and* $e^{\mathbf{A}\tau}$

We have,

$$e^{\mathbf{A}t}e^{\mathbf{A}\tau} = e^{\mathbf{A}(t+\tau)}$$

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Properties (cont.)

Proof

We expand both exponentials in their corresponding series and multiply

$$e^{\mathbf{A}t}e^{\mathbf{A}\tau} = \Big(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots\Big)\Big(\mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} + \frac{\mathbf{A}^3\tau^3}{3!} + \cdots\Big)$$

$$= \left\{\begin{array}{lllllllllll} \mathbf{I} & + & \mathbf{A}\tau & + & \frac{\mathbf{A}^2\tau^2}{2!} & + & \frac{\mathbf{A}^3\tau^3}{3!} & + & \frac{\mathbf{A}^4\tau^4}{4!} & & \cdots \\ & + & \mathbf{A}t & + & \mathbf{A}^2t\tau & + & \frac{\mathbf{A}^3t\tau^2}{2!} & + & \frac{\mathbf{A}^4t\tau^3}{3!} & & \cdots \\ & & & + & \frac{\mathbf{A}^2t^2}{2!} & + & \frac{\mathbf{A}^3t^2\tau}{2!} & + & \frac{\mathbf{A}^4t^2\tau^2}{2!2!} & & \cdots \\ & & & & & + & \frac{\mathbf{A}^3t^3}{3!} & + & \frac{\mathbf{A}^4t^3\tau}{3!} & & \cdots \\ & & & & & & & + & \frac{\mathbf{A}^4t^4}{4!} & & \cdots \\ & & & & & & & & & & \cdots \end{array}\right.$$

$$= \mathbf{I} + \mathbf{A}(t+\tau) + \frac{\mathbf{A}^2}{2!}(t^2 + 2t\tau + \tau^2)^2 + \frac{\mathbf{A}^3}{3!}(t^3 + 3t^2\tau + 3t\tau^2 + \tau^3)$$
$$+ \frac{\mathbf{A}^4}{4!}(t^4 + 4t^3\tau + 6t^2\tau^2 + 4t\tau^3 + \tau^4) + \cdots$$

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Properties (cont.)

$$e^{\mathbf{A}t}e^{\mathbf{A}\tau} = \mathbf{I} + \mathbf{A}(t+\tau) + \frac{\mathbf{A}^2(t+\tau)}{2!} + \frac{\mathbf{A}^3(t+\tau)^3}{3!} + \frac{\mathbf{A}^4(t+\tau)^4}{4!} + \cdots$$

$$\rightsquigarrow \quad = \sum_{k=0}^{\infty}\frac{\mathbf{A}^k(t+\tau)^k}{k!} = e^{\mathbf{A}(t+\tau)}$$

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Properties (cont.)

The previous result is not trivial

In the scalar case, we always have $e^{at}e^{a\tau} = e^{a(t+\tau)}$ or $e^{at}e^{bt} = e^{(a+b)t}$

In the matrix case, it is not necessarily true that $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{(\mathbf{A}+\mathbf{B})t}$

- ⇝ Equality holds if and only if $\mathbf{AB} = \mathbf{BA}$
- ⇝ (If the matrices commute)

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Properties (cont.)

Proposition

Inverse of the state transition matrix

Let $e^{\mathbf{A}t}$ be a state transition matrix

Its inverse $\left(e^{\mathbf{A}t}\right)^{-1}$ is matrix $e^{-\mathbf{A}t}$

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{-\mathbf{A}t}e^{\mathbf{A}t} = \mathbf{I}$$

Proof

Based on the previous proposition, we have

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{\mathbf{A}(t-t)} = e^{\mathbf{A}\cdot 0} = \mathbf{I} + \mathbf{A}\cdot 0 + \frac{\mathbf{A}^2\cdot 0^2}{2!} + \frac{\mathbf{A}^3\cdot 0^3}{3!} + \cdots = \mathbf{I}$$

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Properties (cont.)

A state transition matrix $e^{\mathbf{A}t}$ is always invertible (non-singular)

- Even if $\mathbf{A}$ were singular

The result follows from the previous proposition

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Properties (cont.)

Matrix inverse

Consider a square matrix $\mathbf{A}$ of order n

We define the inverse of $\mathbf{A}$ the square matrix of order n, $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

The inverse of matrix $\mathbf{A}$ exists if and only if $\mathbf{A}$ is non-singular

- When the inverse exists it is unique

Properties (cont.)

Matrix minors

Consider a square matrix $\mathbf{A}$ of order $n \geq 2$

The **minor** (i, j) of matrix $\mathbf{A}$ is a square matrix $\mathbf{A}_{i,j}$ of order $(n-1)$

$$\mathbf{A}_{i,j} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cancel{a_{1,j}} & \cdots & a_{1,p} \\ a_{2,1} & a_{2,2} & \cdots & \cancel{a_{2,j}} & \cdots & a_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \cancel{a_{i,1}} & \cancel{a_{i,2}} & \cdots & \cancel{a_{i,j}} & \cdots & \cancel{a_{i,p}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & \cancel{a_{m,j}} & \cdots & a_{m,p} \end{bmatrix}$$

It is obtained from $\mathbf{A}$ by deleting the i-th row and the j-th column

Properties (cont.)

Matrix determinant

Consider a square matrix $\mathbf{A}$ of order n

The determinant of $\mathbf{A}$ is a real number

$$\det(\mathbf{A}) = |\mathbf{A}|$$

- For $n = 1$, let $\mathbf{A} = [a_{1,1}]$, we have

$$\rightsquigarrow \quad \det(A) = a_{1,1}$$

- For $n \geq 2$, we have

$$\rightsquigarrow \quad \det(\mathbf{A}) = a_{1,1}\hat{a}_{1,1} + a_{2,1}\hat{a}_{2,1} + \cdots + a_{n,1}\hat{a}_{n,1} = \sum_{i=1}^{n} a_{i,1}\hat{a}_{i,1}$$

$\hat{a}_{i,j}$ denotes the **cofactor** of element (i, j), it is a scalar

- It is equal to the determinant of minor $\mathbf{A}_{i,j}$ multiplied by $(-1)^{i+j}$

Sylvester expansion

Sylvester expansion

Sylvester expansion

We determine the analytical expression of the state transition matrix $e^{\mathbf{A}t}$

- (without necessarily calculating the infinite expansion)

The procedure is known as **Sylvester expansion**

- There are also other procedures
- (We discuss them later on)

Sylvester expansion (cont.)

Proposition

The Sylvester expansion

Let $\mathbf{A}$ *be a* $(n \times n)$ *matrix*

The corresponding state transition matrix is $e^{\mathbf{A}t}$

We have,

$$
\begin{aligned}
e^{\mathbf{A}t} &= \sum_{i=0}^{n-1} \beta_i(t) \mathbf{A}^i \\
&= \beta_0(t)\mathbf{I} + \beta_1(t)\mathbf{A} + \beta_2(t)\mathbf{A}^2 + \cdots + \beta_{n-1}(t)\mathbf{A}^{n-1}
\end{aligned}
\quad (3)
$$

The coefficients of the expansion β_i *are appropriate functions of time*

- *They can be determined by solving a set of linear equations*

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Sylvester expansion (cont.)

We discuss how to determine the coefficients of the expansion

We individually consider several cases

- Eigenvalues of $\mathbf{A}$ have multiplicity one
- Eigenvalues of $\mathbf{A}$ have multiplicity larger than one
- Matrix $\mathbf{A}$ has complex eigenvalues (with multiplicity one)

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Sylvester expansion (cont.)

Eigenvalues and eigenvectors

Let $\lambda \in \mathcal{R}$ be some scalar and let $\mathbf{v} \neq \mathbf{0}$ be a $(n \times 1)$ column vector

Consider a square matrix $\mathbf{A}$ of order n

Suppose that the identify holds

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

The scalar λ is called an **eigenvalue** of $\mathbf{A}$

The vector $\mathbf{v}$ is called the associated **eigenvector**

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Sylvester expansion (cont.)

Consider a square matrix $\mathbf{A}$ of order n whose elements are real numbers

Matrix $\mathbf{A}$ has n (not necessarily distinct) eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$

- They can be real numbers or conjugate-complex pairs

If $\lambda_i \neq \lambda_j$ for $i \neq j$, we say that matrix $\mathbf{A}$ has multiplicity one

Sylvester expansion (cont.)

Eigenvalues of triangular and diagonal matrices

Let matrix $\mathbf{A} = \{a_{i,j}\}$ be triangular or diagonal

The eigenvalues of $\mathbf{A}$ are the n diagonal elements $\{a_{i,i}\}$, $i = 1, 2, \dots, n$

Sylvester expansion (cont.)

Characteristic polynomial

The **characteristic polynomial** of a square matrix $\mathbf{A}$ of order n

- The n-order polynomial in the variable s

$$P(s) = \det(s\mathbf{I} - \mathbf{A})$$

Computing eigenvalues and eigenvectors

The eigenvalues of matrix $\mathbf{A}$ of order n solve its characteristic polynomial

$\rightsquigarrow$ The roots of the equation $P(s) = \det(s\mathbf{I} - \mathbf{A}) = 0$

Let λ be an eigenvalue of matrix $\mathbf{A}$

Each eigenvector $\mathbf{v}$ associated to it is a non-trivial solution to the system

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$

$\mathbf{0}$ is a $(n \times 1)$ column-vector whose elements are all zero

Sylvester expansion (cont.)

Proof

An eigenvalue λ and an eigenvector $\mathbf{v}$ must satisfy

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

$(\lambda\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$ follows from this identity

The non-trivial solution $\mathbf{v} \neq \mathbf{0}$ is admissible iff matrix $(\lambda\mathbf{I} - \mathbf{A})$ is singular

$$\rightsquigarrow \quad \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

Thus, λ is root to the characteristic polynomial of matrix $\mathbf{A}$

■

Sylvester expansion (cont.)

Systems of linear equations

Consider a system of n linear equations in n unknowns

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$\rightsquigarrow$ $\mathbf{A}$ is a $(n \times n)$ matrix of **coefficients**

$\rightsquigarrow$ $\mathbf{b}$ is a $(n \times 1)$ vector of **known terms**

$\rightsquigarrow$ $\mathbf{x}$ is a $(n \times 1)$ vector of **unknowns**

If matrix $\mathbf{A}$ is non-singular, the system admits one and only one solution

If $\mathbf{A}$ is singular, let $\mathbf{M} = [\mathbf{A}|\mathbf{b}]$ be a $[n \times (n+1)]$ matrix

- If $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{M})$, system has infinite solutions
- If $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{M})$, system has no solutions

Sylvester expansion (cont.)

Matrix rank

The **rank** of a $(m \times n)$ matrix $\mathbf{A}$ is equal to the number of columns (or rows) of the matrix that are linearly independent

$$\text{rank}(\mathbf{A})$$

Define the minors of matrix $\mathbf{A}$ as any matrix obtained from $\mathbf{A}$ by deleting an arbitrary number of rows and columns

- $\text{rank}(\mathbf{A})$ equals the order of the largest non-singular square minor

Properties (cont.)

Matrix kernel or null space

Consider a $(m \times n)$ matrix $\mathbf{A}$

We define the **null space** or **kernel**

$$\ker(\mathbf{A}) = \{\mathbf{x} \in \mathcal{R}^n | \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

It is all vectors $\mathbf{x} \in \mathcal{R}^n$ that left-multiplied by $\mathbf{A}$ produce the null vector

The set is a vector space, its dimension is called the **nullity** of matrix $\mathbf{A}$

$$\text{null}(\mathbf{A})$$

Sylvester expansion (cont.)

Eigenvalues with multiplicity one

Let matrix $\mathbf{A}$ have distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$e^{\mathbf{A}t} = \sum_{i=0}^{n-1} \beta_i(t)\mathbf{A}^i$$
$$= \beta_0(t)\mathbf{I} + \beta_1(t)\mathbf{A} + \beta_2(t)\mathbf{A}^2 + \cdots + \beta_{n-1}(t)\mathbf{A}^{n-1}$$

The n unknown functions $\beta_i(t)$ are those that solve the system

$$\rightsquigarrow \begin{cases} 1\beta_0(t) + \lambda_1\beta_1(t) + \lambda_1^2\beta_2(t) + \cdots + \lambda_1^{n-1}\beta_{n-1}(t) = e^{\lambda_1 t} \\ 1\beta_0(t) + \lambda_2\beta_1(t) + \lambda_2^2\beta_2(t) + \cdots + \lambda_2^{n-1}\beta_{n-1}(t) = e^{\lambda_2 t} \\ \quad \ldots \\ 1\beta_0(t) + \lambda_n\beta_1(t) + \lambda_n^2\beta_2(t) + \cdots + \lambda_n^{n-1}\beta_{n-1}(t) = e^{\lambda_n t} \end{cases} \quad (4)$$

Sylvester expansion (cont.)

Or, equivalently,

$$\mathbf{V}\boldsymbol{\beta} = \boldsymbol{\eta} \quad (5)$$

- The vector of unknowns

$$\rightsquigarrow \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0(t) & \beta_1(t) & \cdots & \beta_{n-1}(t) \end{bmatrix}^T$$

- The coefficients matrix[1]

$$\rightsquigarrow \quad \mathbf{V} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

- The known vector

$$\rightsquigarrow \quad \boldsymbol{\eta} = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \end{bmatrix}^T$$

[1] A matrix in this form is known as Vandermonde matrix.

Sylvester expansion (cont.)

$$\boldsymbol{\eta} = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} & \cdots & e^{\lambda_n t} \end{bmatrix}^T$$

The components of vector η are functions of time, $e^{\lambda t}$

- Functions $e^{\lambda t}$ are the **modes** of matrix $\mathbf{A}$
- Mode $e^{\lambda t}$ associates with eigenvalue λ

Each element of $e^{\mathbf{A}t}$ is a linear combination of such modes

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Sylvester expansion (cont.)

