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CK0191: Partial evaluation I

Exercise 01 (50%). Consider the following IO representation of a system

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}y(t) + 7\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 32\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 60y(t) = 3\frac{\mathrm{d}}{\mathrm{d}t}u(t) + 3u(t)$$

A. Identify the general properties that characterise a system with this structure (linear or nonlinear, time-invariant or time-variant, dynamical or instantaneous, with or without delay elements, proper or improper). Motivate your answers.

B. Verify that the characteristic polynomial can be written as $P(s) = (s+3)(s^2+4s+20)$. Determine its roots, the corresponding modes of the system and determine whether the modes are stable or not. If possible, sketch the temporal evolution of the modes.

Solution: The differential equation is in the form $\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(m)}(t)$. Thus,

- the system is linear (the differential equation is linear);
- the system is stationary (coefficients a_i and b_i do not depend on time);
- the system is dynamic (the differential equation is of order $n = 3 > 0$);
- the system has no delay elements (there are no time-shifted terms of the form $(t - T)$);
- the system is strictly proper ($n = 3$ and $m = 1$, so $n > m$).

The characteristic polynomial $P(s) = (s+3)(s^2+4s+20) = s^3+7s^2+32s+60$ has roots i) $p_1 = -3$; and ii) $p_{2,3} = \alpha \pm j\omega = -2 \pm j4$. The associated modes are

- $e^{p_1}(t) = e^{-3t}$, which is an aperiodic stable mode with time constant $\tau = -a/p_1 = 1/3$
- $e^{\alpha t}\cos(\omega t) = e^{-2t}\cos(4t)$, which is a pseudo-periodic mode with natural pulsation $\omega_n = \sqrt{\alpha^2+\omega^2} = \sqrt{20} \approx 4.47$ and dumping factor $\zeta = -\alpha/\omega_n = 1/\sqrt{5}$.

Exercise 02 (50%). Consider the following SS representation of a system

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -14 & -16 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 2u(t) \end{cases}$$

A. Identify the general properties that characterise a system with this structure (linear or nonlinear, time-invariant or time-variant, dynamical or instantaneous, with or without delay elements, proper or improper). Motivate your answers.

B. Determine the state transition matrix of the system using the Sylvester expansion (you can use $P(\lambda) = (\lambda+14)(\lambda-10)+144$). Determine the force-free evolution of the state and of the output from the initial state $\mathbf{x}(0) = (2,0)^T$ at time $t_0 = 0$.

Solution: The system is represented in the standard form of n linear differential equations of the first order and a linear transformation of the output

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) + D\mathbf{u}(t) \end{cases}.$$

Thus,

- the system is stationary (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and D do not depend on time);
- the system is dynamic (the system of equations has order $n = 2 > 0$);
- the system has no delay elements (there are no time-shifted terms of the form $(t-T)$);
- the system is proper, but not strictly proper (matrix D is not zero).

The characteristic polynomial $P(\lambda) = (\lambda+14)(\lambda-10)+144$ of matrix $\mathbf{A}$ has roots -2 and -2. Matrix $\mathbf{A}$ has thus a single eigenvalue $\lambda_1 = -2$ of multiplicity 2.

To determine matrix $e^{\mathbf{A}t}$ using the Sylvester expansion, we write

$$\begin{cases} e^{\lambda_1 t} = \alpha_0(t) + \lambda_1\alpha_1(t) \\ t^{\lambda_1 t} = \alpha_1(t) \end{cases} \quad \rightsquigarrow \quad \begin{cases} e^{-2t} = \alpha_0(t) - 2\alpha_1(t) \\ te^{-2t} = \alpha_1(t) \end{cases} \quad \rightsquigarrow \quad \begin{cases} \alpha_0(t) = e^{-2t} + 2te^{-2t} \\ \alpha_1(t) = te^{-2t} \end{cases}.$$

Therefore, we have

$$\begin{aligned} e^{\mathbf{A}t} &= \alpha_0(t)\mathbf{I}_2 + \alpha_0(t)\mathbf{A} = (e^{-2t} + 2te^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-2t}\begin{bmatrix} -14 & -16 \\ 9 & 10 \end{bmatrix} \\ &= \begin{bmatrix} (e^{-2t} - 12te^{-2t}) & -16te^{-2t} \\ 9te^{-2t} & (e^{-2t} + 12te^{-2t}) \end{bmatrix} \end{aligned}.$$

We can compute the force-free evolution of the system (state and output)

$$\begin{aligned} \mathbf{x}_u(t) &= e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} (e^{-2t} - 12te^{-2t}) & -16te^{-2t} \\ 9te^{-2t} & (e^{-2t} + 12te^{-2t}) \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} (2e^{-2t} - 24te^{-2t}) \\ 18te^{-2t} \end{bmatrix} \\ y_u(t) &= \mathbf{C}\mathbf{x}_u(t) = \begin{bmatrix} 0.5 & 0 \end{bmatrix}\begin{bmatrix} (2e^{-2t} - 24te^{-2t}) \\ 18te^{-2t} \end{bmatrix} = (e^{-2t} - 12te^{-2t}) \end{aligned}.$$