Example

Consider the (2×2) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

We want to determine $e^{\mathbf{A}t}$

Matrix $\mathbf{A}$ is triangular, the eigenvalues correspond to the diagonal elements

Matrix $\mathbf{A}$ has 2 distinct eigenvalues

- $\lambda_1 = -1$
- $\lambda_2 = -2$

To determine $e^{\mathbf{A}t}$, we write the system

$$\begin{cases} \beta_0(t) + \lambda_1 \beta_1(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2 \beta_1(t) = e^{\lambda_2 t} \end{cases} \rightsquigarrow \begin{cases} \beta_0(t) - \beta_1(t) = e^{-t} \\ \beta_0(t) - 2\beta_1(t) = e^{-2t} \end{cases}$$

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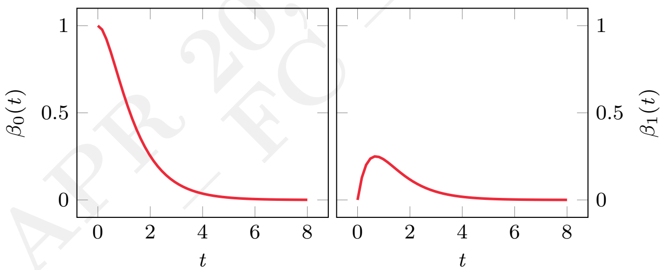
Sylvester expansion (cont.)

By simple manipulation, we get

$$\rightsquigarrow \begin{cases} \beta_0(t) = 2e^{-t} - e^{-2t} \\ \beta_1(t) = e^{-t} - e^{-2t} \end{cases}$$



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Sylvester expansion (cont.)

$$\begin{cases} \beta_0(t) = 2e^{-t} - e^{-2t} \\ \beta_1(t) = e^{-t} - e^{-2t} \end{cases}$$

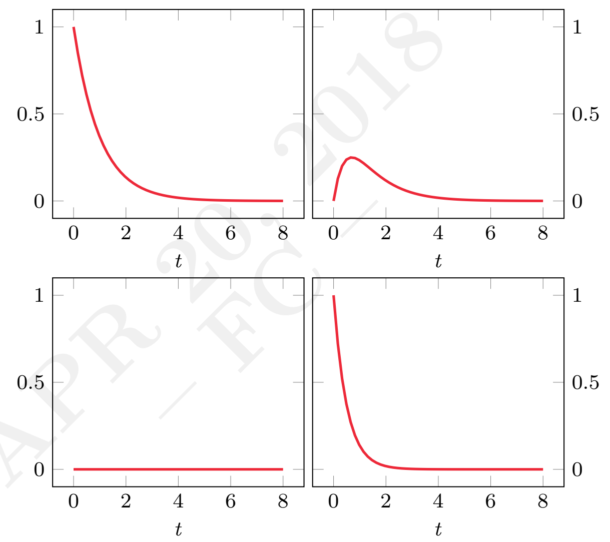
Thus,

$$\begin{aligned} e^{\mathbf{A}t} &= \beta_0(t)\mathbf{I}_2 + \beta_1(t)\mathbf{A} \\ &= (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Each element of matrix $e^{\mathbf{A}t}$ is a linear combination of the two modes

- e^{-t}
- e^{-2t}

Sylvester expansion (cont.)



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Sylvester expansion (cont.)

Eigenvalues with multiplicity larger than one

Let matrix $\mathbf{A}$ have eigenvalues with multiplicity larger than one

As in the previous case, we build a system of equations

Eigenvalues λ of multiplicity ν associate to ν equations

$$\rightsquigarrow \quad \begin{cases} \Big[\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t)\Big] = e^{\lambda t} \\ \dfrac{d}{d\lambda}\Big[\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t)\Big] = \dfrac{d}{d\lambda}e^{\lambda t} \\ \quad\vdots \\ \dfrac{d^{\nu-1}}{d\lambda^{\nu-1}}\Big[\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t)\Big] = \dfrac{d^{\nu-1}}{d\lambda^{\nu-1}}e^{\lambda t} \end{cases} \tag{6}$$

Sylvester expansion (cont.)

That is,

$$\rightsquigarrow \quad \begin{cases} 1\beta_0(t) + \lambda\beta_1(t) + \cdots + \lambda^{n-1}\beta_{n-1}(t) = e^{\lambda t} \\ 1\beta_1(t) + 2\lambda\beta_2(t) + \cdots + (n-1)\lambda^{n-2}\beta_{n-1}(t) = te^{\lambda t} \\ \quad\vdots \\ \dfrac{(\nu-1)!}{0!}\beta_{\nu-1}(t) + \cdots + \dfrac{(n-1)!}{(n-\nu)!}\lambda^{n-\nu}\beta_{n-1}(t) = t^{\nu-1}e^{\lambda t} \end{cases} \tag{7}$$

It is again possible to re-write the linear system in compact form

$$\rightsquigarrow \quad \mathbf{V}\boldsymbol{\beta} = \boldsymbol{\eta}$$

Sylvester expansion (cont.)

$$\mathbf{V}\boldsymbol{\beta} = \boldsymbol{\eta}$$

Consider the eigenvalues λ with multiplicity ν

- They are associated with ν rows in the coefficient matrix[2] $\mathbf{V}$

$$\rightsquigarrow \quad \mathbf{V} = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{\nu-1} & \cdots & \lambda^{n-1} \\ 0 & 1 & 2\lambda & \cdots & (\nu-1)\lambda^{\nu-2} & \cdots & (n-1)\lambda^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (\nu-1)! & \cdots & \dfrac{(n-1)!}{(n-\nu)!}\lambda^{n-\nu} \end{bmatrix}$$

- They are associated with ν rows in the vector of known terms $\boldsymbol{\eta}$

$$\rightsquigarrow \quad \boldsymbol{\eta} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & t^{\nu-1}e^{\lambda t} \end{bmatrix}^T$$

- The vector of unknowns

$$\rightsquigarrow \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0(t) & \beta_1(t) & \cdots & \beta_{n-1}(t) \end{bmatrix}^T$$

■

[2] A matrix of this form is known as confluent Vandermonde matrix.

Sylvester expansion (cont.)

Example

Consider the (3×3) matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 1.5 \\ 0 & 0 & 3 \end{bmatrix}$$

We want to determine $e^{\mathbf{A}t}$

The characteristic polynomial of matrix $\mathbf{A}$

$$P(s) = (s-3)^2(s+1)$$

Matrix $\mathbf{A}$ has two eigenvalues

- ⇝ $\lambda_1 = +3$ (multiplicity 2)
- ⇝ $\lambda_2 = -1$ (multiplicity 1)

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Sylvester expansion (cont.)

We can write the system

$$\begin{cases} \beta_0(t) + \lambda_1\beta_1(t) + \lambda_1^2\beta_2(t) = e^{\lambda_1 t} \\ \beta_1(t) + 2\lambda_1\beta_2(t) = te^{\lambda_1 t} \\ \beta_0(t) + \lambda_2\beta_1(t) + \lambda_2^2\beta_2(t) = e^{\lambda_2 t} \end{cases}$$

$$\rightsquigarrow \begin{cases} \beta_0(t) + 3\beta_1(t) + 9\beta_2(t) = e^{3t} \\ \beta_1(t) + 6\beta_2(t) = te^{3t} \\ \beta_0(t) - \beta_1(t) + \beta_2(t) = e^{-t} \end{cases}$$

We get,

$$\rightsquigarrow \begin{cases} \beta_0(t) = 1/16(7e^{3t} - 12te^{3t} + 9e^{-t}) \\ \beta_1(t) = 1/8(3e^{3t} - 4te^{3t} - 3e^{-t}) \\ \beta_2(t) = 1/16(-e^{3t} + 4te^{3t} + e^{-t}) \end{cases}$$

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Sylvester expansion (cont.)

Thus,

$$e^{\mathbf{A}t} = \beta_0(t)\mathbf{I}_3 + \beta_1(t)\mathbf{A} + \beta_2(t)\mathbf{A}^2$$

$$= \begin{bmatrix} e^{3t} & 0 & te^{3t} \\ (0.5e^{3t} - 0.5e^{-t}) & e^{-t} & (0.25e^{3t} + 0.5te^{3t} - 0.25e^{-t}) \\ 0 & 0 & e^{3t} \end{bmatrix}$$

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Sylvester expansion (cont.)

Complex eigenvalues

Let matrix $\mathbf{A}$ have complex eigenvalues

We can still determine the coefficients β of the Sylvester expansion

It is convenient to modify the procedure

- ⇝ To avoid computations that involve complex numbers

We only discuss only the case of eigenvalues with multiplicity one

Sylvester expansion (cont.)

Let matrix $\mathbf{A}$ have distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

The n unknown functions $\beta_i(t)$ are those that solve the system

$$\rightsquigarrow \begin{cases} \beta_0(t) + \lambda_1 \beta_1(t) + \lambda_1^2 \beta_2(t) + \cdots + \lambda_1^{n-1} \beta_{n-1}(t) = e^{\lambda_1 t} \\ \beta_0(t) + \lambda_2 \beta_1(t) + \lambda_2^2 \beta_2(t) + \cdots + \lambda_2^{n-1} \beta_{n-1}(t) = e^{\lambda_2 t} \\ \quad \vdots \\ \beta_0(t) + \lambda_n \beta_1(t) + \lambda_n^2 \beta_2(t) + \cdots + \lambda_n^{n-1} \beta_{n-1}(t) = e^{\lambda_n t} \end{cases} \tag{8}$$

Suppose that two of the n eigenvalues of $\mathbf{A}$ are complex-conjugate

$\rightsquigarrow$ $\lambda, \lambda' = \alpha \pm j\omega$

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Sylvester expansion (cont.)

In the resulting system, there should appear the two equations

$$\rightsquigarrow \begin{cases} 1\beta_0(t) + \lambda \beta_1(t) + \lambda^2 \beta_2(t) + \cdots + \lambda^{n-1} \beta_{n-1}(t) \\ \qquad = e^{\lambda t} = e^{\alpha t} e^{j\omega t} \\ 1\beta_0(t) + \lambda' \beta_1(t) + (\lambda')^2 \beta_2(t) + \cdots + (\lambda')^{n-1} \beta_{n-1}(t) \\ \qquad = e^{\lambda' t} = e^{\alpha t} e^{-j\omega t} \end{cases} \tag{9}$$

We can substitute these equations with two equivalent ones

$$\rightsquigarrow \begin{cases} \beta_0(t) + \text{Re}(\lambda)\beta_1(t) + \text{Re}(\lambda^2)\beta_2(t) + \cdots + \text{Re}(\lambda^{n-1})\beta_{n-1}(t) \\ \qquad = e^{\alpha t} \cos(\omega t) \\ \text{Im}(\lambda)\beta_1(t) + \text{Im}(\lambda^2)\beta_2(t) + \cdots + \text{Im}(\lambda^{n-1})\beta_{n-1}(t) \\ \qquad = e^{\alpha t} \sin(\omega t) \end{cases} \tag{10}$$

$\rightsquigarrow$ $\text{Re}(\lambda) = \alpha$

$\rightsquigarrow$ $\text{Im}(\lambda) = \omega$

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Sylvester expansion (cont.)

$$\begin{cases} 1\beta_0(t) + \lambda \beta_1(t) + \lambda^2 \beta_2(t) + \cdots + \lambda^{n-1} \beta_{n-1}(t) \\ \qquad = e^{\lambda t} = e^{\alpha t} e^{j\omega t} \\ 1\beta_0(t) + \lambda' \beta_1(t) + (\lambda')^2 \beta_2(t) + \cdots + (\lambda')^{n-1} \beta_{n-1}(t) \\ \qquad = e^{\lambda' t} = e^{\alpha t} e^{-j\omega t} \end{cases}$$

The first equation, is obtained by summing the two equations above

- Then, by dividing by 2

The second one, by subtracting the second equation from the first one

- Then, by dividing by $2j$

$$\rightsquigarrow \begin{cases} \beta_0(t) + \text{Re}(\lambda)\beta_1(t) + \text{Re}(\lambda^2)\beta_2(t) + \cdots + \text{Re}(\lambda^{n-1})\beta_{n-1}(t) \\ \qquad = e^{\alpha t} \cos(\omega t) \\ \text{Im}(\lambda)\beta_1(t) + \text{Im}(\lambda^2)\beta_2(t) + \cdots + \text{Im}(\lambda^{n-1})\beta_{n-1}(t) \\ \qquad = e^{\alpha t} \sin(\omega t) \end{cases}$$

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Sylvester expansion (cont.)

Sine and cosine terms on the RHS are from Euler formulæ

As λ and λ' are conjugate-complex, so are λ^k and $(\lambda')^k$

Thus,

$$\lambda^k + (\lambda')^k = 2\text{Re}(\lambda^k)$$
$$\lambda^k - (\lambda')^k = 2j\text{Im}(\lambda^k)$$

Sylvester expansion (cont.)

Example

Consider a state-space system with (2×2) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$$

We are interested in the state transition matrix $e^{\mathbf{A}t}$

Matrix $\mathbf{A}$ has characteristic polynomial

$$P(s) = s^2 - 2\alpha s + (\alpha^2 + \omega^2)$$

Matrix $\mathbf{A}$ has distinct eigenvalues

$\rightsquigarrow$ $\lambda, \lambda' = \alpha \pm j\omega$

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Sylvester expansion (cont.)

To determine the state-transition matrix $e^{\mathbf{A}t}$, we write the system

$$\begin{cases} \beta_0(t) + \text{Re}(\lambda)\beta_1(t) = e^{\alpha t}\cos(\omega t) \\ \text{Im}(\lambda)\beta_1(t) = e^{\alpha t}\sin(\omega t) \end{cases}$$

$$\rightsquigarrow \begin{cases} \beta_0(t) + \alpha\beta_1(t) = e^{\alpha t}\cos(\omega t) \\ \omega\beta_1(t) = e^{\alpha t}\sin(\omega t) \end{cases}$$

We obtain,

$$\begin{cases} \beta_0(t) &= e^{\alpha t}\cos(\omega t) - \dfrac{\alpha e^{\alpha t}}{\omega}\sin(\omega t) \\ \beta_1(t) &= \dfrac{e^{\alpha t}}{\omega}\sin(\omega t) \end{cases}$$

Thus,

$$e^{\mathbf{A}t} = \beta_0(t)\mathbf{I}_2 + \beta_1(t)\mathbf{A} = e^{\alpha t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

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Lagrange formula

We can now prove the solution to the analysis problem for MIMO systems

- Lagrange formula

Lagrange formula (cont.)

Theorem

Lagrange formula

Consider the SS representation of a stationary linear system of order n

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- $\mathbf{x}(t)$, *state vector* (n *components*)
- $\dot{\mathbf{x}}(t)$, *derivative of the state vector* (n *components*)
- $\mathbf{u}(t)$, *input vector* (r *components*)
- $\mathbf{y}(t)$, *output vector* (p *components*)

The solution for $t \geq t_0$, for an initial state $\mathbf{x}(t_0)$ and an input $\mathbf{u}(t|t \geq t_0)$

$$\begin{cases} \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t) \end{cases} \quad (11)$$

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Lagrange formula (cont.)

Proof

Multiply the state equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ by $e^{-\mathbf{A}t}$

We get,

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

The resulting state equation can be rewritten,

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Then, by using the result on the derivative of the state transition matrix[3],

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-\mathbf{A}t}\mathbf{x}(t)\Big] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

[3] Derivative of the state transition matrix

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-\mathbf{A}t}\mathbf{x}(t)\Big] &= e^{-\mathbf{A}t}\Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t)\Big] + \Big[\frac{\mathrm{d}}{\mathrm{d}t}e^{\mathbf{A}t}\Big]\mathbf{x}(t) \\ &= e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) \end{aligned} \quad (12)$$

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Lagrange formula (cont.)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[e^{-\mathbf{A}t}\mathbf{x}(t)\Big] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

By integrating between t_0 and t, we obtain

$$\Big[e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\Big]_{t_0}^{t} = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau$$

That is,

$$e^{\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t)$$

Thus,

$$e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t)$$

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Lagrange formula (cont.)

$$e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t_0}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(t)$$

The first Lagrange formula is obtained by multiplying both sides by $e^{\mathbf{A}\mathbf{t}}$

$$\rightsquigarrow \quad \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau$$

The second formula is obtained by substituting $\mathbf{x}(t)$ in the output equation

$$\begin{aligned} &\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \\ &\rightsquigarrow \mathbf{C}\Big[e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau\Big] + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

■

Force-free and forced evolution

Lagrange formula

Force-free and forced evolution

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\mathbf{x}_u(t)} + \underbrace{\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau}_{\mathbf{x}_f(t)}$$

We can write the state solution (for $t \geq t_0$) as the sum of two terms

$$\mathbf{x}(t) = \mathbf{x}_u(t) + \mathbf{x}_f(t)$$

- ⇝ The force-free evolution of the state, $\mathbf{x}_u(t)$
- ⇝ The forced evolution of the state, $\mathbf{x}_f(t)$

Force-freee and forced evolution (cont.)

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\text{force-free evolution } \mathbf{x}_u(t)} + \underbrace{\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau}_{\text{forced evolution } \mathbf{x}_f(t)}$$

The force-free evolution of the state, from the initial condition $\mathbf{x}(t_0)$

$$\rightsquigarrow \quad \mathbf{x}_l(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) \tag{13}$$

- ⇝ $e^{\mathbf{A}(t-t_0)}$ indicates the transition from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$
- ⇝ In the absence of contribution from the input

The forced evolution of the state

$$\rightsquigarrow \quad \mathbf{x}_f(t) = \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau = \int_{0}^{t-t_0} e^{\mathbf{A}t}\mathbf{B}\mathbf{u}(t-\tau)\mathrm{d}\tau \tag{14}$$

- ⇝ The contribution of $\mathbf{u}(\tau)$ to state $\mathbf{x}(t)$
- ⇝ Thru a weighting function, $e^{\mathbf{A}(t-\tau)}\mathbf{B}$

Force-free and forced evolution (cont.)

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\text{force-free evolution } \mathbf{y}_u(t)} + \underbrace{\mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t)}_{\text{forced evolution } \mathbf{y}_f(t)}$$

We can write the output solution (for $t \geq t_0$) as the sum of two terms

$$\mathbf{y}(t) = \mathbf{y}_l(t) + \mathbf{y}_f(t)$$

- ⇝ The force-free evolution of the output, $\mathbf{y}_u(t)$
- ⇝ The forced evolution of the output, $\mathbf{y}_f(t)$

Free and forced evolution (cont.)

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)}_{\text{free evolution } \mathbf{y}_u(t)} + \underbrace{\mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t)}_{\text{forced evolution } \mathbf{y}_f(t)}$$

The **force-free evolution** of the output, from initial condition $\mathbf{y}(t_0) = \mathbf{C}\mathbf{x}(t_0)$

$$\rightsquigarrow \quad \mathbf{y}_u(t) = \mathbf{C}e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) = \mathbf{C}\mathbf{x}_u(t) \tag{15}$$

The **forced-evolution** of the output

$$\rightsquigarrow \quad \mathbf{y}_f(t) = \mathbf{C}\int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t) = \mathbf{C}\mathbf{x}_f(t) + \mathbf{D}\mathbf{u}(t) \tag{16}$$

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Free and forced evolution (cont.)

Note that for $t_0 = 0$, we have

$$\rightsquigarrow \begin{cases} \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \\ \mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t) \end{cases}$$

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Free and forced evolution (cont.)

Example

Consider a system with the SS representation,

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \tag{17}$$

We want to determine the state and the output evolution for $t \geq 0$

We consider the input signal $u(t)$

$$u(t) = 2\delta_{-1}(t)$$

We consider the initial state $\mathbf{x}(0)$

$$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

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Free and forced evolution (cont.)

The state transition matrix for this SS representation,

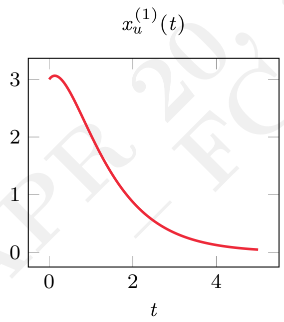
$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

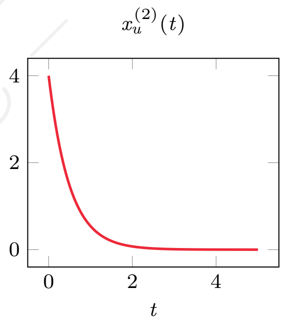
We computed it earlier

Free and forced evolution (cont.)

The force-free evolution of the state, for $t \geq 0$

$$\rightsquigarrow \quad \mathbf{x}_u(t) = e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (7e^{-t} - 4e^{-2t}) \\ 4e^{-2t} \end{bmatrix}$$

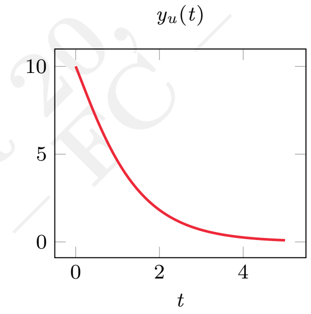




Free and forced evolution (cont.)

The force-free evolution of the output, for $t \geq 0$

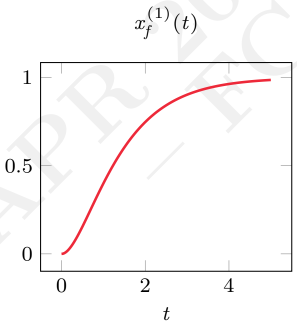
$$\rightsquigarrow \quad y_u(t) = \mathbf{C}\mathbf{x}_u(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} (7e^{-t} - 4e^{-2t}) \\ 4e^{-2t} \end{bmatrix} = 14e^{-t} - 4e^{-2t}$$

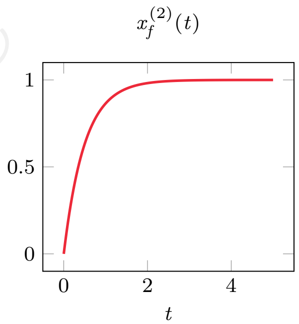


Free and forced evolution (cont.)

The forced evolution of the state, for $t \geq 0$

$$\begin{aligned} \rightsquigarrow \quad \mathbf{x}_f(t) &= \int_0^t e^{\mathbf{A}t}\mathbf{B}u(t-\tau)\mathrm{d}\tau = \int_0^t \begin{bmatrix} e^{-\tau} & (e^{-\tau} - e^{-2\tau}) \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2\mathrm{d}\tau \\ &= 2\int_0^t \begin{bmatrix} (e^{-\tau} - e^{-2\tau}) \\ e^{-2\tau} \end{bmatrix} \mathrm{d}\tau = 2 \begin{bmatrix} \int_0^t (e^{-\tau} - e^{-2\tau})\mathrm{d}\tau \\ \int_0^t e^{-2t}\mathrm{d}\tau \end{bmatrix} \\ &= 2 \begin{bmatrix} (1 - e^{-t}) - 1/2(1 - e^{-2t}) \\ 1/2(1 - e^{-2t}) \end{bmatrix} = \begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix} \end{aligned}$$

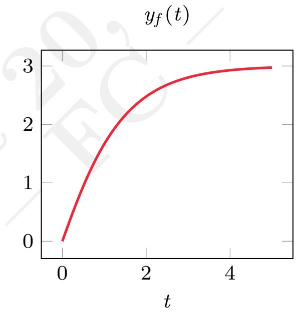




Free and forced evolution (cont.)

Since $\mathbf{D} = \mathbf{0}$, the forced evolution of the output for $t \geq 0$

$$\begin{aligned} \rightsquigarrow \quad y_f(t) = \mathbf{C}\mathbf{x}_f(t) &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix} \\ &= 3 - 4e^{-t} + e^{-2t} \end{aligned}$$



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Impulse response

We discussed the impulse response for systems in IO representation

- The forced response due to a unit impulse

We complete the presentation for systems in SS representation

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Impulse response (cont.)

Proposition

Impulse response

Consider the SS representation of a SISO system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + D\mathbf{u}(t) \end{cases}$$

The ***impulse response***

$$w(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + D\delta(t) \quad (18)$$

Proof

The impulse response is the forced response due to a unit impulse

Let $u(t) = \delta(t)$ and substitute it in the Lagrange formula

$$w(t) = \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathrm{d}\tau + D\delta(t)$$

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Impulse response (cont.)

Consider a continuous function f of t

By the properties of the Dirac function, we have that $f(t-\tau)\delta(\tau) = f(t)\delta(\tau)$

Thus, we have

$$w(t) = \mathbf{C}\int_0^t e^{\mathbf{A}t}\mathbf{B}\delta(\tau)\mathrm{d}\tau + D\delta(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\underbrace{\int_0^t \delta(\tau)\mathrm{d}\tau}_{1} + D\delta(t)$$

■

Impulse response (cont.)

$$w(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + D\delta(t)$$

If the system is strictly proper, we have that $D = 0$

- $w(t)$ is a linear combination of modes
- Through matrix $e^{\mathbf{A}t}$

If the system is not strictly proper, we have $D \neq 0$

- $w(t)$ is a linear combination of modes
- Plus, an impulse term

Impulse response (cont.)

The **forced response** can calculated using Lagrange formula

It corresponds to the what derived by the Durhamel's integral

$$\rightsquigarrow \quad y_f(t) = \int_0^t w(t-\tau)u(\tau)\mathrm{d}\tau = \int_0^t \Big[\mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B} + D\delta(t-\tau)\Big]u(\tau)\mathrm{d}\tau$$
$$= \int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\mathrm{d}\tau + \int_0^t D\delta(\tau-t)u(\tau)\mathrm{d}\tau$$
$$= \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\mathrm{d}\tau + Du(t)$$

Similarity transformation

State-space representation

Similarity tranformation

The form of the state space representation depends on the choice of states

- The choice is not unique

There is an infinite number of different representations of the same system

- They are all related by a **similarity transformation**

We define the concept of similarity transformation

Similarity tranformation (cont.)

The main advantage of the similarity transformation procedure is flexibility

- We can change to easier system representations

The state matrix can be set in **canonical form**

⇝ **Diagonal form**

⇝ **Jordan form**

There are other canonical forms

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Similarity tranformation (cont.)

Definition

Similarity transformation

Consider the SS representation of a linear stationary system of order n

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

- $\mathbf{x}(t)$, *state vector* (n *components*)
- $\mathbf{u}(t)$, *input vector* (r *components*)
- $\mathbf{y}(t)$, *output vector* (p *components*)

Let vector $\mathbf{z}(t)$ *be related to* $\mathbf{x}(t)$ *by a linear transformation* $\mathbf{P}$

$$\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t) \tag{19}$$

$\mathbf{P}$ *is any* $(n \times n)$ *non-singular matrix of constants*

- *Thus, the inverse of* $\mathbf{P}$ *always exists*
- *We have* $\mathbf{z}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$

Transformation/matrix $\mathbf{P}$ *is called* ***similarity transformation****/****matrix***

■

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Similarity tranformation (cont.)

Proposition

Similar representation

Consider the SS representation of a linear stationary system of order n

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{20}$$

Let $\mathbf{P}$ *be some transformation matrix such that* $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$

Vector $\mathbf{z}(t)$ *satisfies the new SS representation*

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}'\mathbf{z}(t) + \mathbf{B}'\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}'\mathbf{z}(t) + \mathbf{D}'\mathbf{u}(t) \end{cases} \tag{21}$$

⇝ $\mathbf{A}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

⇝ $\mathbf{B}' = \mathbf{P}^{-1}\mathbf{B}$

⇝ $\mathbf{C}' = \mathbf{C}\mathbf{P}$

⇝ $\mathbf{D}' = \mathbf{D}$

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Similarity tranformation (cont.)

Proof

Take the time-derivative of $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$

We get,

$$\rightsquigarrow \quad \dot{\mathbf{x}}(t) = \mathbf{P}\dot{\mathbf{z}}(t) \tag{22}$$

Substitute $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ into the SS representation

We get,

$$\rightsquigarrow \quad \begin{cases} \mathbf{P}\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{P}\mathbf{z}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{P}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Pre-multiply the state equation by $\mathbf{P}^{-1}$, to complete the proof

■

Similarity tranformation (cont.)

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}'\mathbf{z}(t) + \mathbf{B}'\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}'\mathbf{z}(t) + \mathbf{D}'\mathbf{u}(t) \end{cases}$$

We obtained a different SS representation of the same system

- Input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$ are unchanged
- The new state is indicated by $\mathbf{z}(t)$

There is an infinite number of non-singular matrixes $\mathbf{P}$

⇝ An infinite number of equivalent representations

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Similarity tranformation (cont.)

Example

Consider a system with SS representation $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\mathbf{B}} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1.5 \\ 0 \end{bmatrix}}_{\mathbf{D}} u(t) \end{cases}$$

Consider the similarity transformation

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{P}} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

What is the $\{\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'\}$ SS representation corresponding to state $\mathbf{z}(t)$

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Similarity tranformation (cont.)

We have,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \rightsquigarrow \quad \mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since $\mathbf{z}(t) = \mathbf{P}^{-1}\mathbf{x}(t)$, we have

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_1(t) - x_2(t) \end{bmatrix}$$

⇝ The first component of $\mathbf{z}(t)$ is the second component of $\mathbf{x}(t)$

⇝ The second component of $\mathbf{z}(t)$ is the difference between the first and the second component of $\mathbf{x}(t)$

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Similarity tranformation (cont.)

In addition,

$$\mathbf{A}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{B}' = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{C}' = \mathbf{C}\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\mathbf{D}' = \mathbf{D} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

■

Similarity tranformation (cont.)

Proposition

Similarity and state transition matrix

Consider the matrix $\mathbf{A}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

The state transition matrix,

$$e^{\mathbf{A}'t} = \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}$$

Proof

Note that

$$(\mathbf{A}')^k = \underbrace{(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \cdot (\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \cdots (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})}_{k \text{ times}}$$

$$= \mathbf{P}^{-1}\underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k \text{ times}}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}$$

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Similarity tranformation (cont.)

Thus, by definition

$$e^{\mathbf{A}'t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}')^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(\mathbf{P}^{-1}\mathbf{A}^k\mathbf{P})t^k}{k!}$$

$$\rightsquigarrow \quad = \mathbf{P}^{-1}\Big(\sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}\Big)\mathbf{P} = \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}$$

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Similarity tranformation (cont.)

We show how two similar representations describe the same IO relation

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Similarity tranformation (cont.)

Proposition

Invariance of the IO relationship by similarity

Consider two similar SS representations of the same LTI system

$\rightsquigarrow$ $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ *and* $\{\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}'\}$

$\rightsquigarrow$ $\mathbf{P}$ *is the transformation matrix*

Let the system be subjected to some input $\mathbf{u}(t)$

The two representations produce the same forced response

$\rightsquigarrow$ $\mathbf{y}_f(t)$

Similarity tranformation (cont.)

Proof

Consider the SS representation of the system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Consider the SS representation of the system

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{A}'\mathbf{z}(t) + \mathbf{B}'\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}'\mathbf{z}(t) + \mathbf{D}'\mathbf{u}(t) \end{cases}$$

- ⇝ $\mathbf{A}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$
- ⇝ $\mathbf{B}' = \mathbf{P}^{-1}\mathbf{B}$
- ⇝ $\mathbf{C}' = \mathbf{C}\mathbf{P}$
- ⇝ $\mathbf{D}' = \mathbf{D}$

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Similarity tranformation (cont.)

Consider the Lagrange formula

The forced response of the second representation due to input $\mathbf{u}(t)$

$$\begin{aligned} \mathbf{y}_f(t) &= \mathbf{C}' \int_{t_0}^{t} e^{\mathbf{A}'(t-\tau)}\mathbf{B}'\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t) \\ &= \mathbf{C}\mathbf{P} \int_{t_0}^{t} \mathbf{P}^{-1} e^{\mathbf{A}(t-\tau)}\mathbf{P}\mathbf{P}^{-1}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t) \\ &= \mathbf{C} \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

This response corresponds to that of the first SS representation

$$\mathbf{y}_f(t) = \mathbf{C} \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t)$$

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Similarity tranformation (cont.)

Proposition

Invariance of the eigenvalues under similarity transformations

Matrix $\mathbf{A}$ *and* $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ *have the same characteristic polynomial*

Proof

The characteristic polynomial of matrix $\mathbf{A}'$

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}') &= \det(\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \det(\lambda \underbrace{\mathbf{P}^{-1}\mathbf{P}}_{\mathbf{I}} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \\ &= \det[\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}] = \det(\mathbf{P}^{-1})\det(\lambda\mathbf{I} - \mathbf{A})\det(\mathbf{P}) \\ &= \det(\lambda\mathbf{I} - \mathbf{A}) \end{aligned}$$

The last equality is obtained from $\det(\mathbf{P}^{-1})\det(\mathbf{P}) = 1$

$\mathbf{A}$ and $\mathbf{A}'$ share the same characteristic polynomial

- ⇝ Thus, also the eigenvalues are the same

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Similarity tranformation (cont.)

Two similar SS representations have the same modes

- The modes characterise the dynamics

The modes are independent of the representation

- ⇝ This is important

Similarity tranformation (cont.)

Example

Consider two similar SS representations of the same LTI system

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

The similarity transformation matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We are interested in the eigenvalues and modes of the system

Matrix **A** and **A** have two eigenvectors

- $\lambda_1 = -1$ and $\lambda_2 = -2$

The system modes are e^{-t} and e^{-2t}

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Diagonalisation

We consider a special similarity transformation **P**

- We seek for a diagonal matrix $\mathbf{A}'$

$\rightsquigarrow$ $\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

A SS representation with diagonal state matrix

- **Diagonal canonical form**

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Diagonalisation (cont.)

Consider a SISO LTI system characterised by the following state equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

The evolution of the i-th component of the state vector

$$\rightsquigarrow \quad \dot{x}_i(t) = \lambda_i x_i(t) + b_i u(t)$$

State derivatives are not related to other components

Diagonalisation (cont.)

We think of a system with diagonal matrix **A** as a collection of sub-systems

- ⇝ Each sub-system is described by a single state component
- ⇝ Each state component evolves independently
- ⇝ The representation is **decoupled**
- ⇝ n first-order subsystems

The characteristic polynomial of the system for the i-th component

$$\rightsquigarrow \quad P_i(s) = (s - \lambda_i)$$

This subsystem has mode $e^{-\lambda_i t}$

Diagonalisation (cont.)

A special similarity transformation to get a representation in diagonal form

- A special similarity matrix

Diagonalisation (cont.)

Definition

Modal matrix

Consider a system in state space representation with $(n \times n)$ matrix $\mathbf{A}$

- *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a set of the eigenvectors of matrix $\mathbf{A}$*
- *Suppose that they correspond to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$*

Suppose that eigenvectors in this set are linearly independent

*We define the **modal matrix** of $\mathbf{A}$ as the $(n \times n)$ matrix $\mathbf{V}$*

$$\mathbf{V} = [\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_n]$$

■

Diagonalisation (cont.)

Example

Consider the state-space representation of a system with matrix **A**

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

We are interested in the modal matrix **V** of **A**

The eigenvalues and eigenvectors of **A**

- ⇝ $\lambda_1 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$
- ⇝ $\lambda_2 = 5$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$

The modal matrix **V**,

$$\mathbf{V} = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Diagonalisation (cont.)

The eigenvectors are determined up to a scaling constant

- (Plus, the ordering of the eigenvalues is arbitrary)

It is clear that there can be more than one modal matrix

These two modal matrices of matrix $\mathbf{A}$ are equivalent

$$\mathbf{V}' = \big[\mathbf{v}_2 | \mathbf{v}_1\big] = \begin{bmatrix} 2 & 3 \\ -2 & 9 \end{bmatrix}$$

$$\mathbf{V}'' = \big[2\mathbf{v}_1 | 3\mathbf{v}_2\big] = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

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Diagonalisation (cont.)

If a matrix $\mathbf{A}$ has n distinct eigenvalues, then its modal matrix exists

- As its n eigenvectors are linearly independent

Distinct eigenvalues

Let $\mathbf{A}$ be a n-order matrix whose n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct

Then, there is a set of n linearly independent eigenvectors

- Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a basis for $\mathcal{R}^n$

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Diagonalisation (cont.)

Consider a matrix $\mathbf{A}$ whose eigenvalues have multiplicity larger than one

- The modal matrix exists if and only if to each eigenvalue λ with multiplicity ν is possible to associate ν linearly independent eigenvectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\nu$$

This is not always possible

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Diagonalisation (cont.)

Example

Consider the state space representation of a system with matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Its eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $\big[\lambda\mathbf{I} - \mathbf{A}\big]\mathbf{v} = \mathbf{0}$

$$\big[2\mathbf{I} - \mathbf{A}\big]\mathbf{v} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$$

We can choose any two linearly independent eigenvectors for λ

- As the equation is satisfied for any value of a and b

The modal matrix by choosing the eigenvectors from the canonical basis

$$\rightsquigarrow \quad \mathbf{V} = \big[\mathbf{v}_1 | \mathbf{v}_2\big] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Diagonalisation (cont.)

Example

Consider the state space representation of a system with matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Its eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $\big[\lambda\mathbf{I} - \mathbf{A}\big]\mathbf{v} = \mathbf{0}$

$$[2\mathbf{I} - \mathbf{A}]\mathbf{v} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} -b = 0 \\ 0 = 0 \end{cases}$$

As $b = 0$, we can choose only one linearly independent eigenvector for λ

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix $\mathbf{A}$ does not admit a modal matrix

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Diagonalisation (cont.)

But, ...

If a matrix admits a modal matrix, then it can be diagonalised

- (This is what matters to us)

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Diagonalisation (cont.)

Proposition

Diagonalisation

Consider the state space representation of a system with matrix $\mathbf{A}$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ *be its eigenvalues*

Let $\mathbf{V} = \big[\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_n\big]$ *be one of its modal matrices*

Matrix $\mathbf{\Lambda}$ *from this similarity transformation is diagonal*

$$\rightsquigarrow \quad \mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

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Diagonalisation (cont.)

Proof

$$\mathbf{V} = \big[\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_n\big]$$

Note that the modal matrix is non-singular and can be inverted

- Its columns are linearly independent, by definition

By the definition of eigenvalue and eigenvector, we have

$$\lambda_i \mathbf{v}_i = \mathbf{A}\mathbf{v}_i, \text{ for } i = 1, \dots, n$$

By combining these expressions, we have

$$\rightsquigarrow \quad \big[\lambda_1\mathbf{v}_1 | \lambda_2\mathbf{v}_2 | \cdots | \lambda_n\mathbf{v}_n\big] = \big[\mathbf{A}\mathbf{v}_1 | \mathbf{A}\mathbf{v}_2 | \cdots | \mathbf{A}\mathbf{v}_n\big]$$

Diagonalisation (cont.)

We can rewrite this identity,

$$[\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} = \mathbf{A}[\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]$$

That is,

$$\mathbf{V}\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} = \mathbf{AV}$$

By left-multiplying both sides by $\mathbf{V}^{-1}$, we have

$$\rightsquigarrow \quad \mathbf{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} = \mathbf{V}^{-1}\mathbf{AV}$$

■

Diagonalisation (cont.)

Example

Consider a system with SS representation $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$

$$\begin{cases}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t)\\ \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1.5\\ 0\end{bmatrix}u(t)\end{cases}$$

We are interested in a diagonal representation by similarity

The eigenvalues and eigenvectors of $\mathbf{A}$

- $\lambda_1 = -1$ and $\mathbf{v}_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$
- $\lambda_2 = -2$ and $\mathbf{v}_2 = \begin{bmatrix}1\\ -1\end{bmatrix}$

Diagonalisation (cont.)

The modal matrix and its inverse

$$\mathbf{V} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

Thus,

$$\mathbf{A}' = \mathbf{\Lambda} = \mathbf{V}^{-1}\mathbf{AV} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & -2\\ 0 & 2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$$

$$\mathbf{B}' = \mathbf{V}^{-1}\mathbf{B} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\mathbf{C}' = \mathbf{CV} = \begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 0 & -2\end{bmatrix}$$

$$\mathbf{D}' = \mathbf{D} = \begin{bmatrix}1.5\\ 0\end{bmatrix}$$

■

State transition matrix by diagonalisation

Diagonalisation

State transition matrix by diagonalisation

An alternative to Sylvester expansion to compute the state transition matrix

We consider a SS representation whose matrix $\mathbf{A}$ can be diagonalised

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Transition matrix by diagonalisation (cont.)

Proposition

State transition matrix by diagonalisation

Consider a $(n \times n)$ *matrix* $\mathbf{A}$ *and let* $\lambda_1, \lambda_2, \dots, \lambda_n$ *be the eigenvalues*

Suppose that $\mathbf{A}$ *admits the modal matrix* $\mathbf{V}$

We have, the state transition matrix

$$e^{\mathbf{A}t} = \mathbf{V} e^{\mathbf{\Lambda}t} \mathbf{V}^{-1} = \mathbf{V} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \mathbf{V}^{-1} \qquad (23)$$

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State transition matrix by diagonalisation (cont.)

Proof

We have shown (similarity and state transition matrices[4])

$$e^{\mathbf{\Lambda}t} = \mathbf{V}^{-1} e^{\mathbf{A}t} \mathbf{V}$$

To complete, multiply both sides by $\mathbf{V}$ on the left and by $\mathbf{V}^{-1}$ on the right

■

[4] Given $\mathbf{A}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, we have $e^{\mathbf{A}'t} = \mathbf{P}^{-1} e^{\mathbf{A}t} \mathbf{P}$.

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State transition matrix by diagonalisation (cont.)

Example

Consider a system with SS representation $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

We are interested in the state transition matrix $e^{\mathbf{A}t}$

State transition matrix by diagonalisation (cont.)

We already computed the modal matrix and its inverse

$$\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Thus, we have

$$e^{\mathbf{A}t} = \mathbf{V} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \mathbf{V}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

This is the same expression we determined by using the Sylvester expansion

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Complex eigenvalues

The diagonalisation procedure applies to matrices with complex eigenvalues

- ⇝ The corresponding eigenvectors are conjugate-complex
- ⇝ The modal matrix and the state matrix are complex

We prefer to choose a similarity matrix that differs from the modal matrix

- The objective is a real canonical form
- With some desirable properties

To each pair of conjugate-complex eigenvalues associates a order 2 real block

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Complex eigenvalues (cont.)

Consider a system with state space representation with matrix $\mathbf{A}$

Suppose that $\mathbf{A}$ has a pair of complex conjugate eigenvalues

⇝ $\lambda, \lambda' = \alpha \pm j\omega$

Suppose that the remaining eigenvalues are real and distinct

⇝ $\lambda_1, \lambda_2, \cdots, \lambda_R$

The eigenvectors $\mathbf{v}$ and $\mathbf{v}'$ associated to λ and λ'

$$\mathbf{v} = \text{Re}(\mathbf{v}) + j\text{Im}(\mathbf{v}) = \mathbf{u} + j\boldsymbol{\omega}$$

$$\mathbf{v}' = \text{Re}(\mathbf{v}') + j\text{Im}(\mathbf{v}') = \mathbf{u} - j\boldsymbol{\omega}$$

They are also conjugate complex

Complex eigenvalues (cont.)

We want to show that $\mathbf{u}$ and $\boldsymbol{\omega}$ are linearly independent

- They are also linearly independent of other eigenvectors
- Those associated to the other eigenvalues

By the definition of eigenvalue/eigenvector, we have

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$
$$\mathbf{A}(\mathbf{u} + j\boldsymbol{\omega}) = (\alpha + j\omega)(\mathbf{u} + j\boldsymbol{\omega})$$

We consider the real and the imaginary part individually

We have,

$$\mathbf{A}\mathbf{u} = (\alpha\mathbf{u} - \omega\boldsymbol{\omega})$$
$$\mathbf{A}\boldsymbol{\omega} = (\omega\mathbf{u} + \alpha\boldsymbol{\omega})$$

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Complex eigenvalues (cont.)

Choose a particular similarity matrix $\tilde{\mathbf{V}}$

Columns associated to real eigenvalues are the corresponding eigenvectors

- (as with the modal matrix)

We associate columns $\mathbf{u}$ and $\mathbf{v}$ to the pair of conjugate complex eigenvalues

$$\left[\lambda_1 \mathbf{v}_1 | \lambda_2 \mathbf{v}_2 | \cdots | \lambda_R \mathbf{v}_R | \alpha\mathbf{u} - \omega\boldsymbol{\omega} | \omega\mathbf{u} + \alpha\boldsymbol{\omega}\right] = \left[\mathbf{A}\mathbf{v}_1 | \mathbf{A}\mathbf{v}_2 | \cdots | \mathbf{A}\mathbf{v}_R | \mathbf{A}\mathbf{u} | \mathbf{A}\boldsymbol{\omega}\right]$$

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Complex eigenvalues (cont.)

We can re-write this equation,

$$\rightsquigarrow \quad \left[\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_R | \mathbf{u} | \boldsymbol{\omega}\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_R & 0 & 0 \\ 0 & 0 & \cdots & 0 & \alpha & \omega \\ 0 & 0 & \cdots & 0 & -\omega & \alpha \end{bmatrix} = \mathbf{A}\left[\mathbf{v}_1 | \mathbf{v}_2 | \cdots | \mathbf{v}_R | \mathbf{u} | \boldsymbol{\omega}\right]$$

That is,

$$\tilde{\mathbf{A}} = \tilde{\mathbf{V}}^{-1}\mathbf{A}\tilde{\mathbf{V}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_R & 0 & 0 \\ 0 & 0 & \cdots & 0 & \alpha & \omega \\ 0 & 0 & \cdots & 0 & -\omega & \alpha \end{bmatrix}$$

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Complex eigenvalues (cont.)

We associated to the pair of eigenvalues $\lambda, \lambda' = \alpha \pm j\omega$ to a block

The block represents the eigenvalues in matrix form

$$\rightsquigarrow \quad \mathbf{H} = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$$

Complex eigenvalues (cont.)

Consider a matrix $\mathbf{A}$ that has R distinct real roots (λ_i, $i = 1, \dots, R$) and S pairs of distinct conjugate complex roots (λ, λ', $i = R+1, \dots, R+S$)

Matrix $\mathbf{A}$ can be written in a canonical quasi-diagonal form using matrix $\tilde{\mathbf{V}}$

$$\tilde{\boldsymbol{\Lambda}} = \tilde{\mathbf{V}}^{-1}\mathbf{A}\tilde{\mathbf{V}}$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & \lambda_2 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_R & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{R+1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{H}_{R+2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_{R+S} \end{bmatrix} \quad (24)$$

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Complex eigenvalues (cont.)

To pairs of conjugate complex roots $\lambda, \lambda' = \alpha \pm j\omega$ associates a real block

The block that represents the pair in matrix form

$$\rightsquigarrow \quad \mathbf{H}_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$$

Complex eigenvalues (cont.)

Example

Consider a system in state-space representation with matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ -3 & -2 & -4 \end{bmatrix}$$

We are interested in a (quasi-) diagonal representation

The characteristic polynomial of matrix $\mathbf{A}$

$$P(s) = s^3 + 6s^2 + 13s + 20$$

The eigenvalues and the eigenvectors

$\rightsquigarrow$ $\lambda_1 = -4$ and $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\rightsquigarrow$ $\lambda_2, \lambda_2' = 1 \pm j2$ and $\mathbf{v}_2, \mathbf{v}_2' = \mathbf{u}_2 \pm j\boldsymbol{\omega}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \pm j \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Complex eigenvalues (cont.)

Consider the matrix $\tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_2 & \boldsymbol{\omega}_2 \end{bmatrix}$

We have,

$$\tilde{\boldsymbol{\Lambda}} = \tilde{\mathbf{V}}^{-1}\mathbf{A}\tilde{\mathbf{V}} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

■

Complex eigenvalues (cont.)

$$\tilde{\mathbf{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & \lambda_2 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_R & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{R+1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{H}_{R+2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_{R+S} \end{bmatrix}$$

Computing the matrix exponential of a matrix in this form is straightforward

- (We derived a proposition)
- $\tilde{\mathbf{\Lambda}}$ is a block-matrix

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Complex eigenvalues (cont.)

$$e^{\tilde{\mathbf{\Lambda}}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & e^{\lambda_2 t} & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_R t} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & e^{\mathbf{H}_{R+1}t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & e^{\mathbf{H}_{R+2}t} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & e^{\mathbf{H}_{R+S}t} \end{bmatrix}$$

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Complex eigenvalues (cont.)

Let $\lambda_i, \lambda'_i = \alpha_i \pm j\omega_i$ be a pair of complex-conjugate roots

For each, there is a canonical block

$$\mathbf{H}_i = \begin{bmatrix} \alpha_i & \omega_i \\ -\omega_i & \alpha_i \end{bmatrix}$$

$\mathbf{H}_i$ represents the pair λ, λ' in matrix form

The matrix exponential for this specific matrix

$$\rightsquigarrow \quad e^{\mathbf{H}_i t} = e^{\alpha_i t} \begin{bmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix}$$

The state transition matrix for matrix $\mathbf{A}$,

$$\rightsquigarrow \quad e^{\mathbf{A}t} = \tilde{\mathbf{V}} e^{\tilde{\mathbf{\Lambda}}t} \tilde{\mathbf{V}}^{-1}$$

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Complex eigenvalues (cont.)

Example

Consider a system with SS representation with matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ -3 & -2 & -4 \end{bmatrix}$$

We are interested in its matrix exponential $e^{\mathbf{A}t} = \tilde{\mathbf{V}} e^{\tilde{}t}\tilde{\mathbf{V}}^{-1}$

- From its (quasi-) diagonal form $\tilde{\mathbf{V}}$

Matrix $\mathbf{A}$ can be written in quasi-diagonal form

$$\tilde{\mathbf{\Lambda}} = \tilde{\mathbf{V}}^{-1}\mathbf{A}\tilde{\mathbf{V}} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -2 \end{bmatrix}$$

Complex eigenvalues (cont.)

Thus, we obtain

$$e^{\tilde{\mathbf{\Lambda}} t} = \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ 0 & -e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix}$$

We also have,

$$e^{\mathbf{A}t} = \tilde{\mathbf{V}} e^{\tilde{\mathbf{\Lambda}} t} \tilde{\mathbf{V}}^{-1} \begin{bmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) & 0 \\ -e^{-t}\sin(2t) & e^{-t}\cos(2t) & 0 \\ e^{-4t} - e^{-t}\cos(2t) & -e^{-t}\sin(2t) & e^{-t} \end{bmatrix}$$

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Jordan form

State-space representation

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Jordan form

Consider a state-space representation of a system with $(n \times n)$ matrix $\mathbf{A}$

Let its eigenvalues have multiplicity larger than one

The existence of n linearly independent eigenvectors cannot be guaranteed

⇝ Needed for the construction of the modal matrix

We cannot necessarily go to a diagonal form by similarity transformation

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Jordan form (cont.)

We can still find a set of n linearly independent **generalised eigenvectors**

- We need to extend the concept of eigenvector

Generalised eigenvectors are used to build a **generalised modal matrix**

⇝ By similarity, we obtain $\mathbf{J} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$

⇝ A block-diagonal canonical form

⇝ A **Jordan form**

Jordan form (cont.)

Definition

Jordan block of order p

Let $\lambda \in \mathcal{C}$ be a complex number and let $p \geq 1$ be a integer number

*The $(p \times p)$ matrix is a order p **Jordan block** associated to λ*

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

Diagonal entries equal λ, entries of the first upper band equal 1

- *(All the other entries are zero)*

λ is an eigenvalue (multiplicity p) of this Jordan block

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Jordan form (cont.)

Definition

Jordan form

*Matrix $\mathbf{J}$ is said to be in **Jordan form** if it is in block-diagonal form*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_p \end{bmatrix}$$

Each block $\mathbf{J}_i$ along the diagonal is a Jordan block

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Jordan form (cont.)

More than one Jordan block can be associated to the same eigenvalue

The Jordan form generalises the conventional diagonal form

- (With order 1 blocks along the diagonal)

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Jordan form (cont.)

Example

Matrix $\mathbf{J}_1$, $\mathbf{J}_2$ and $\mathbf{J}_2$ are all in Jordan form

$$\mathbf{J}_1 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{J}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{J}_3 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Jordan form (cont.)

Eigenvalues $\lambda_1 = 2$ (multiplicity 4) and $\lambda_2 = 3$ (multiplicity 2)

$$\mathbf{J}_1 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\lambda_1 = 2$ associates with two Jordan blocks (order 3 and 1)

$\lambda_2 = 3$ associates with a single Jordan block (order 2)

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Jordan form (cont.)

Eigenvalues $\lambda_1 = 2$ (multiplicity 2) and $\lambda_2 = 3$ (multiplicity 1)

$$\mathbf{J}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda_1 = 2$ associates with two Jordan blocks (order 1)

$\lambda_2 = 3$ associates with a single Jordan block (order 1)

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Jordan form (cont.)

Eigenvalues $\lambda_1 = 2$ (multiplicity 2) and $\lambda_2 = 0$ (multiplicity 1)

$$\mathbf{J}_3 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda_1 = 2$ associates with a single Jordan blocks (order 2)

$\lambda_2 = 0$ associates with a single Jordan block (order 1)

■

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Jordan form (cont.)

Proposition

Jordan form

A square matrix $\mathbf{A}$ *can always be written in a Jordan canonical form* $\mathbf{J}$

- *This can be done by using a similarity transformation*

The resulting form is unique, up to block permutations

■

Proposition

Jordan form

Let λ *be an eigenvalue with multiplicity* ν *for* $\mathbf{A}$

- *Let* μ *be its geometric multiplicity*[5]
- *Let* p_i *be the order of* i*-th block*

We have,

$$\sum_{i=1}^{\mu} p_i = \nu$$

■

[5] The number of linearly independent eigenvectors associated to it ($1 \leq \mu \leq \nu$).

Jordan form (cont.)

Multiplicity

Consider a square matrix $\mathbf{A}$ or order n

Suppose that $\mathbf{A}$ has $r \leq n$ distinct eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_r$$

$\rightsquigarrow$ $\lambda_i \neq \lambda_j$, for $i \neq j$

The characteristic polynomial can be written in the form

$$P(s) = (s-\lambda_1)^{\nu_1}(s-\lambda_2)^{\nu_2}\cdots(s-\lambda_r)^{\nu_r}, \quad \sum_{i=1}^{r} \nu_i = n$$

$\rightsquigarrow$ $\nu_i \in \mathcal{N}^+$ (**algebraic multiplicity**)

Define the **geometric multiplicity** of the eigenvalue λ_i

- Number ν_i of linearly independent eigenvectors associated to it

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Jordan form (cont.)

Definition

Eigenvalue index

Let $\mathbf{A}$ be a matrix that can be written in Jordan form $\mathbf{J}$

Let λ be an eigenvalue with multiplicity ν

Let π be the order of the Jordan block in $\mathbf{J}$ associated with eigenvalue λ

$\rightsquigarrow$ *π is the* ***eigenvalue index*** *of λ*

$$1 \leq \pi \leq \nu$$

■

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Jordan form (cont.)

Knowledge of eigenvalues and their algebraic and geometric multiplicity

- It is sufficient to determine the Jordan form
- (And, thus the index of the eigenvalues)

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Jordan form (cont.)

Example

Consider the 3-order matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & -2 \\ -2 & -2 & 0 \end{bmatrix}$$

We are interested in its Jordan form

The characteristic polynomial

$$P(s) = s^3 - 4s^2 + 4s = s(s-2)^2$$

Its eigenvalues and eigenvectors

$\rightsquigarrow$ $\lambda_1 = 0$, multiplicity $\nu_1 = 1$

$\rightsquigarrow$ $\lambda_2 = 2$, multiplicity $\nu_2 = 2$

Jordan form (cont.)

Eigenvalue with multiplicity one has unit geometric multiplicity and index

- ⇝ (λ_1, with $\nu_1 = 1$)
- ⇝ $\mu_1 = 1$
- ⇝ $\pi_1 = 1$

λ_1 associates with a single 1-order block

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Jordan form (cont.)

As for the geometric multiplicity of the second eigenvalue, we have

$$\begin{aligned}\mu_2 &= \text{null}(\lambda_2 \mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda_2 \mathbf{I} - \mathbf{A}) \\ &= 3 - \text{rank}\left(\begin{bmatrix} -1 & -1 & -2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}\right) \\ &= 3 - 2 = 1\end{aligned}$$

λ_2 associates with a single 2-order block

- ⇝ $\pi_2 = 2$

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Jordan form (cont.)

The resulting Jordan form,

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Equivalently, by block-permutation

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

■

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Jordan form (cont.)

There are cases eigenvalues and their algebraic and geometric multiplicity is not sufficient to characterise neither the Jordan form nor eigenvalues' index

Jordan form (cont.)

Example

Consider some (5×5) matrix $\mathbf{A}$

Let λ_1 and λ_2 be its eigenvalues

- λ_1, multiplicity $\nu_1 = 4$
- λ_2, multiplicity $\nu_2 = 1$

We are interested in its Jordan form

Let eigenvalue λ_2 associate to a Jordan block of order 1

To eigenvalue λ_1 can be associated one or more blocks

- Depending on its geometric multiplicity
- $\mu_1 \leq \nu_1 = 4$

We can consider four possible cases

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Jordan form (cont.)

$\mu_1 = 4$

The eigenvalue associates with as many Jordan blocks as its multiplicity

- Each of which has order 1

The index of eigenvalue is $\pi_1 = 1$

The resulting diagonasable form

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

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Jordan form (cont.)

$\mu_1 = 3$

The eigenvalue associates with three Jordan blocks

- The order of the blocks is $p_1 = 2, p_2 = 1, p_3 = 1$
- (As $p_1 + p_2 + p_3 = \nu_1 = 4$)

The index of eigenvalue is $\pi_1 = 2$

The resulting form

$$\mathbf{J}_2 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

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Jordan form (cont.)

$\mu_1 = 2$

The eigenvalue associates with two Jordan blocks

- The order of the blocks is p_1, p_2
- (As $p_1 + p_2 = \nu_1 = 4$)

Two resulting Jordan structures are possible

- The index of eigenvalue is $\pi_1 = 2$

$$\mathbf{J}_3 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

- The index of eigenvalue is $\pi_1 = 3$

$$\mathbf{J}_4 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

Jordan form (cont.)

$\mu_1 = 1$

The eigenvalue associates with a single Jordan block of order 4

The index of eigenvalue is $\pi_1 = 4$

The resulting (non-derogatory) form

$$\mathbf{J}_5 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

■

Jordan form (cont.)

The general way to determine the Jordan form $\mathbf{J}$ of a matrix $\mathbf{A}$

- We must compute the generalised modal matrix
- It generates the Jordan form, by similarity

We describe this procedure (not a fundamental read)

Basis of generalised eigenvectors

Jordan form

Basis of generalised eigenvectors

We have introduced informally the concept of generalised eigenvector

- We provide a formal definition

We determine a set of n linearly independent generalised eigenvectors

- A set that is a basis for $\mathcal{R}$

Basis of generalised eigenvectors (cont.)

Definition

Generalised eigenvector

Consider a $(n \times n)$ *matrix* $\mathbf{A}$

Let $\mathbf{v}$ *be vector in* $\mathcal{R}^n$

Suppose that the following holds true

$$\begin{cases} (\lambda \mathbf{I} - \mathbf{A})^k \mathbf{v} = \mathbf{0} \\ (\lambda \mathbf{I} - \mathbf{A})^{k-1} \mathbf{v} \neq \mathbf{0} \end{cases} \qquad (25)$$

$\mathbf{v}$ *is a* ***generalised eigenvector*** *of order* k *associated to eigenvalue* λ

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Basis of generalised eigenvectors (cont.)

An eigenvector is thus a special generalised eigenvector
⇝ $k = 1$

That is,

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$
$$\mathbf{v} \neq \mathbf{0}$$

The equations are satisfied by $\mathbf{v}$ and λ

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Basis of generalised eigenvectors (cont.)

Example

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

We are interested in the existence of a generalised eigenvector

The characteristic polynomial

$$P(s) = \det(s\mathbf{I} - \mathbf{A}) = (s-3)^4$$

One single eigenvalue $\lambda = 3$

- Multiplicity $\nu = 4$

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Basis of generalised eigenvectors (cont.)

We have,

$$(3\mathbf{I} - \mathbf{A}) = \begin{bmatrix} -2 & 0 & 0 & -4 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Moreover,

$$(3\mathbf{I} - \mathbf{A})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(3\mathbf{I} - \mathbf{A})^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of generalised eigenvectors (cont.)

Let $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ be a generalised eigenvector

We must have

$$(3\mathbf{I} - \mathbf{A})^3 \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$(3\mathbf{I} - \mathbf{A})^2 \mathbf{v} = \begin{bmatrix} 0 \\ a + 2d \\ 0 \\ 0 \end{bmatrix} \neq \mathbf{0}$$

- $\rightsquigarrow$ The first system is satisfied for any a, b, c, d
- $\rightsquigarrow$ The second system is satisfied by $a + 2d \neq 0$

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Basis of generalised eigenvectors (cont.)

$$a + 2d \neq 0$$

Let $a = 1$ and $d = 0$, we have

$$\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

Let $a = 0$ and $d = 1$, we have

$$\mathbf{v}'_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

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Basis of generalised eigenvectors (cont.)

Proposition

Chain of generalised eigenvectors

Consider a square matrix $\mathbf{A}$

Let $\mathbf{v}_k$ *be a* k*-order generalised eigenvector associated to eigenvalue* λ

For $j = 1, \dots, k-1$, *the* ***j-order generalised eigenvector***

$$\mathbf{v}_j = -(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_{j+1} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{j+1} \tag{26}$$

The ***k-long chain of generalised eigenvectors***

$$\mathbf{v}_k \rightarrow \mathbf{v}_{k-1} \rightarrow \cdots \rightarrow \mathbf{v}_1$$

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Basis of generalised eigenvectors (cont.)

Proof

We need to show that each vector in the chain is a generalised eigenvector

If $\mathbf{v}_j = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{j+1}$, for $j = 1, \dots, k-1$, then we have

$$\rightsquigarrow \mathbf{v}_j = (\mathbf{A} - \lambda\mathbf{I})^{\mathbf{v}_{k-j}} \mathbf{v}_k$$

If $\mathbf{v}_k$ is a k-order generalised eigenvector, then we have

$$\begin{cases} (\mathbf{A} - \lambda\mathbf{I})^k \mathbf{v}_k = \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})^{k-1} \mathbf{v}_k \neq \mathbf{0} \end{cases} \quad \rightsquigarrow \begin{cases} (\mathbf{A} - \lambda\mathbf{I})^j \mathbf{v}_j = \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})^{j-1} \mathbf{v}_j \neq \mathbf{0} \end{cases}$$

Vector $\mathbf{v}_k$ is thus a j-order generalised eigenvector

■

Basis of generalised eigenvectors (cont.)

Example

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial

$$P(s) = \det(s\mathbf{I} - \mathbf{A}) = (s-3)^4$$

One eigenvalue $\lambda = 3$, multiplicity $\nu = 4$

Basis of generalised eigenvectors (cont.)

$\mathbf{v}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ is a generalised eigenvector of order 3

We can construct the chain of length 3

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} \rightarrow \mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We have that $\mathbf{v}_1$ is an eigenvector of $\mathbf{A}$

Basis of generalised eigenvectors (cont.)

$\mathbf{v}'_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$ is a generalised eigenvector of order 3

We can construct the chain of length 3

$$\mathbf{v}'_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \mathbf{v}'_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}'_3 = \begin{bmatrix} 4 \\ 1 \\ -2 \\ -2 \end{bmatrix} \rightarrow \mathbf{v}'_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}'_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

We have that $\mathbf{v}'_1$ is an eigenvector of $\mathbf{A}$

Basis of generalised eigenvectors (cont.)

$\mathbf{v}_3$ and $\mathbf{v}'_3$ are linearly independent, $\mathbf{v}_2$ and $\mathbf{v}'_2$ (and $\mathbf{v}_1$ and $\mathbf{v}'_1$) are not

- They differ by a multiplicative constant

■

Basis of generalised eigenvectors (cont.)

Proposition

The structure of generalised eigenvectors

Consider a $(n \times n)$ matrix $\mathbf{A}$

Let λ be an eigenvalue with multiplicity ν and geometric multiplicity μ

*It is possible to assign to such an eigenvalue λ a **structure** of ν linearly independent eigenvectors consisting of μ chains*

$$\begin{cases} \mathbf{v}_{p_1}^{(1)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(1)} \rightarrow \mathbf{v}_1^{(1)}, & \textit{chain } 1 \\ \mathbf{v}_{p_2}^{(2)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(2)} \rightarrow \mathbf{v}_1^{(2)}, & \textit{chain } 2 \\ \quad \vdots \\ \mathbf{v}_{p_\mu}^{(\mu)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(\mu)} \rightarrow \mathbf{v}_1^{(\mu)}, & \textit{chain } \mu \end{cases}$$

Let p_i be the length of the generic chain i

We have,

$$\sum_{i=1}^{\mu} p_i = \nu$$

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Basis of generalised eigenvectors (cont.)

Proof

The theorem can be proved in a constructive way

- An algorithm to determine the structure
- (For a specific eigenvalue)

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Basis of generalised eigenvectors (cont.)

Start by noticing that each chain terminates with an eigenvector

$$\begin{cases} \mathbf{v}_{p_1}^{(1)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(1)} \rightarrow \mathbf{v}_1^{(1)}, & \text{chain } 1 \\ \mathbf{v}_{p_2}^{(2)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(2)} \rightarrow \mathbf{v}_1^{(2)}, & \text{chain } 2 \\ \quad \vdots \\ \mathbf{v}_{p_\mu}^{(\mu)} \rightarrow \cdots \rightarrow \mathbf{v}_2^{(\mu)} \rightarrow \mathbf{v}_1^{(\mu)}, & \text{chain } \mu \end{cases}$$

The number of chains of an eigenvalue equals the geometric multiplicity μ

- The number of linearly independent eigenvectors associated to it

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Basis of generalised eigenvectors (cont.)

Consider the structure of generalised eigenvectors from some eigenvalue

It corresponds to the Jordan block structure from that eigenvalue

In the Jordan form there are μ blocks (one per chain)

- ⇝ The length of the longest chain associated with λ
- ⇝ It equals the index of that eigenvalue
- ⇝ $\pi = \max{(p_1, p_2, \dots, p_\mu)}$

Basis of generalised eigenvectors (cont.)

Consider some $(n \times n)$ matrix $\mathbf{A}$

Let λ be one of its eigenvalues

- Multiplicity ν

Consider the matrix $(\lambda\mathbf{I} - \mathbf{A})$ and its nullity

$$\rightsquigarrow \quad \alpha_1 = \text{null}(\lambda\mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda\mathbf{I} - \mathbf{A})$$

This is the dimensionality of the vector subspace

$$\rightsquigarrow \quad \ker(\lambda\mathbf{I} - \mathbf{A}) = \{\mathbf{x} \in \mathcal{R}^n | (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}\}$$

Number of linearly independent vectors $\mathbf{x}$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

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Basis of generalised eigenvectors (cont.)

Parameter α_1 corresponds to the geometric multiplicity μ of eigenvalue λ

The geometric multiplicity has two important meanings

- Number of linearly independent generalised eigenvectors of $\mathbf{A}$ from λ
- As each chain of generalised eigenvectors ends with an eigenvector

$\rightsquigarrow$ (Number of chains that can be associated with λ)

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Basis of generalised eigenvectors (cont.)

Consider matrix $(\lambda\mathbf{I} - \mathbf{A})$ and its nullity

$$\rightsquigarrow \quad \alpha_2 = n - \text{rank}(\lambda\mathbf{I} - \mathbf{A})^2$$

This is the dimensionality of the vector subspace

$$\rightsquigarrow \quad \ker(\lambda\mathbf{I} - \mathbf{A})^2 = \{\mathbf{x} \in \mathcal{R}^n | (\lambda\mathbf{I} - \mathbf{A})^2\mathbf{x} = \mathbf{0}\}$$

The number of linearly independent vectors $\mathbf{x}$ such that $(\lambda\mathbf{I} - \mathbf{A})^2\mathbf{x} = \mathbf{0}$

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Basis of generalised eigenvectors (cont.)

If $\mathbf{x} = \ker(s\mathbf{I} - \mathbf{A})$, then $\mathbf{x} \in \ker(s\mathbf{I} - \mathbf{A})$

- We have, $\alpha_1 < \alpha_2$

α_2 equals the number of linearly independent generalised eigenvectors of order 2 that can be chosen linearly independent of the α_1 eigenvectors

Basis of generalised eigenvectors (cont.)

By the same token, consider matrix $(\lambda \mathbf{I} - \mathbf{A})^h$ and its nullity

$$\rightsquigarrow \quad \alpha_h = n - \text{rank}(\lambda \mathbf{I} - \mathbf{A})^h = \nu$$

In this case, we have $\alpha_1 < \alpha_2 < \cdots < \alpha_h$

Thus, there are ν generalised eigenvectors of $\mathbf{A}$ that are linearly independent

$\rightsquigarrow$ Their order is smaller or equal to h

Moreover, $\beta_h = \alpha_h - \alpha_{h-1}$ of them are of order h

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Basis of generalised eigenvectors (cont.)

Consider the case in which β_{i+1} $(i = 1, 2, \ldots, h-1)$

The number of eigenvectors of order i is such that $\beta_i \geq \beta_{i+1}$

- For each generalised eigenvector of order $i+1$, it is possible to determine a generalised eigenvector of order i
- (We proved a proposition about this fact)

The difference $\gamma_i = \beta_i \beta_{i+1}$ indicates the number of new chains of order i

- They originate from a generalised eigenvector of order i

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Basis of generalised eigenvectors (cont.)

Computing a set of linearly independent generalised eigenvalues

Given a $(n \times n)$ matrix $\mathbf{A}$ and one of its eigenvalues λ with multiplicity ν

1. Compute $\alpha_i = n\text{rank}(\lambda \mathbf{I} - \mathbf{A})^i$ for $i = 1, \ldots, h$ until $\alpha_h = \nu$
2. Build the table

i	1	2	$\cdots$	$h-1$	h
α_i	α_1	α_2	$\cdots$	α_{h-1}	α_h
β_i	α_1	$\alpha_2 - \alpha_1$	$\cdots$	$\alpha_{h-1} - \alpha_{h-2}$	$\alpha_h - \alpha_{h-1}$
γ_i	$\beta_1 - \beta_2$	$\beta_2 - \beta_3$	$\cdots$	$\beta_{h-1} - \beta_h$	β_h

$\rightsquigarrow$ α_i is the nullity of $(\lambda \mathbf{I} - \mathbf{A})^i$
$\rightsquigarrow$ β_i is the number of linearly independent generalised eigenvectors of order i of matrix $\mathbf{A}$ ($\beta_1 = \alpha_1$, and $\beta_i = \alpha_i - \alpha_{i-1}$ for $i = 2, \cdots, h$)
$\rightsquigarrow$ γ_i is the number of chains of generalised eigenvectors of length i of matrix $\mathbf{A}$ ($\gamma_i = \beta_i - \beta_{i-1}$, for $i = 1, \cdots, h-1$ and $\gamma_h = \beta_h$)

3. If $\gamma_i > 0$, determine γ_i linearly independent generalised eigenvectors of order i and compute for each of them the chain of length i

The algorithm determines $\sum_{i=1}^{h} \gamma_i = \alpha_1$ chains, a number that equals the geometric multiplicity of λ, an total of $\sum_{i=1}^{h} i\gamma_i = \nu$ generalised eigenvectors

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Basis of generalised eigenvectors (cont.)

Example

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

One eigenvalue $\lambda = 3$, multiplicity $\nu = 4$

We have,

$$\alpha_1 = n - \text{rank}(3\mathbf{I} - \mathbf{A}) = 4 - 2 = 2$$
$$\alpha_2 = n - \text{rank}(3\mathbf{I} - \mathbf{A})^2 = 4 - 1 = 3$$
$$\alpha_3 = n - \text{rank}(3\mathbf{I} - \mathbf{A})^3 = 4 - 0 = 4$$

As $\alpha_3 = 4 = \nu$, we have $h = 3$

Basis of generalised eigenvectors (cont.)

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

We can build the table

i	1	2	3
α_i	2	3	4
β_i	2	1	1
γ_i	1	0	1

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Basis of generalised eigenvectors (cont.)

As $\gamma_3 = 1$, we must choose a generalised eigenvector of order 3

- It will generate a chain of length 3

We denote by (1) at the exponent all vectors belonging to such a chain

Choose the generalised eigenvector of order 3, $\mathbf{v}_3^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$

We get,

$$\mathbf{v}_3^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{v}_2^{(1)} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} \rightarrow \mathbf{v}_1^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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Basis of generalised eigenvectors (cont.)

As $\gamma_2 = 0$, we do not determine other generalised eigenvectors of order 2

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Basis of generalised eigenvectors (cont.)

As $\gamma_1 = 1$, we must choose a generalised eigenvector of order 1

- A conventional eigenvector

This is the fourth vector we get

We denote by (2) at exponent vectors belonging to such a chain of length 1

Choose the eigenvector $\mathbf{v} = \begin{bmatrix} a & b & c & d \end{bmatrix}^T \neq \mathbf{0}$

We get,

$$(3\mathbf{I} - \mathbf{A})\mathbf{v} = \begin{bmatrix} -2a - 4d \\ -a - d \\ a + 2d \\ a + d \end{bmatrix} = \mathbf{0}$$

We can have that $a = d = 0$

We could choose $b = 1$ and $c = 0$ or $b = 0$ and $c = 1$

Basis of generalised eigenvectors (cont.)

Suppose that we choose $b = 1$ and $c = 0$, we get $\mathbf{v}_1^{(1)}$

Suppose that we choose $b = 0$ and $c = 1$, we get

$$\mathbf{v}_1^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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Basis of generalised eigenvectors (cont.)

It is possible to associate to an eigenvalue λ and multiplicity ν a structure

- ν linearly independent generalised eigenvectors

This extends to generalised eigenvectors a classical theorem

A matrix with n distinct eigenvalues has n linearly independent eigenvectors

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Basis of generalised eigenvectors (cont.)

Proposition

The generalised eigenvectors associated to distinct eigenvalues are linearly independent

Proposition

Consider a $(n \times n)$ *matrix* $\mathbf{A}$

$\mathbf{A}$ *possesses* n *linearly independent generalised eigenvectors*

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Suppose we have determined n linearly independent generalised eigenvectors

We can use them to build a non-singular matrix

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Generalised modal matrix (cont.)

Definition

Generalised modal matrix

Consider a $(n \times n)$ *matrix* $\mathbf{A}$

Consider a set of linearly independent generalised eigenvectors of $\mathbf{A}$

Suppose that to eigenvalue λ *correspond* μ *chains of generalised eigenvectors*

$\rightsquigarrow$ *Lengths* $p_1, p_2, \ldots, p_\mu$

We can sort the generalised eigenvectors of λ *and build a matrix* $\mathbf{V}_\lambda$

$$\Big[\underbrace{\big[\mathbf{v}_1^{(1)}|\mathbf{v}_2^{(1)}|\cdots|\mathbf{v}_{p_1}^{(1)}\big]}_{\text{chain } 1}\;\underbrace{\big[\mathbf{v}_1^{(2)}|\mathbf{v}_2^{(2)}|\cdots|\mathbf{v}_{p_2}^{(2)}\big]}_{\text{chain } 2}\;\cdots\;\underbrace{\big[\mathbf{v}_1^{(\mu)}|\mathbf{v}_2^{(\mu)}|\cdots|\mathbf{v}_{p_\mu}^{(\mu)}\big]}_{\text{chain } \mu}\Big]$$

Suppose that matrix $\mathbf{A}$ *has* r *distinct eigenvalues* λ_i $(i = 1, \ldots, r)$

We define the $(n \times n)$ *generalised modal matrix of* $\mathbf{A}$

$$\mathbf{V} = \big[\mathbf{V}_{\lambda_1}|\mathbf{V}_{\lambda_2}|\cdots|\mathbf{V}_{\lambda_r}\big]$$

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Generalised modal matrix (cont.)

Consider the definition of generalised modal matrix $\mathbf{V}$

- The ordering of the chain is not essential
- The choice is arbitrary

It is important however that the columns that are associated to the generalised eigenvectors belonging to the same chain are positioned side-by-side

- Moreover, they must ordered
- From the eigenvector to the generalised eigenvector of maximum order

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Generalised modal matrix (cont.)

Example

Consider the (4×4) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial $P(s) = \det(s\mathbf{I} - \mathbf{A}) = (s-4)^4$

- Eigenvalue $\lambda = 3$, multiplicity $\nu = 4$

To this eigenvalue correspond two chains of generalised eigenvalues

- Lengths 3 and 1

Generalised modal matrix (cont.)

There is a single distinct eigenvalue

Hence, the modal matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1^{(1)} & \mathbf{v}_2^{(1)} & \mathbf{v}_3^{(1)} & \mathbf{v}_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

By swapping the order of the chains, we obtain a different modal matrix

$$\mathbf{V}' = \begin{bmatrix} \mathbf{v}_1^{(2)} & \mathbf{v}_1^{(1)} & \mathbf{v}_2^{(2)} & \mathbf{v}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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Generalised modal matrix (cont.)

We thus have,

$$\mathbf{J} = {}^{-1}\mathbf{AV} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The index of eigenvalue $\lambda = 3$ is $\pi = 3$

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Generalised modal matrix (cont.)

Proposition

Consider a square matrix $\mathbf{A}$ *and let* $\mathbf{V}$ *be its generalised modal matrix*

Matrix $\mathbf{J}$ *from similarity transformation* $\mathbf{J} = {}^{-1}\mathbf{AV}$ *is in Jordan form*

There are μ *chains of generalised eigenvectors correspond to eigenvalue* λ

$\rightsquigarrow$ *Lengths* $p_1, p_2, \ldots, p_\mu$

Thus, μ *Jordan blocks of order* $p_1, p_2, \ldots, p_\mu$

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Generalised modal matrix (cont.)

Proof

The columns of the generalised modal matrix are linearly independent

- The generalised modal matrix is non-singular
- It can be inverted

Consider the j-th chain of length p associated to λ

By definition,

$$\lambda \mathbf{v}_1^{(j)} = \mathbf{A}\mathbf{v}_1^{(j)}$$

For the i-th (generalised eigen-) vector (of order $i > 1$) $\mathbf{v}_i^{(j)}$

$$\mathbf{v}_{i-1}^{(j)} = (\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_i^{(j)} \rightsquigarrow \lambda \mathbf{v}_i^{(j)} + \mathbf{v}_{i-1}^{(j)} = \mathbf{A}\mathbf{v}_i^{(j)}$$

Generalised modal matrix (cont.)

By combining equations, let the j-th chain contributes the first p columns

$$\left[\lambda\mathbf{v}_1^{(j)}\middle|\lambda\mathbf{v}_2^{(j)}+\mathbf{v}_1^{(j)}\middle|\cdots\middle|\lambda\mathbf{v}_p^{(j)}+\mathbf{v}_{p-1}^{(j)}\middle|\cdots\right] = \left[\mathbf{A}\mathbf{v}_1^{(j)}\middle|\mathbf{A}\mathbf{v}_2^{(j)}\middle|\cdots\middle|\mathbf{A}\mathbf{v}_p^{(j)}\middle|\cdots\right]$$

That is,

$$\left[\mathbf{v}_1^{(j)}\middle|\mathbf{v}_2^{(j)}\middle|\cdots\middle|\mathbf{v}_{p-1}^{(j)}\middle|\mathbf{v}_p^{(j)}\middle|\cdots\right]\begin{bmatrix}\lambda & 1 & \cdots & 0 & 0 & \cdots\\ 0 & \lambda & \cdots & 0 & 0 & \cdots\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 1 & \cdots\\ 0 & 0 & \cdots & 0 & \lambda & \cdots\\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots\end{bmatrix} = \mathbf{A}\left[\mathbf{v}_1^{(j)}\middle|\mathbf{v}_2^{(j)}\middle|\cdots\middle|\mathbf{v}_{p-1}^{(j)}\middle|\mathbf{v}_p^{(j)}\middle|\cdots\right]$$

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Generalised modal matrix (cont.)

$$\mathbf{J} = \begin{bmatrix}\lambda & 1 & \cdots & 0 & 0 & \cdots\\ 0 & \lambda & \cdots & 0 & 0 & \cdots\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \lambda & 1 & \cdots\\ 0 & 0 & \cdots & 0 & \lambda & \cdots\\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots\end{bmatrix}$$

That is, we have

$$\mathbf{VJ} = \mathbf{AV}$$

The chain of length p associates to a block of order p in $\mathbf{J}$

To complete the proof, left-multiply this equation by $\mathbf{V}^{-1}$

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Generalised modal matrix (cont.)

Example

Consider the (4×4) matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix}5 & 0 & 0 & 4\\ 1 & 3 & 0 & 1\\ -1 & 0 & 3 & -1\\ -1 & 0 & 0 & 1\end{bmatrix}$$

The characteristic polynomial $P(s) = \det(s\mathbf{I} - \mathbf{A}) = (s-4)^4$

- Eigenvalue $\lambda = 3$, multiplicity $\nu = 4$

To this eigenvalue correspond two chains of generalised eigenvalues

- Lengths 3 and 1

The matrix can be written in Jordan form by similarity

- To blocks, order 3 and 1, to eigenvalue $\lambda = 3$

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Generalised modal matrix (cont.)

We can choose a generalised modal matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1^{(1)} & \mathbf{v}_2^{(1)} & \mathbf{v}_3^{(1)} & \mathbf{v}_1^{(2)}\end{bmatrix} = \begin{bmatrix}0 & -2 & 1 & 0\\ 1 & -2 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}$$

Its inverse

$$\mathbf{V}' = \begin{bmatrix}0 & 1 & 0 & 1\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 2\\ 0 & 0 & 1 & -1\end{bmatrix}$$

We have,

$$\mathbf{J} = \mathbf{V}^{-1}\mathbf{AV} = \begin{bmatrix}3 & 1 & 0 & 0\\ 0 & 3 & 1 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

The index of the eigenvalue $\lambda = 3$ is $\pi = 3$

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Transition matrix by Jordan

Jordan form

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Transition matrix by Jordan

A formula for computing the matrix exponential of a matrix in Jordan form

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Transition matrix by Jordan (cont.)

Proposition

Consider a matrix in Jordan form

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \mathbf{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_q \end{bmatrix}$$

Its matrix exponential

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{\mathbf{J}_1 t} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{J}_2 t} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

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Transition matrix by Jordan (cont.)

Let $\mathbf{J}_i$ be the generic block of order p

$$\mathbf{J}_i = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$$

Its matrix exponential

$$e^{\mathbf{J}_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{p-3}}{(p-3)!}e^{\lambda t} & \frac{t^{p-2}}{(p-2)!}e^{\lambda t} & \frac{t^{p-1}}{(p-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{p-4}}{(p-4)!}e^{\lambda t} & \frac{t^{p-3}}{(p-3)!}e^{\lambda t} & \frac{t^{p-2}}{(p-2)!}e^{\lambda t} \\ 0 & 0 & e^{\lambda t} & \cdots & \frac{t^{p-5}}{(p-5)!}e^{\lambda t} & \frac{t^{p-4}}{(p-4)!}e^{\lambda t} & \frac{t^{p-3}}{(p-3)!}e^{\lambda t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & 0 & \cdots & 0 & 0 & e^{\lambda t} \end{bmatrix}$$

Transition matrix by Jordan (cont.)

Proof

Matrix $\mathbf{J}$ is in block-diagonal form, hence the form of its exponential

For the second result, determine the k-th power of block $\mathbf{J}_i$

- λ is the associated eigenvalue

We have,

$$\mathbf{J}_i^k = \begin{bmatrix} \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{k-p+2}\lambda^{k-p+2} & \binom{k}{p-1}\lambda^{k-p+1} \\ 0 & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{k-p+3}\lambda^{k-p+2} & \binom{k}{p-2}\lambda^{k-p+2} \\ 0 & 0 & \binom{k}{0}\lambda^k & \cdots & \binom{k}{k-p+4}\lambda^{k-p+4} & \binom{k}{p-3}\lambda^{k-p+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{k}{0}\lambda^k \end{bmatrix}$$

We used the definition of binomial coefficient

$$\begin{cases} \binom{k}{j} = \dfrac{k!}{j!(k-j)!}, & \text{for } j \leq k \\ \binom{k}{j} = 0, & \text{for } j > k \end{cases}$$

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Transition matrix by Jordan (cont.)

The generic element of matrix $e^{\mathbf{J}_i t}$ is on the upper-diagonal

- Starting from element $1, j+1$, for $j = 0, \ldots, p-1$

$$\sum_{k=0}^{\infty} \frac{k=0}{\infty}\binom{k}{j}\lambda^{k-j} = \sum_{k=j}^{\infty} \frac{t^k}{j!(k-j)!}\lambda^{k-j} = \frac{t^j}{j!}\Big(\sum_{k=j}^{\infty} \frac{t^{k-j}}{(k-j)!}\lambda^{k-j}\Big)$$
$$= \frac{t^j}{j!}\Big(\sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda^k\Big) = \frac{t^j}{j!}e^{\lambda t}$$

This is because we have

$$e^{\mathbf{J}_i t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{J}_i^k$$

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Transition matrix by Jordan (cont.)

Proposition

Consider a matrix $\mathbf{A}$ of order n and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

Let $\mathbf{V}$ be a generalised modal matrix to get a Jordan form

$$\mathbf{J} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

We have,

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{J}t}\mathbf{V}^{-1} \tag{27}$$

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Transition matrix by Jordan (cont.)

Example

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Consider the generalised modal matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1^{(1)} & \mathbf{v}_2^{(1)} & \mathbf{v}_3^{(1)} & \mathbf{v}_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We can write $\mathbf{A}$ in Jordan form

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Transition matrix by Jordan (cont.)

We have,

$$e^{\mathbf{J}t} = \begin{bmatrix} e^{3t} & te^{3t} & \frac{t^2}{2}e^{3t} & 0 \\ 0 & e^{3t} & te^{3t} & 0 \\ 0 & 0 & e^{3t} & 0 \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

We thus have,

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{V}t}\mathbf{V}^{-1} = \begin{bmatrix} e^{3t}+2e^{3t} & 0 & 0 & 4te^{3t} \\ te^{3t}+0.5t^2e^{3t} & e^{3t} & 0 & te^{3t}+t^2e^{3t} \\ -te^{3t} & 0 & e^{3t} & -2te^{3t} \\ -te^{3t} & 0 & 0 & e^{3t}-2te^{3t} \end{bmatrix}$$

■

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Transition matrix by Jordan (cont.)

Consider a matrix $\mathbf{A}$ with conjugate complex eigenvalues

⇝ Its Jordan form is not real

We can modify the diagonalisation procedure

- A modified modal matrix

We get a real canonical quasi Jordan form

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Transition matrix and modes

The modes are function that characterise the dynamical behaviour

- We studied them for IO representations

We establish a similar concept also for SS representations

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Minimum polynomial and modes

Consider a matrix $\mathbf{J}$ in Jordan canonical form

- Let $e^{\mathbf{J}t}$ be the state transition matrix

Consider a given block of order p associated to eigenvalue λ

$$\mathbf{J}_i = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$$

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Minimum polynomial and modes (cont.)

In the block of the matrix exponential, we will have the functions

$$e^{\lambda t}, te^{\lambda t}, \cdots, t^{p-1}e^{\lambda t}$$

Functions of time to be multiplied by appropriate coefficients

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Minimum polynomial and modes (cont.)

Definition

Minimum polynomial

Consider a matrix $\mathbf{A}$ with r distinct eigenvalues λ_i

- *Let π_i be the indexes of the eigenvalues*

*We define the **minimum polynomial***

$$P_{min}(s) = \prod_{i=1}^{r} (s - \lambda_i)^{\pi_i}$$

Consider the roots λ_i of the minimum polynomial of multiplicity π_i

- *To them we can associate the π_i functions of time*
- *We call them **modes***

$$e^{\lambda_i t}, te^{\lambda_i t}, \ldots, t^{\pi_i - 1}e^{\lambda_i t}$$

Each element of state transition matrix is a linear combination of modes

$$\rightsquigarrow \quad e^{\mathbf{A}t}$$

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Minimum polynomial and modes (cont.)

Minimum and characteristic polynomial coincide in nonderogatory matrices

$\rightsquigarrow$ (Special case of eigenvalues with multiplicity one)

Minimum polynomial and modes (cont.)

Example

Consider a system with SS representation

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

The state matrix $\mathbf{A}$ has two eigenvalues, both with multiplicity one

- $\lambda_1 = -1$
- $\lambda_2 = -2$

The index is unitary, too

The minimum polynomial of $\mathbf{A}$ and the characteristic polynomial match

$$P_{\min}(s) = P(s) = (s+1)(s+2)$$

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Minimum polynomial and modes (cont.)

The modes are e^{-t} and e^{-2t}

We have,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Each element is a linear combination of the modes

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Minimum polynomial and modes (cont.)

Example

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 4 \\ 1 & 3 & 0 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

One eigenvalue $\lambda = 3$, multiplicity $\nu = 4$, index $\pi = 3$

The characteristic and the minimum polynomial

$$P(s) = (s - \lambda)^{\nu} = (s-3)^4$$
$$P_{\min}(s) = (s - \lambda)^{\pi} = (s-3)^3$$

The modes

$$e^{3t}, te^{3t}, t^2 e^{3t}$$

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Minimum polynomial and modes (cont.)

The generalised modal matrix $\mathbf{V}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1^{(1)} & \mathbf{v}_2^{(1)} & \mathbf{v}_3^{(1)} & \mathbf{v}_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Jordan form of matrix $\mathbf{A}$

$$\mathbf{J} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Minimum polynomial and modes (cont.)

Each element of matrix $e^{\mathbf{A}t}$ is a linear combination of the modes

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{J}t}\mathbf{V}^{-1} = \begin{bmatrix} e^{3t}+2e^{3t} & 0 & 0 & 4te^{3t} \\ te^{3t}+0.5t^2e^{3t} & e^{3t} & 0 & te^{3t}+t^2e^{3t} \\ -te^{3t} & 0 & e^{3t} & -2te^{3t} \\ -te^{3t} & 0 & 0 & e^{3t}-2te^{3t} \end{bmatrix}$$

There is no mode in the form $t^{\nu-1}e^{\lambda t} = t^3e^{3t}$

- Though there is a $\lambda = 3$, with $\nu = 4$

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On the eigenvectors

Consider the state-space representation of a system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

We give an interpretation to the real eigenvectors of $\mathbf{A}$

We start with a general result, valid for all eigenvectors

- Both real and complex eigenvectors

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On the eigenvectors (cont.)

Proposition

Let $\mathbf{v}$ be an eigenvector of matrix $\mathbf{A}$

- *λ is the associated eigenvalue*

We have,

$$e^{\mathbf{A}t}\mathbf{v} = e^{\lambda t}\mathbf{v}$$

That is, $\mathbf{v}$ is an eigenvector of matrix $e^{\mathbf{A}t}$

$\rightsquigarrow$ *$e^{\lambda t}$ is the associated eigenvalue*

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On the eigenvectors (cont.)

Proof

Let $\mathbf{v}$ be an eigenvector of matrix $\mathbf{A}$

- λ is the associated eigenvalue

We thus have,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

By pre-multiplying both sides by $\mathbf{A}$, we get

$$\mathbf{A}^2\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}$$

The operation can be repeated, we get

$$\mathbf{A}^k\mathbf{v} = \lambda^k\mathbf{v}, \text{ for } k \in \mathcal{N}$$

We obtain,

$$e^{\mathbf{A}t}\mathbf{v} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{A}^k\mathbf{v} = \sum_{k=0}^{\infty}\frac{t^k}{k!} = e^{\lambda t}\mathbf{v}$$

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On the eigenvectors (cont.)

Consider a linear system with SS representation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

We are interested in its time evolution, from different initial conditions

Consider the initial state $\mathbf{x}(t_0)$ at time t_0, we have

- $\mathbf{x}_u(t)$ defines a parameterised curve
- The curve lies in the state space
- Time t is the parameter of $\mathbf{x}_u(t)$

The curve is called **state evolution**

The set of points along the curve defines the **trajectory** of the evolution

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On the eigenvectors (cont.)

We can embed a physical interpretation to the real eigenvectors of $\mathbf{A}$

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On the eigenvectors (cont.)

Suppose that $\mathbf{x}_0$ corresponds to an eigenvector of matrix $\mathbf{A}$

- (λ is the associated eigenvalue)

By using Lagrange formula and $e^{\mathbf{A}t}\mathbf{v} = e^{\lambda t}\mathbf{v}$, we have

$$\rightsquigarrow \quad \mathbf{x}_u(t) = e^{\mathbf{A}t}\mathbf{x}_0 = e^{\lambda t}\mathbf{x}_0$$

The state vector $\mathbf{x}_u(t)$ keeps in time the direction of $\mathbf{x}_0$

$\rightsquigarrow$ Its magnitude changes according to the mode $e^{\lambda t}$

- (It goes with the associated eigenvalue)

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On the eigenvectors (cont.)

Suppose that the system has a state matrix $\mathbf{A}$ of order n

Suppose that $\mathbf{A}$ has n linearly independent eigenvectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$$

- (The associated eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$)

On the eigenvectors (cont.)

Suppose that $\mathbf{x}_0$ does not coincide with $\mathbf{v}_i$

We can always write,

$$\rightsquigarrow \quad \mathbf{x}_0 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

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On the eigenvectors (cont.)

The initial condition is a linear combination of the basis of eigenvectors

- Through appropriate coefficients α_i

We have,

$$\mathbf{x}_u(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \sum_{i=1}^{n} \alpha_i e^{\mathbf{A}t}\mathbf{v}_i = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}\mathbf{v}_i$$

Time evolution is a linear combination of evolutions, along eigenvectors

- Through the same coefficients α_i

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On the eigenvectors (cont.)

Example

Consider a system with state-space representation $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

The state matrix $\mathbf{A}$ has the eigenvalues and eigenvectors

$\rightsquigarrow$ λ_1 and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\rightsquigarrow$ λ_2 and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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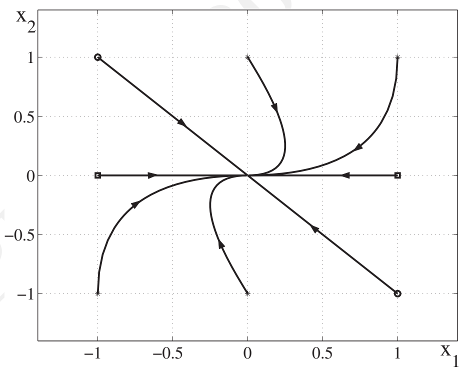
On the eigenvectors (cont.)

The force-free evolution on the (x_1, x_2)-plane for different cases

Each trajectory corresponds to a different initial condition

- t increases according to the arrow

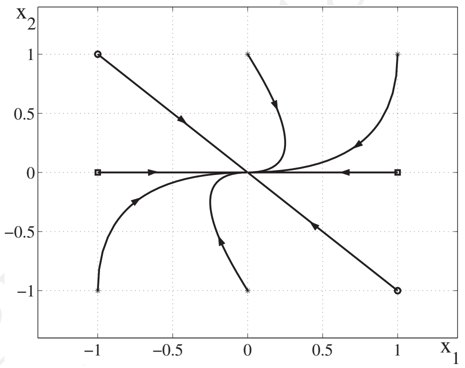


Two initial conditions are placed along the eigenvector $\mathbf{v}_1$

$\rightsquigarrow$ $\mathbf{x}_u(t)$ keeps the same direction

$\rightsquigarrow$ Its modulo decreases, e^{-t} is stable

On the eigenvectors (cont.)



Two initial conditions are placed along the eigenvector $\mathbf{v}_2$

- $\rightsquigarrow$ $\mathbf{x}_u(t)$ keeps the same direction
- $\rightsquigarrow$ Its modulo decreases, e^{-2t} is stable

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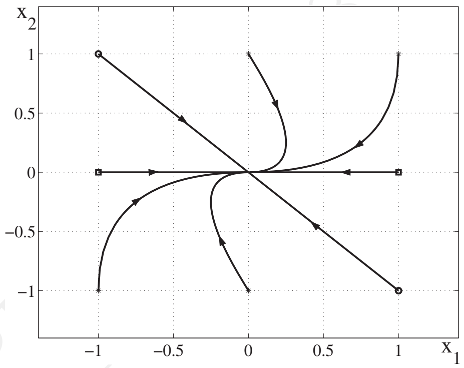
On the eigenvectors (cont.)



Two initial conditions are placed along a combination of eigenvectors

- $\rightsquigarrow$ $\mathbf{x}_u(t)$ keeps a curved direction, tend to zero
- $\rightsquigarrow$ Components evolve along different modes
- $\rightsquigarrow$ e^{-2t} is (extinguishes) faster

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On the eigenvectors (cont.)

Example

Consider the SS representation of a system with state matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$$

The eigenvalues

- $\rightsquigarrow$ $\lambda, \lambda' = \alpha \pm j\omega = -1 \pm j2$

We have,

$$e^{\mathbf{A}t} = e^{-t} \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$$

We want to study the force-free evolution

- From initial condition $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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On the eigenvectors (cont.)

We have,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\mathbf{A}t}\mathbf{x}_0 = \begin{bmatrix} e^{-t}\cos(2t) \\ -e^{-t}\sin(2t) \end{bmatrix}$$

The solution determines a vector in the (x_1, x_2) plane

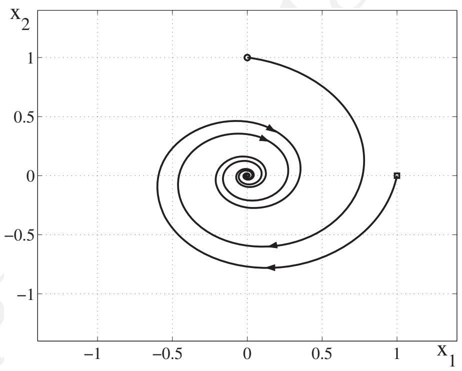
- The vector rotates clockwise
- The angular speed $\omega = 2$

The magnitude decreases according to mode e^{-t}

- A spiral

On the eigenvectors (cont.)

The trajectory is the spiral starting at $\square$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



All trajectories have qualitatively similar behaviour

- Whatever the initial condition

⇝ Starting at $\bigcirc$, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$